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Structural Properties of Pairwise Difference lindelöf Spaces: Statistical Applications in Data Analysis

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ABSTRACT: This paper formalizes D-Lindelöf spaces in bitopological spaces, extending Lindelöf concept through D-sets. Key contributions include proving countable spaces are generally pairwise D-Lindelöf; establishing that pairwise D-Lindelöfness implies η_1-Lindelöfness; demonstrating closure of D-sets under finite intersections; and analyzing preservation under pairwise continuous surjections, D-irresolute functions, and perfect functions. The inheritance conditions for η_1-closed subspaces are determined, with limitations clarified by counterexamples. Beyond theory, we demonstrate statistical applications: D-set classification for mixed feature data, convergence analysis under dual topologies, D-compactness in probability measures, and parameter identification modeling.

Key Words: Topological space, $D-$set, $D-$Lindelöf space, locally indiscrete space, $D-$Lindelöf perfect functions.

Contents

1. Introduction and Preliminaries

The interplay of covering properties and separation axioms is an essential feature of general topology, providing deep understanding into the structural and behavioral characteristics of topological spaces. This paper systematically investigates the concepts of pairwise D-Lindelöf spaces and D-Lindelöf perfect functions in the complex structure of bitopological spaces. By strategically employing D-sets, our work extends classical Lindelöf theory into asymmetric topological environments. This approach effectively addresses significant gaps in our current understanding of bitopological compactness and its functional invariants.

Our research builds upon the seminal contributions of Tong [19] concerning D-sets and their corresponding separation axioms (D_i, for $i = 0, 1, 2$). In this theoretical context, we we introduce and rigorously define the concept of pairwise D-Lindelöfness. The new covering property requires every pairwise D-cover of a space to admit a countable subcover. This property synthesizes the interactions between two distinct topologies, offering a significant generalization of traditional Lindel's spaces. Furthermore, it addresses long-standing questions about compactness in non-Hausdorff and mixed-topology settings.

While recent scholarly efforts by Alrababah et al. [1] on difference paracompactness, Atoom et al. [9] on difference ParaLindelöf, exploring separation axioms and limit points in $[d, e]$-Lindelöfness spaces through the use of $[d, e]$-open covers [7], and Mahmoud et al. [15] on soft bitopological D-sets have advanced related areas, a comprehensive theoretical treatment of D-Lindelöfness within bitopological spaces

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has remained conspicuously absent, see more [2,5,12,13,16,17,18,20] . This study aims to bridge this critical gap by making the following key contributions: We establish the fundamental properties of pairwise D-Lindelöf spaces. We define D-Lindelöf perfect functions and characterize their preservation properties in section 2. analyze the behavior of this inheritance characteristics within subspaces in section 3. Establishes conditions under which pairwise D-Lindelöfness is preserved in subspaces of bitopological spaces in section4. And in section 5, explores the potential applications of pairwise D-Lindelöf bitopological spaces and their structural properties to statistical theory and data analysis.

To ensure the clarity and self-contained nature of this exposition, we begin by recalling a few key definitions that will be used frequently throughout the following sections.

Definition 1.1 *[19] A subset A of a topological space $(\mathcal{L}, \eta)$ is designated as a D-set if there is open sets U and V such that $U \neq \mathcal{L}$ and $A = U \setminus V$. In this context, we state that A is a D-set generated by U and V.*

It is noteworthy that any open set U distinct from $\mathcal{L}$ can be considered a D-set by setting $V = \emptyset$.

Definition 1.2 *[8] A collection $\tilde{D} = \{D_{\mathfrak{k}} : \mathfrak{k} \in \mathtt{K}\}$ of subsets of a topological space $(\mathcal{L}, \eta)$ is referred to as a D-cover if each $D_{\mathfrak{k}}$ is a D-set for all $\mathfrak{k} \in \mathtt{K}$.*

Definition 1.3 *[8] A topological space $(\mathcal{L}, \eta)$ is termed D-Lindelöf if every D-cover of $(\mathcal{L}, \eta)$ admits a countable subcover.*

Definition 1.4 *[6] A topological space $(\mathcal{L}, \eta)$ is classified as D-countably compact if every countable D-cover of $(\mathcal{L}, \eta)$ admits a finite subcover.*

