Bol. Soc. Paran. Mat. (3s.) **v. 2026 (44)** : 1–9.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.78335

State Space Analysis of Measurable Functions in Symmetric Algebra via Approach Structure

Boushra Y. Hussein

ABSTRACT: This paper investigates the role of approach structures in the study of algebraic systems, with particular attention to the state spaces of measurable functions in symmetric algebras. By integrating techniques from Banach algebra and approach theory, the research provides new insights into the structural properties of measurable functions and their significance in functional analysis. The findings expand the theoretical foundations of symmetric algebras, extend classical approaches to state space analysis, and demonstrate the effectiveness of approach structures in linking measure-theoretic and algebraic perspectives. In addition, the results point to potential applications in operator algebras and related areas of modern mathematics.

Key Words: Approach space, contraction, approach group, Ψ-approach normed, symmetric Banach algebra.

Contents

1. Introduction

Recent advancements in functional analysis and topology have led to the development of various generalized structures extending the classical Banach and normed spaces. Hussein and Kreeam [17,9] introduced a new structure of random approach vector spaces and random approach normed space via Banach space, offering novel insights into the relationship between probabilistic methods and functional analysis. In the context of topological structures. Furthermore, Abed Ali and Hussein [4] proposed a new type of vector space based on S-proximity structures, while Hussein and Washaych [10] explored Q-bounded functions in symmetric Ψ-Banach algebras. Hussein and Wshyayeh [11,8] further extended the study of state spaces of measurable functions in symmetric Banach algebras, presenting multiple new results. In another contribution, Saeed and Hussein [25] investigated normed approach spaces via β-approach structures, providing new analytical tools. The works of Kadhim and Hussein [14,13] on proximit structures in topological vector and metric spaces further expanded the theoretical landscape and proved proximit structure is equivalent to approach structure. Foundational concepts from Jameson [12], Kaplansky [16], Wang [26], Fleming and Jamison [6], and Kadison and Ringrose [15] underpin much of this research, as well as Hussein and Fahim [7], explored fuzzy soft ordered Banach algebras and Neamah and Hussein contributed multiple studies [22,23,24] on t^w-normed approach spaces and Ψ-character in symmetric t^w-Banach algebras, providing completion results and structural characterizations.

Additional advancements include Abbas and Hussein's works [1,2,3] on topological approach vector spaces and completion results for normed approach spaces. These studies are supported by the foundational theories of approach spaces developed by Baekeland and Lowen [5] and Lowen [19], Lowen et al. [20,21], and Li and Zhang [18]. Together, these contributions form a comprehensive foundation for the ongoing exploration of generalized Banach-type structures, approach spaces, and their associated topological and algebraic frameworks.

2010 *Mathematics Subject Classification:* 46E22, 46K15.
Submitted August 09, 2025. Published January 20, 2026

2. Preliminaries

We start this work by definition of important notion, namely Ψ-approach distance.

Definition 2.1 *[9] Let $\mathfrak{X}$ be a non-empty set. A function $\Psi : \mathfrak{X} \times 2^{\mathfrak{X}} \to [0, \infty]$ is called distance on $\mathfrak{X}$ if the following properties are satisfies:*

D1. $\forall \mathtt{d} \in \mathfrak{X} : \Psi(\mathtt{d}, \{\mathtt{d}\}) = 0,$

D2. $\forall \mathtt{d} \in \mathfrak{X} : \Psi(\mathtt{d}, \emptyset) = \infty,$

D3. $\forall \mathtt{d} \in \mathfrak{X} : \forall \mathcal{G}, P \in 2^{\mathfrak{X}} : \Psi(\mathtt{d}, \mathcal{G} \bigcup P) = \min \Psi\{(\mathtt{d}, \mathcal{G}), \Psi(\mathtt{d}, P)\},$

D4. $\forall \mathtt{d} \in \mathfrak{X} : \forall \mathcal{G} \in 2^{\mathfrak{X}}, \forall \varepsilon \in [0, \infty] : \Psi(\mathtt{d}, \mathcal{G}) \leq \Psi\left(\mathtt{d}, \mathcal{G}^{(\varepsilon)}\right) + \varepsilon.$

$\mathcal{G}$ pair $(\mathfrak{X}, \Psi)$ where Ψ is a distance is called an approach space and denoted by appr-spaces. Instead of $(D4')$ for all $\mathtt{d} \in \mathfrak{X}$ and $\mathcal{G}, P \in 2^{\mathfrak{X}}$, $\Psi(\mathtt{d}, \mathcal{G}) \leq \Psi(\mathtt{d}, P) + \sup_{b \in P} \Psi(b, \mathcal{G})$ $(D4')$ is equivalent to $(D4)$.

x

Definition 2.2 *[17] Let $(\aleph, \Psi)$ and (Y, Ψ) are appr- spaces. A function $g : \aleph \to Y$ is known as Ψ-contraction if for all $\omega < \infty$, and for all $\mathcal{G}, P \in 2^{\aleph}$, $\Psi'(g(\mathtt{d}), g(P)) \leq \Psi(\mathtt{d}, P)$,*

Definition 2.3 *[1] Let $(\aleph, \Psi)$ be Ψ- appr-space, a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ in the Ψ-appr- space $\aleph$ is said to be convergent sequence to $\mathtt{d} \in \aleph$ if $\lim_{n\to\infty} \inf_{\mathtt{d} \in P} \Psi(\mathtt{d}_n, P) = 0$ and $\lim_{n\to\infty} \sup_{\mathtt{d} \in P,} \Psi(\mathtt{d}_n, P) = 0$.*

Definition 2.4 *[3] A set $P \in 2^{\aleph}$ is said to be a cluster point in a APS $(\aleph, \Psi)$ if there exists disjoint sequence $\{\mathtt{d}_n\}_{n=1}^{\infty}$ in $\aleph$ such that $\inf_{\mathtt{d} \in P} \Psi(\mathtt{d}_n, P) = 0$, which is written by $\mathtt{d}_{n}{}_{n=1}^{\infty} \to \mathtt{d}$. We denoted the set of all cluster point in APS $\Gamma(\aleph)$.*

Definition 2.5 *[2] A sequence $\mathtt{d}_{n}{}_{n=1}^{\infty}$ in $\aleph$ is said to be Cauchy sequence in APS (Ψ–Cauchy sequence) if for every cluster point $P, \lim_{n\to\infty} \inf_{\mathtt{d} \in P, \upsilon_i \in \mathcal{G}_i} \Psi(\mathtt{d}_n, P) = 0$, a sequence $\{\mathtt{d}_n\}_{n=1}^{\infty}$ in $\aleph$ is said to be Ψ-convergent sequence in APS if there exist $\mathtt{d} \in \aleph$ for all $P \in \Gamma(\aleph), \Psi(\mathtt{d}_n, P) = 0$.*

Definition 2.6 *[1] A triple $(\aleph, \Psi, *)$ is known as Ψ- appr- semi group if and only if*

i. $(\aleph, \Psi)$ *is Ψ- appr- space;*

ii. $(\aleph, *)$ *is a semi group;*

iii. $* : \aleph \otimes \aleph \to \aleph : (a, b) \mapsto a * b$ *is Ψ-contraction.*

Definition 2.7 *[2] The triple $(\aleph, \Psi, *)$ is known as Ψ-appr- group if satisfy the following:*

i. $(\aleph, \Psi)$ *is Ψ- appr- space;*

ii. $(\aleph, *)$ *is a group;*

iii. $* : \aleph \otimes \aleph \to \aleph : (a, b) \mapsto a * b$ *is Ψ-contraction;*

iv. $\aleph \to \aleph : a \mapsto -a$ *is Ψ-contraction.*

Definition 2.8 *[3] Let F be a field and let $\aleph$ be a non-empty set with two binary operations: an addition and a scalar multiplication, Ψ is an appr- distance on $2^{\aleph}$, then, a quadruple $(\aleph, \Psi, *, \odot)$ is said to be Ψ-vector approach space if satisfy the following:*

1. $(\aleph, \Psi, *)$ *is Ψ- appr-group;*

2. $(\aleph, \Psi, \odot)$ *is Ψ-appr- semi group;*

3. $\mathfrak{m}.\mathtt{d} \in \aleph$*;*

4. $\mathfrak{m}\,(\mathtt{d}+y) = \mathfrak{m}\mathtt{d} + \mathfrak{m}\ y$ *for all* $\mathfrak{m}\ \in\ F$, *for all* $\mathtt{d}, y \in \aleph$;

5. $(\mathtt{d}+y)\,\mathfrak{m} = \mathtt{d}\mathfrak{m} + y\mathfrak{m}$ *for all* $\mathfrak{m} \in\ F$, *for all* $\mathtt{d}, y \in \aleph$;

6. $(\lambda.\mathfrak{m})\,.\mathtt{d} = \lambda.(\mathfrak{m}.\mathtt{d})$, *for all* $\mathtt{d} \in \aleph$ *and* $\lambda, \mathfrak{m} \in F$;

7. $\odot : F \times \aleph \to \mathfrak{X}, \odot(\mathfrak{m}, \mathtt{d}) = \mathfrak{m}.\mathtt{d}$ *is* Ψ*-contraction;*

8. $\mathtt{d} = \mathtt{d}$, *for all* $\mathtt{d} \in \aleph$.

Example 2.9 *Let* $\mathbb{D}$ *be a set of all real numbers, then, for all* $\omega < \infty$, *a quadruple* $(\mathbb{D}, \Psi, +, \odot)$ *with a usual addition* $(+)$ *and scalar multiplication* $(\odot)$ *is* Ψ*-appr. Vector space such that*

$$\Psi(\mathbb{G}, P) = \begin{cases} \infty & \mathbb{G} \neq \emptyset, P = \emptyset \\ 0 & \mathbb{G} \neq \emptyset, P \neq \emptyset,\ \mathbb{G} = P \\ \inf_{\varrho \in \mathbb{G}, \mathtt{d} \in P} |\varrho - \mathtt{d}| & \mathbb{G} \neq \emptyset, P \neq\ \emptyset \end{cases}$$

Proposition 2.10 *Let* $(V, \Psi, +, \odot)$ *is* Ψ*- appr- vector space, then* $(V, \Psi, +, \odot)$ *is a vector space.*

Proof: The proof is direct according to definition of t^{ω}- vector appr-space, $(V, \Psi, +, \odot)$ satisfy the condition of vector space. □

Remark 2.11 *The convers of Proposition 2.10 is not true, we show that by the following example.*

Example 2.12 *Let* $(\mathbb{D}, +, \bullet)$ *be a vector space of real numbers with an usual addition and a scalar multiplication*

$$\Psi(\mathbb{G}, P) = \begin{cases} 0 & if\ \mathtt{d}\ \in\ P \\ 5 & \mathbb{G}\ if\ \mathtt{d} \notin\ P \end{cases}$$

Obvious, $(\mathbb{D}, \Psi)$ *is not* Ψ*-appr-space .*

3. Ψ- Approach Banach Algebra

Definition 3.1 *Let* $\aleph$ *is a* Ψ*-approach linear space over the field* F. Ψ*- norm on* $\aleph$ *is a function* $\Psi : \aleph \to R$ *for all* $\mathtt{d}, y \in \aleph, \Psi$ *is an approach distance on* $\aleph$, $\mathbb{G}$ *pair* $(\aleph, \|.\|_{\Psi})$ *is called* Ψ*- approach normed space if the following conditions:*

1. $\|u\|_{\Psi} \geq 0$.