Definition 1.5 *[11] A function $\phi\colon (\mathcal{L}, \eta) \to (\mathfrak{O}, \xi)$ is defined as D-perfect if ϕ is continuous, closed, and for every $o \in \mathfrak{O}$, the preimage $\phi^{-1}(o)$ is D-compact.*

Definition 1.6 *[14] A space $(\mathcal{L}, \eta)$ is characterized as locally indiscrete if every open set in $\mathcal{L}$ is also clopen.*

Definition 1.7 *[3] A function $\phi\colon (\mathcal{L}, \eta) \to (\mathfrak{O}, \xi)$ is termed a strongly function if for every open cover $\mathcal{U} = \{U_{\mathfrak{k}} : \mathfrak{k} \in \mathtt{K}\}$ of $\mathcal{L}$, there exists an open cover $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ of $\mathfrak{O}$ such that:*

$$\phi^{-1}(V) \subseteq \bigcup \{U_{\mathfrak{k}} : \mathfrak{k} \in \mathtt{K}'\} \quad \forall V \in \mathcal{V},$$

where $\mathtt{K}'$ is a countable subset of $\mathtt{K}$.

Definition 1.8 *[4] In a bitopological space $(\mathcal{L}, \eta_1, \eta_2)$, a subset $A \subseteq \mathcal{L}$ is defined as a pairwise D-set if there exist $U \in \eta_1$ and $V \in \eta_2$ such that $U \neq \mathcal{L}$ and $A = U \setminus V$. In this case, we assert that A is generated by U and V.*

Definition 1.9 *[4] A collection $\mathcal{D} = \{D_{\mathfrak{k}}\}_{\mathfrak{k} \in \mathtt{K}}$ consisting of pairwise D-sets is designated as a pairwise D-cover of $\mathcal{L}$ if $\bigcup_{\mathfrak{k} \in \mathtt{K}} D_{\mathfrak{k}} = \mathcal{L}$.*

2. Pairwise D-Lindelöf Spaces

This section discusses pairwise D-Lindelö of bitopological spaces, which have countable subcovers of pairwise D-covers. It determines cardinality-based adequate conditions, the preservation under continuous and irresolute functions, and closure properties of pairwise (D)-sets. Counterexamples indicated limitations in set operations and non-preservation of property in unions.

Definition 2.1 *A bitopological space $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf if every pairwise D-cover of $\mathcal{L}$ admits a countable subcover.*

Example 2.2 *Consider* $(\mathbb{N}, \eta_{dis}, \eta_{cof})$ *is pairwise D-sets; For any* $U \in \eta_{dis}$ *with* $U \neq \mathbb{N}$ *and* $V \in \eta_{cof}$, $D = U \setminus V$.

To determine that is pairwise D-Lindelöf; let $\mathcal{D} = \{D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}\}_{\mathfrak{k}\in I}$ *be a pairwise D-cover of* $\mathbb{N}$. *Since* $\{U_{\mathfrak{k}}\}_{\mathfrak{k}\in I}$ *is a* η_{dis}*-open cover and* $\mathbb{N}$ *is countable, there is a countable subcover* $\{U_{\mathfrak{k}_n}\}_{n=1}^{\infty}$. *Let* $F = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} D_{\mathfrak{k}_n}$. *For every* $l \in F$, *if* $l \in U_{\mathfrak{k}_n}$ *for some* n, *then* $l \in V_{\mathfrak{k}_n}$. *Thus* $F \subseteq \bigcup_{n=1}^{\infty} V_{\mathfrak{k}_n}$. *Since* $\mathbb{N}$ *is countable,* F *is countable. For each* $l \in F$, *select* $D_{\beta_l} \in \mathcal{D}$ *with* $l \in D_{\beta_l}$. *Then* $\{D_{\mathfrak{k}_n}\}_{n=1}^{\infty} \cup \{D_{\beta_l}\}_{l\in F}$ *is a countable subcover.*

Example 2.3 $(\mathbb{Q}, \eta_{Euclidean}, \eta_{dis})$ *is pairwise D-sets; For* $U \in \eta_{Euc}$ $(U \neq \mathbb{Q})$ *and* $V \in \eta_{dis}$, $D = U \setminus V$.