2. $\|\lambda u\|_{s} \leq \|u\|_{\Psi} \quad |\lambda| \leq 1$.

3. $\lim_{\lambda \to 0} \|\lambda u\|_{s} = 0$

4. $\|\lambda u + \mu v\|_{\Psi} \leq |\lambda + \mu|\,(\|u\|_{\Psi} + \|v\|_{\Psi}), for\ all\ \ |\lambda|, |\mu| \geq 1$.
   The pair $(\Omega, \|\cdot\|_{\Psi})$ *is called* Ψ*-normed space.*

5. $\Psi(\mathtt{d}, P) = \sup_{P \in 2^{\aleph}} \inf_{a \in P} \|\mathtt{d}\ -\ a\|_{\Psi}$

Definition 3.2 *Let* $\aleph$ *be a* Ψ*- approach normed space , then* $\aleph$ *is complete* Ψ*- approach space if every Cauchy sequence is convergent.*

Definition 3.3 *Let* $(\aleph, \Psi)$ *be* Ψ*- appr-space over the field, then* $(\aleph, \Psi, +, \cdot)$ *is called* Ψ*-approach linear algebra if satisfies the following conditions:*

1. $(\aleph, \Psi, +, \cdot)$ *is* Ψ*-approach linear space;*

2. *The operation of multiplicative defined on* $\aleph$ *satisfies the conditions for all* $\mathtt{d}, y, z \in \aleph$ *and* $\varrho \in F$*:*

i. $\varrho.(\mathtt{d}.y) = (\varrho.\mathtt{d}).y = \mathtt{d}.(\varrho.y)$;

ii. $(\mathtt{d}.y)\cdot z = \mathtt{d}\cdot(y.z)$;

iii. $(\mathtt{d}+y)\cdot z = (\mathtt{d}\cdot z + y\cdot z)$ *and* $\mathtt{d}\cdot(y+z) = \mathtt{d}\cdot y + \mathtt{d}\cdot z$;

If any $\mathtt{d}, y \in \aleph$ *satisfies* $\Psi(\mathtt{d}.y, \mathcal{G}) = \Psi(y.\mathtt{d}, \mathcal{G})$, *then* $(\aleph, \Psi, +, .)$ *is called commutative* Ψ *-approach linear algebra Let* $\aleph_s \subseteq \aleph$, *then* $\aleph_s$ *is said to be* Ψ *–approach subalgebra if satisfies the same conditions of* Ψ *–approach linear algebra.*

Definition 3.4 *Ψ-approach linear space $\aleph$ is said to be Ψ- appr- normed linear algebra if:*

1. *$\aleph$ is Ψ- approach normed space;*
2. *$\aleph$ is Ψ-appr- linear algebra;*
3. *For all , $b \in \aleph$, $\|ab\|_\Psi \leq \|a\|_\Psi \|b\|_\Psi$;*
4. *If $\aleph$ with identity ,then $\|e\|_\Psi = 1$.*

Definition 3.5 *Let $\aleph$ be a Ψ-approach normed algebra, then $\aleph$ is a complete Ψ-approach algebra if every Ψ -Cauchy sequence is convergent.*

Definition 3.6 *$\mathcal{G}$ complete Ψ- approach algebra is called a Ψ- approach Banach algebra.*

Definition 3.7 *Ψ- Banach algebra $\aleph$ is called unital if it has a unit element e such that $\Psi(e.\mathtt{d}, \mathcal{G}) = \Psi(\mathtt{d}.e, \mathcal{G}) = \Psi(\mathtt{d}, \mathcal{G})$, for all $\mathtt{d}, e \in \aleph$, $\mathcal{G} \in 2^{\aleph}$.*

Definition 3.8 *Let $\mathcal{S}$ is an algebra. We say that $\mathcal{S}$ is symmetric algebra if:*

1. *$\mathcal{S}$ is algebra .*
2. *an operation is defined in $\mathcal{S}$ which assigns to each element $\mathfrak{r}$ in $\mathcal{S}$ the element $\mathfrak{r}^*$ in $\mathcal{S}$ in such way that the following conditions are satisfied:*

   i. $(\mathfrak{r}^*)^* = \mathfrak{r}$.

   ii. $(\Psi\mathfrak{r} + \mu\mathfrak{f})^* = \overline{\Psi}\mathfrak{r}^* + \overline{\mu}\mathfrak{f}^*$.

   iii. $(\mathfrak{r}\mathfrak{f})^* = \mathfrak{r}^*\mathfrak{f}^*$.

The operation $\mathfrak{r} \to \mathfrak{r}^*$ will be called involution on an algebra $\mathcal{S}$ (if satisfies (i) – (iii)) and the elements $\mathfrak{r}^*$ and $\mathfrak{r}$ will be said to be adjoint to each other.

***Notation**: Let Inv $(\mathcal{S})$ symbols to the set of invertible elements.*

Definition 3.9 *A complete Ψ- approach normed algebra is called symmetric Ψ- approach Banach algebra.*

Definition 3.10 *[12] Let $\aleph$ be a symmetric Banach algebra. A functional T in $\aleph$ is said to be p-state functional if T is the positive linear functional.*

4. Main Result

Definition 4.1 *Let $(\aleph, \mathcal{F}, \mu)$ be a measure space, and define $L^p((\aleph, \mathcal{F}, \mu)) = L^p(\mu)$ to be the space of all measurable functions with the usual convention about identifying functions equal almost everywhere. Then a new space can be defined on $L^p(\mu)$ is called p-state space of measurable functions and denoted by $S_{\mathbb{C}}(L^p(\mu))$ such that $S_{\mathbb{C}}(L^p(\mu)) = \{p | p : L^p(\mu) \to \mathbb{C}$ is a state functional$\}$.*

Therefore $L^p(\mu)$ is Ψ- approach normed space, so, it is a symmetric Ψ - approach Banach algebra.

Theorem 4.2 *Let $S_{\mathbb{C}}(L^p(\mu))$ be the set of all p-state functional in $L^p(\mu)$, with the usual distance $\Psi : 2^{S_{\mathbb{C}}(L^p(\mu))} \times 2^{S_{\mathbb{C}}(L^p(\mu))} \to [0, \infty]$ defined on $S_{\mathbb{C}}(L^p(\mu))$ and for all $\forall\omega < \infty$, $p_1, p_2 \in S_{\mathbb{C}}(L^p(\mu))$, $\mathcal{G}, P \in 2^{S_{\mathbb{C}}(L^p(\mu))}$,*

$$\Psi(\mathcal{G}, P) = \begin{cases} \inf_{p_1 \in \mathcal{G}, p_2 \in P, \mathtt{d} \neq 0} \frac{|p_1(\varrho)_{(\mathtt{d})} - p_2(\varrho)_{(\mathtt{d})}|}{|\mathtt{d}|} & \text{if } \mathcal{G} \text{ and } P \neq \emptyset \\ \infty & \text{if } \mathcal{G} \text{ or } P \neq \emptyset \end{cases}$$

Proof: First: (I) we must prove that $L^p(\mu), \Psi, \mathtt{d}_\epsilon)$ is Ψ -appr- space.

a. Let $\Psi(\mathcal{D},\mathcal{T}) = 0$ then $\sup_{\varrho_1\in\mathcal{D}}\inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) = 0$ implies that $\varrho_1(\jmath) = \varrho_2(\jmath)$ so there exist $a \in \varrho_1(\jmath) \cap \varrho_2(\jmath)$ hence $\varrho_1(\jmath) \cap \varrho_2(\jmath) \neq \emptyset$.

b. If $\mathcal{D} = \emptyset$ then $\Psi(\mathcal{D},\mathcal{T}) = \Psi(\emptyset,\mathcal{T}) = \infty$. If $\mathcal{T} = \emptyset$ implise $\Psi(\mathcal{D},\mathcal{T}) = \Psi(\mathcal{D},\emptyset) = \infty$.

c. Let $\mathcal{D},\mathcal{T},\mathcal{C} \in 2^{L^\circ(\mu)}$.

$$\begin{aligned}\Psi(\mathcal{D},\mathcal{T}\cap\mathcal{C}) = \Psi(\mathcal{D},\emptyset) = \infty = \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \ \mathrm{V}\ \infty \\ = \max \Psi(\mathcal{D},\mathcal{T}), \Psi(\mathcal{D},\mathcal{C})\end{aligned}$$

*If $\mathcal{T} = \emptyset, \mathcal{C} = \emptyset$ then

$$\Psi(\mathcal{D},\mathcal{T}\cap\mathcal{C}) = \Psi(\mathcal{D},\emptyset) = \infty = \max \infty,\infty = \max \Psi(\mathcal{D},\mathcal{T}), \Psi(\mathcal{D},\mathcal{C})$$

*If $\mathcal{T} \neq \emptyset, \mathcal{C} \neq \emptyset$ then

$$\begin{aligned}\Psi(\mathcal{D},\mathcal{T}\cap\mathcal{C}) &= \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}\cap\mathcal{C}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \\ &= \sup_{\varrho_1\in\mathcal{D}}\inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \mathrm{V} \sup_{\varrho_1\in\mathcal{D}}\inf_{\varrho_2\in\mathcal{C}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \\ &= \max \sup_{\varrho_1\in\mathcal{D}}\inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \mathrm{V} \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \\ &= \max \Psi(\mathcal{D},\mathcal{T}), \Psi(\mathcal{D},\mathcal{C})\end{aligned}$$

d. Let $\mathcal{D},\mathcal{T} \in 2^{L^p(\mu)}$ and $\epsilon,\gamma \in \mathtt{R}^+$.

$$\begin{aligned}\Psi(\mathcal{D}^\epsilon,\mathcal{T}) &= \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \\ &= \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) \\ &\leq \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) + \epsilon + \gamma \\ &= \sup_{\varrho_1\in\mathcal{D}} \inf_{\varrho_2\in\mathcal{T}} \int \frac{|\varrho_1(\jmath)-\varrho_2(\jmath)|}{1+|\varrho_1(\jmath)-\varrho_2(\jmath)|} d\mu(\jmath) + \epsilon + \gamma \\ &= \Psi(\mathcal{D},\mathcal{T}) + \epsilon + \gamma.\end{aligned}$$