The space is pairwise D-Lindelöf; Let $\mathcal{D} = \{D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}\}_{\mathfrak{k}\in I}$ *be a pairwise D-cover. Since* $\{U_{\mathfrak{k}}\}_{\mathfrak{k}\in I}$ *is a* η_{Euc}*-open cover and* $\mathbb{Q}$ *is second-countable (hence Lindelöf), there exists a countable subcover* $\{U_{\mathfrak{k}_n}\}_{n=1}^{\infty}$. *Let* $F = \mathbb{Q} \setminus \bigcup_{n=1}^{\infty} D_{\mathfrak{k}_n}$. *Then* $F \subseteq \bigcup_{n=1}^{\infty} V_{\mathfrak{k}_n}$, *since if* $l \in U_{\mathfrak{k}_n}$ *but* $l \notin D_{\mathfrak{k}_n}$, *then* $l \in V_{\mathfrak{k}_n}$. *As* $\mathbb{Q}$ *is countable,* F *is countable. For each* $l \in F$, *choose* $D_{\beta_l} \in \mathcal{D}$ *containing* l. *Then* $\{D_{\mathfrak{k}_n}\}_{n=1}^{\infty} \cup \{D_{\beta_l}\}_{l\in F}$ *is a countable subcover.*

Cardinality has a significant impact on covering characteristics; countability alone ensures pairwise D-Lindelöfness, regardless of topology.

Theorem 2.4 *If* $\mathcal{L}$ *is a countable set, then* $(\mathcal{L}, \eta_1, \eta_2)$ *is pairwise D-Lindelöf for any* (η_1, η_2) *on* $\mathcal{L}$.

Proof: Consider $\mathcal{L} = \{\mathcal{L}_n\}_{n=1}^{\infty}$ be enumerated. Given a pairwise D-cover $\mathcal{D} = \{D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}\}_{\mathfrak{k}\in \mathtt{K}}$ with $U_{\mathfrak{k}} \in \eta_1$, $V_{\mathfrak{k}} \in \eta_2$, $U_{\mathfrak{k}} \neq \mathcal{L}$, define a selection function: For each $n \in \mathbb{N}$, choose $\mathfrak{k}_n \in \mathtt{K}$ such that $l_n \in D_{\mathfrak{k}_n}$. $\{D_{\mathfrak{k}_n}\}_{n=1}^{\infty}$ is a countable subcover:

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \{l_n\} \subseteq \bigcup_{n=1}^{\infty} D_{\mathfrak{k}_n}$$

□

This cardinality-based result extends trivially from finite to countable spaces:

Remark 2.5 *Theorem 2.4 applies pairwise D-Lindelöfness to countable spaces through countable subcovers, proving that countability is sufficient for pairwise D-Lindelöfness regardless of topology.*

Cardinality decides on covering properties in bitopological spaces, with finite and countable sets demonstrating universal Lindelöf characteristics:

Theorem 2.6 *For any finite or countable set* $\mathcal{L}$ *and bitopology* (η_1, η_2):

1. $(\mathcal{L}, \eta_1, \eta_2)$ *is pairwise D-Lindelöf*
2. $(\mathcal{L}, \eta_1, \eta_2)$ *is pairwise Lindelöf*

Proof: Part (1): Let $\mathcal{D} = \{D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}\}_{\mathfrak{k}\in \mathtt{K}}$ be a pairwise D-cover. Enumerate $\mathcal{L} = \{l_n\}_{n=1}^{N}$ (finite) or $\mathcal{L} = \{l_n\}_{n=1}^{\infty}$ (countable). For each l_n, select $\mathfrak{k}_n \in \mathtt{K}$ such that $l_n \in D_{\mathfrak{k}_n}$. Then $\{D_{\mathfrak{k}_n}\}$ is a countable (or finite) subcover since:

$$\bigcup_n D_{\mathfrak{k}_n} \supseteq \bigcup_n \{l_n\} = \mathcal{L}.$$

Part (2): Let $\mathcal{U} = \{U_\beta\}_{\beta\in B}$ be a pairwise open cover $(U_\beta \in \eta_1 \cup \eta_2)$. For each $l_n \in \mathcal{L}$, choose $U_{\beta_n} \ni l_n$. Then $\{U_{\beta_n}\}$ is a countable subcover. □

The rational numbers illustrate Theorem 2.6 in non-trivial topologies:

Example 2.7 *Let $\mathcal{L} = \mathbb{Q}$ with $\eta_1 = \eta_{std}$, $\eta_2 = \eta_{cof}$. By Theorem 2.6, $\mathcal{L}$ is pairwise D-Lindelöf. For example: The pairwise D-cover $\mathcal{D} = \{D_q\}_{q\in\mathbb{Q}}$ where*

$$D_q = \{q\} = (\mathbb{R} \setminus \{p\}) \cap \mathbb{Q} \setminus [(\mathbb{R} \setminus \{q, p\}) \cap \mathbb{Q}] \quad (p \neq q)$$

admits the countable subcover $\mathcal{D}$ itself since $|\mathbb{Q}| = \aleph_0$.