Then $(L^p(\mu), \Psi, \mathtt{d}_\epsilon)$ is Ψ-approach sp Let $\omega < \infty, \varrho \in L^p(\mu), \upsilon_i \in \aleph, i = 1,\dots,n, \mathcal{D},\mathcal{T} \in 2^{L^p(\mu)} : \varrho_1 \in A, \varrho_2 \in \mathcal{T}$ □

Theorem 4.3 *Let $L^p(\mu)$ to be the space of all measurable functions with the usual convention about identifying functions equal almost everywhere and $S_{\mathbb{C}}(L^p(\mu))$ be a space of all p-state on L^p. Then $S_{\mathbb{C}}(L^p(\mu))$ is Ψ-appr-linear space*

Proof: We prove that $(L^p(\mu), \Psi)$ be Ψ-appr- linear apace. The operations $+, \cdot$ are addition and multiplication defined as:

$$(p_1 + p_2)(\varrho_{(\mathtt{d})}) = p_1(\varrho)_{(\mathtt{d})} + p_2(\varrho)_{(\mathtt{d})};$$

$$(p_1.p_2)(\varrho_{(\mathtt{d})}) = p_1(\varrho)_{(\mathtt{d})}.p_2(\varrho)_{(\mathtt{d})};$$

With

$$\|p(\varrho)_{(\mathtt{d})}\|_{\Psi} = \int \frac{|p(\varrho)_{(\mathtt{d})}|}{1+|p(\varrho)_{(\mathtt{d})}|}\, d\mu(\mathtt{d}).$$

It is very clear that $(L^p(\mu),+)$ is a group.

Now, we prove $+ : L^p(\mu) \oplus L^p(\mu) \to L^p(\mu) : (\varrho_1,\varrho_2) \to \varrho_1+\varrho_2$ is $\Psi-$ contraction for all $\omega < \infty, \varrho_1,\varrho_2,\varrho_3,\varrho_4 \in L^p(\mu), \mathbb{G},\mathbb{P},C,D \in 2^{L^p(\mu)} : \varrho_1 \in \mathbb{G}, \varrho_2 \in \mathbb{P}, \varrho_3 \in C, \varrho_4 \in D$.

$$\begin{aligned}
\Psi'(g(\mathbb{G},\mathbb{P}),g(C,D)) &= \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P},\varrho_3\in C,\varrho_4\in D}\sum_{i=1}^{n}\left(\frac{|(\varrho_1+\varrho_2)(\upsilon_i)-(\varrho_3+\varrho_4)(\upsilon_i)|}{1+|(\varrho_1+\varrho_2)(\upsilon_i)-(\varrho_3+\varrho_4)(\upsilon_i)|}\right)\\
&= \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P},\varrho_3\in C,\varrho_4\in D}\sum_{i=1}^{n}\left(\frac{|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-(\varrho_3(\upsilon_i)+\varrho_4(\upsilon_i))|}{1+|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-(\varrho_3(\upsilon_i)+\varrho_4(\upsilon_i))|}\right)\\
&\le \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P},\varrho_3\in C,\varrho_4\in D}\left(\sum_{i=1}^{n}\left(\frac{|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_3(\upsilon_i)|}{1+|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_3(\upsilon_i)|}\right)\right.\\
&\left.+\sum_{i=1}^{n}\left(\frac{|\varrho_4(\upsilon_i)-(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))|}{1+|\varrho_4(\upsilon_i)-(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))|}\right)\right)\\
&\le \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P},\varrho_3\in C}\sum_{i=1}^{n}\left(\frac{|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_3(\upsilon_i)|}{1+|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_3(\upsilon_i)|}\right)\\
&+\inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P},\varrho_4\in D}\sum_{i=1}^{n}\left(\frac{|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_4(\upsilon_i)|}{1+|(\varrho_1(\upsilon_i)+\varrho_2(\upsilon_i))-\varrho_4(\upsilon_i)|}\right)\\
&\le \inf_{\varrho_1\in\mathbb{G},\varrho_3\in C}\sum_{i=1}^{n}\left(\frac{|\varrho_1(\upsilon_i)-\varrho_3(\upsilon_i)|}{1+|\varrho_1(\upsilon_i)-\varrho_3(\upsilon_i)|}\right)\\
&+\inf_{\varrho_2\in\mathbb{P},\varrho_4\in D}\sum_{i=1}^{n}\left(\frac{|\varrho_2(\upsilon_i)-\varrho_4(\upsilon_i)|}{1+|\varrho_2(\upsilon_i)-\varrho_4(\upsilon_i)|}\right)\\
&= \Psi(\mathbb{G},C)+\Psi(\mathbb{P},D) \le \Psi(\mathbb{G},C)+\Psi(\mathbb{P},D),
\end{aligned}$$

$$\begin{aligned}
\Psi'(g(\mathbb{G}),\ g(\mathbb{P})) &= \Psi'(-\mathbb{G},-\mathbb{P})\\
&= \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P}}\sum_{i=1}^{n}\left(\frac{|-\varrho_1(\upsilon_i)-(-\varrho_2(\upsilon_i))|}{1+|-\varrho_1(\upsilon_i)-(-\varrho_2(\upsilon_i))|}\right)\\
&= \inf_{\varrho_1\in\mathbb{G},\varrho_2\in\mathbb{P}}\sum_{i=1}^{n}\left(\frac{|\varrho_1(\upsilon_i)-\varrho_2(\upsilon_i)|}{1+|\varrho_1(\upsilon_i)-\varrho_2(\upsilon_i)|}\right) = \Psi(\mathbb{G},\mathbb{P}),
\end{aligned}$$

Then $(L^p(\mu),\Psi,+)$ is Ψ- approach group. it is clear that $(\varrho_1(\mathtt{d})+\varrho_2(\mathtt{d})),(\varrho f)(\mathtt{d}),\varrho_1(\mathtt{d})\varrho_2(\mathtt{d})$ are measurable functions . Then $(S_{\mathbb{C}}(L^p(\mu)),\Psi,+,)$ is Ψ-approach linear space. □

Theorem 4.4 *Let $S_{\mathbb{C}}(L^p(\mu))$ be the set of all p-state functional in $L^p(\mu)$. Then $S_{\mathbb{C}}(L^p(\mu))$ is symmetric and complete space.*

Proof: First: To prove $S_{\mathbb{C}}(L^p(\mu))$ symmetric Let $* : S_{\mathbb{C}}(L^p(\mu)) \to S_{\mathbb{C}}(L^p(\mu))$ defined by $*\left(p(\varrho)_{(\mathtt{d})}\right) = \overline{p(\varrho)_{(\mathtt{d})}}$, $*$ is satisfying involution axioms such that $\|\, p\,\|_{\Psi} = \int |p| d\mu(\mathtt{d})$

1. $\left(p(\varrho)_{(\mathtt{d})}\right)^{**} = (p(\varrho)_{(\mathtt{d})})$, since $\left(p(\varrho)_{(\mathtt{d})}\right)^{**} = ((p(\varrho)_{(\mathtt{d})})^*)^*$, We obtain $p^*(\varrho)_{(\mathtt{d})} = \left(\overline{\left(p(\varrho)_{(\mathtt{d})}\right)}\right)^* = \overline{\overline{\left(p(\varrho)_{(\mathtt{d})}\right)}} = \left(p(\varrho)_{(\mathtt{d})}\right)$;

2. $\left[\varrho p_1(\varrho)_{(\mathtt{d})}+\beta p_2(\varrho)_{(\mathtt{d})}\right]^*=\bar{\varrho}\overline{p_1(\varrho)_{(\mathtt{d})}}+\bar{\beta}\,\overline{p_2(\varrho)_{(\mathtt{d})}}$

$=\bar{\varrho}p_1^*(\varrho)_{(\mathtt{d})}+\bar{\beta}p_2^*(\varrho)_{(\mathtt{d})}$;

3. $[p_1.p_2]^*(\varrho)_{(\mathtt{d})}=\frac{1}{2}\left[\overline{(p_1+p_2)^2(\varrho)_{(\mathtt{d})}}-\overline{p_1{}^2(\varrho)_{(\mathtt{d})}}-\overline{p_2{}^2(\varrho)_{(\mathtt{d})}}\right]$
$=\frac{1}{2}\left[\overline{\left(p_1(\varrho)_{(\mathtt{d})}+p_2(\varrho)_{(\mathtt{d})}\right)^2}-\overline{p_1{}^2(\varrho)_{(\mathtt{d})}}-\overline{p_2{}^2(\varrho)_{(\mathtt{d})}}\right]=p_1^*(\varrho)_{(\mathtt{d})}.p_2^*(\varrho)_{(\mathtt{d})}$.

Now, we must prove The function $*:S_{\mathbb{C}}(L^p(\mu))\to S_{\mathbb{C}}(L^p(\mu)):p\to\ p^*$ is $\Psi-contraction for all \omega<\infty, f\in\ L^p(\mu),\mathbb{G},\mathbb{P}\in 2^{S_{\mathbb{C}}(L^p(\mu))}$.

$$\begin{aligned}\Psi'(g(\mathbb{G}),\ g(\mathbb{P}))&=\Psi'\left(\mathbb{G}^*,\mathbb{P}^*\right)=\ \inf_{p_1\in\mathbb{G},\ p_2\in\mathbb{P}}|p_1^*(\varrho)_{(\mathtt{d})}-\ p_2^*(\varrho)_{(\mathtt{d})}|\\&=\inf_{p_1\in\mathbb{G},\ p_2\in\mathbb{P}}\left|\overline{(p_1)^*(\varrho)_{(\mathtt{d})}}-\overline{(p_2)^*(\varrho)_{(\mathtt{d})}}\right|=\inf_{p_1\in\mathbb{G},\ p_2\in\mathbb{P}}\left|\overline{\overline{\left(p_1(\varrho)_{(\mathtt{d})}\right)}}-\overline{\overline{\left(p_2(\varrho)_{(\mathtt{d})}\right)}}\right|\\&=\inf_{p_1\in\mathbb{G},p_2\in\mathbb{P}}\left|p_1(\varrho)_{(\mathtt{d})}-p_2(\varrho)_{(\mathtt{d})}\right|=t^\omega(\mathbb{G},\mathbb{P})\le kt^\omega(\mathbb{G},\mathbb{P})\ ,\ for\ some\ k\in[0,1].\end{aligned}$$

Thus, $(S_{\mathbb{C}}(L^p(\mu)),\|.\|_\Psi,\Psi_{\|.\|_\Psi})$ is symmetric Ψ - approach space.

Second: Now, we must prove that $(S_{\mathbb{C}}(L^p(\mu)),\|.\|_\Psi,\ \Psi_{\|.\|_\Psi})$ is a complete Ψ-approach normed algebra.

Now, let $\mathbb{G}_n,\mathbb{P}\in 2^{S_{\mathbb{C}}(L^p(\mu))}$ and let $\{\mathbb{G}_n\}_{n=1}^\infty$ be a Ψ-Cauchy sequence in $(S_{\mathbb{C}}(L^p(\mu))),\Psi),\{\mathbb{G}_n\}_{n=1}^\infty\to\mathbb{P}$, we denoted the set of all cluster points in Ψ-approach space by $\Gamma(S_{\mathbb{C}}(L^p(\mu))),\mathbb{P}\in\Gamma(S_{\mathbb{C}}(L^p(\mu)))$ and

$$\Psi(\mathbb{G},\mathbb{P})=\begin{cases}\inf_{p_i(\varrho)\in\mathbb{G}_i,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\frac{|p_i(\varrho)_{(\mathtt{d})}-p(\varrho)_{(\mathtt{d})}|}{|\mathtt{d}|},i=1,\dots n\in Z^+ & if\ \mathbb{G}_n\ and\ \mathbb{P}\ \neq\emptyset\\ \infty & if\ \{\mathbb{G}_n\}=\emptyset\ or\ \mathbb{P}=\emptyset\end{cases}$$

This means, for all $\mathbb{P}\in\Gamma\left(2^{S_{\mathbb{C}}(L^p(\mu))}\right),\lim_{n\to\infty}\inf_{p_i(\varrho)\in\mathbb{G}_i,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\Psi(\mathbb{G}_n,\mathbb{P})=0,i=1,\dots,n\ \in Z^+$, so, by Definition 2.3,

$$\lim_{n\to\infty}\inf_{\mathbb{P}\in\Gamma\left(2^{(S_{\mathbb{C}}(L^p(\mu)))}\right)}\inf_{p_n(\varrho)\in\mathbb{G}_n,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\frac{|p_n(\varrho)_{(\mathtt{d})}-p(\varrho)_{(\mathtt{d})}|}{|\mathtt{d}|}=0,$$

and

$$\lim_{n\to\infty}\sup_{\mathbb{P}\in\Gamma\left(2^{(S_{\mathbb{C}}(L^p(\mu)))}\right)}\inf_{p_n(\varrho)\in\mathbb{G}_n,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\frac{|p_n(\varrho)_{(\mathtt{d})}-p(\varrho)_{(\mathtt{d})}|}{|\mathtt{d}|}=0.$$

Thus, $\{\mathbb{G}_n\}_{n=1}^\infty$ is convergent in $(S_{\mathbb{C}}(L^p(\mu))),\Psi)$.

Now, we must prove that $(S_{\mathbb{C}}(L^p(\mu))),\|.\|_\Psi,\Psi_{\|.\|_\Psi})$ is complete Ψ -approach normed algebra. Let $\{\mathbb{G}_n\}_{n=1}^\infty$ be a Ψ -Cauchy sequence in $(S_{\mathbb{C}}(L^p(\mu)),\Psi),\mathbb{G}_n,\mathbb{P}\in 2^{S_{\mathbb{C}}(L^p(\mu))}$ and let $p(\varrho)\in\mathbb{P}$,This means, for all $\mathbb{P}\in\ \Gamma\left(2^{S_{\mathbb{C}}(L^p(\mu))}\right),\lim_{n\to\infty}\inf_{p_i(\varrho)\in\mathbb{G}_i,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\Psi(\mathbb{G}_n,\mathbb{P})=0,i=1,\dots,n\in Z^+$

$$\begin{aligned}0&=\lim_{n\to\infty}\inf_{p_i(\varrho)\in\mathbb{G}_i,p(\varrho)\in\mathbb{P},\mathtt{d}\neq0}\Psi(\mathbb{G}_n,\mathbb{P})=\lim_{n\to\infty}\inf_{p_i(\varrho)\in\mathbb{G}_i,p(\varrho)\in\mathbb{P}}d(p_i(\varrho),p(\varrho))\\&=\lim_{n\to\infty}\inf_{p_j(\varrho)\in\mathbb{G}_j\subseteq\mathbb{P}}\inf_{p_i(\varrho)\in\mathbb{G}_i}d(p_i(\varrho),p_j(\varrho))\\&=\ \lim_{n\to\infty}\inf_{p_j(\varrho)\in\mathbb{G}_j\subseteq\mathbb{P}}\inf_{p_i(\varrho)\in\mathbb{G}_i}\|p_i(\varrho)-p_j(\varrho)\|_\Psi;\end{aligned}$$

$j=1,\dots,n\in Z^+$;

that is,

$$\begin{aligned}
&\lim_{n\to\infty} \inf_{p_i(\varrho)\in\mathbb{G}_i, p(\varrho)\in\mathrm{P}} \|p_i(\varrho) - p(\varrho)\|_\Psi \\
&= \lim_{n\to\infty} \inf_{p_i(\varrho)\in\mathbb{G}_i, p(\varrho)\in\mathrm{P}} \int \frac{|p_i(\varrho)(\mathtt{d}) - p(\varrho)(\mathtt{d})|}{1 + |p_i(\varrho)(\mathtt{d}) - p(\varrho)(\mathtt{d})|}\, d\mu(\mathtt{d}) \\
&\leq \lim_{n\to\infty} \inf_{p_i(\varrho)\in\mathbb{G}_i, p(\varrho)\in\mathrm{P}} \int \frac{|p_i(\varrho)(\mathtt{d})|}{1 + |p_i(\varrho)(\mathtt{d})|}\, d\mu(\mathtt{d}) \\
&+ \lim_{n\to\infty} \inf_{p_i(\varrho)\in\mathbb{G}_i, p(\varrho)\in\mathrm{P}} \int \frac{|p(\varrho)(\mathtt{d})|}{1 + |p(\varrho)(\mathtt{d})|}\, d\mu(\mathtt{d}) = 0.
\end{aligned}$$

□

Then, for all $p_i(\varrho), p(\varrho) \in S_{\mathbb{C}}(L^p(\mu))$ $\lim_{n\to\infty} \inf_{p_i(\varrho)\in\mathbb{G}_i, p(\varrho)\in\mathrm{P}} \|p_i(\varrho) - p(\varrho)\|_\Psi = 0$

Thus, $(S_{\mathbb{C}}(L^p(\mu)), \|\|_\Psi, \Psi_{\|.\|_\Psi})$ is complete Ψ- approach normed algebra.

Therefore, $S_{\mathbb{C}}(L^p(\mu))$ is Ψ - Ψ- approach normed space, so, it is a symmetric Ψ- approach Banach algebra.

5. Conclusion

In this paper, we introduced some important definitions and we construct that the Ψ - approach normed space $(S_{\mathbb{C}}(L^p(\mu)), \|.\|_\Psi, \Psi_{\|.\|_\Psi})$ are complete Ψ- approach normed algebra, also, they are symmetric Ψ- approach Banach algebra.

References

1. R. K. Abbas and B. Y. Hussein, *New results of completion normed approach space*, AIP Conference Proceedings **2845** (2023), 050036.
2. R.K. Abbas and Boushra Y. Hussein, *New results of normed approach space*, Iraqi Journal of Science **63** (2022), no. 5, 2103–2113.
3. R.K. Abbas and B.Y. Hussein, *A new kind of topological vector space: Topological approach vector space*, AIP Conference Proceedings **2386** (2022), no. 1, 060008.
4. Safia S. Abed Ali and Boushra Y. Hussein, *New kind of vector space via ş-proximit structure*, E3S Web of Conferences **508** (2024), 04012.
5. R. Baekeland and R. Lowen, *Measures of lindelof and separability in approach spaces*, International Journal of Mathematics and Mathematical Sciences **17** (1994), no. 3, 597–606.
6. R.J. Fleming and J.E. Jamison, *Hermitian operators on c (x, e) and the banach-stone theorem*, Mathematische Zeitschrift **170** (1980), 77–84.
7. B. Y. Hussein and S. Fahim, *Characterization on fuzzy soft ordered banach algebra*, Boletim da Sociedade Paranaense de Matemática **40** (2022), 1–8, Series 3.
8. B. Y. Hussein and H. A. Wshayeh, *On state space of measurable function in symmetric Δ-banach algebra with new results*, AIP Conference Proceedings **2398** (2022), no. 1, 060064.
9. Boushra Y. Hussein and A. A. Kream, *A random approach vector space: New structure*, Boletim da Sociedade Paranaense de Matemática **43** (2025), 1–10.
10. Boushra Y. Hussein and Hadeer A. Wshayeh, *New results of q-bounded functional in symmetric δ-banach algebra*, AIP Conference Proceedings **2845** (2023), no. 1, 050040.
11. B.Y. Hussein and Huda A.A. Wshayeh, *On state space of measurable function in symmetric δ-banach algebra with new results*, AIP Conference Proceedings **2398** (2022), no. 1, 060064.
12. G. J. O. Jameson, *Topology and normed spaces*, Chapman and Hall, London, 2009.
13. D. A. Kadhim and B. Y. Hussein, *A new type of metric space via proximity structure*, Journal of Interdisciplinary Mathematics **26** (2023), no. 6, 1053–1064.
14. D.A. Kadhim and B.Y. Hussein, *New kind of topological vector space via proximity structure*, Journal of Interdisciplinary Mathematics **26** (2023), no. 6, 1065–1075.
15. R.V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras: Elementary theory*, Academic Press, Inc., London, 1983.

16. I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, American Mathematical Society, Providence, RI, USA, 1970.

17. A.A. Kream and B.Y. Hussein, *A new structure of random approach normed space via banach space*, Iraqi Journal of Science **65** (2024), no. 10, 5617–5628.

18. W. Li and D. Zhang, *Sober metric approach spaces*, Topology and its Applications **233** (2018), 67–88.

19. R. Lowen, *Approach spaces: The missing link in the topology—uniformity—metric triad*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1997.

20. R. Lowen, D. Vaughan, and M. Sioen, *Completing quasi metric spaces: an alternative approach*, Houston Journal of Mathematics **29** (2003), no. 1, 113–136.

21. R. Lowen and S. Verwulgen, *Approach vector spaces*, Houston Journal of Mathematics **30** (2004), no. 4, 1127–1142.

22. M. A. Neamah and B. Y. Hussein, *On δ-character in symmetric t^{ω}-Δ-algebra*, Journal of Discrete Mathematical Sciences and Cryptography **26** (2023), no. 7, 1939–1947.

23. M.A. Neamah and B.Y. Hussein, *New result of t^w normed approach space*, Journal of Interdisciplinary Mathematics **26** (2023), no. 6, 1109–1211.

24. Maysoon A Neamah and Boushra Y Hussein, *Some new results of completion t^{ω}-normed approach space*, Periodicals of Engineering and Natural Sciences (PEN) **10** (2022), no. 5, 82–89.

25. S. Saeed and B. Y. Hussein, *New results of normed approach space via β-approach structure*, Iraqi Journal of Science (2023), 3538–3550.

26. J.K. Wang, *Multipliers of commutative banach algebra*, Pacific Journal of Mathematics **11** (1961), 1131–1149.

Boushra Y. Hussein,

Department of Mathematics,

College of Education, University of Al-Qadisiyah,

Iraq.

E-mail address: `boushra.alshebani@qu.edu.iq`