Although pairwise D-Lindelöfness conveys Lindelöfness in one topology, it does not ensure entirely pairwise Lindelöfness.

Theorem 2.8 *If $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf, then it is η_1-Lindelöf.*

Proof: Let $\mathcal{U} = \{U_\mathfrak{k}\}_{\mathfrak{k}\in K}$ be a η_1-open cover. Every $U_\mathfrak{k}$ is a pairwise D-set:

$$U_\mathfrak{k} = U_\mathfrak{k} \setminus \emptyset \quad \text{with} \quad U_\mathfrak{k} \in \eta_1, \ \emptyset \in \eta_2.$$

Thus $\mathcal{U}$ is a pairwise D-cover. By pairwise D-Lindelöfness, there exists a countable subcover $\{U_{\mathfrak{k}_n}\}_{n\in\mathbb{N}}$. Hence, $\mathcal{L}$ is η_1-Lindelöf. □

Proposition 2.9 *If $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf and every η_2-open set is η_1-closed, then $\mathcal{L}$ is η_2-Lindelöf.*

Proof: Consider $\mathcal{V} = \{V_\beta\}_{\beta\in B}$ be a η_2-open cover of $\mathcal{L}$. each V_β is η_1-closed. For all $l \in \mathcal{L}$, define:

$$C_l = \bigcap\{V_\beta : l \in V_\beta\}.$$

Each C_l is η_1-closed, and $l \in C_l$. Thus $\{C_l\}_{l\in\mathcal{L}}$ is a cover of $\mathcal{L}$ by η_1-closed sets.

By 2.8, $\mathcal{L}$ is η_1-Lindelöf. Hence, there is a countable set $\{l_n\}_{n\in\mathbb{N}}$ such that:

$$\mathcal{L} = \bigcup_{n\in\mathbb{N}} C_{l_n}.$$

For each n, choose V_{β_n} such that $l_n \in V_{\beta_n}$. Then $C_{l_n} \subseteq V_{\beta_n}$ since C_{l_n} is the intersection of all V_β containing l_n. Therefore:

$$\mathcal{L} = \bigcup_{n\in\mathbb{N}} C_{l_n} \subseteq \bigcup_{n\in\mathbb{N}} V_{\beta_n},$$

proving $\mathcal{L}$ is η_2-Lindelöf. □

Our initially result demonstrates that pairwise D-Lindelöfness admires continuous functions.

Theorem 2.10 *Consider $\phi : (\mathcal{L}, \eta_1, \eta_2) \overset{onto}{\to} (\mathfrak{O}, \xi_1, \xi_2)$ is a pairwise continuous. If $\mathcal{L}$ is pairwise D-Lindelöf, then $\mathfrak{O}$ is pairwise D-Lindelöf.*

Proof: Let $\mathcal{D} = \{D_\mathfrak{k}\}_{\mathfrak{k}\in K}$ be a pairwise D-cover of $\mathfrak{O}$, where each $D_\mathfrak{k} = U_\mathfrak{k} \setminus V_\mathfrak{k}$ for ξ_1-open $U_\mathfrak{k} \neq \mathfrak{O}$ and ξ_2-open $V_\mathfrak{k}$. Since ϕ is pairwise continuous, $\phi^{-1}(U_\mathfrak{k})$ is η_1-open and $\phi^{-1}(V_\mathfrak{k})$ is η_2-open in $\mathcal{L}$. Define

$$E_\mathfrak{k} = \phi^{-1}(D_\mathfrak{k}) = \phi^{-1}(U_\mathfrak{k}) \setminus \phi^{-1}(V_\mathfrak{k}).$$

Then $\mathcal{E} = \{E_\mathfrak{k}\}_{\mathfrak{k}\in K}$ is a pairwise D-cover of $\mathcal{L}$. As $\mathcal{L}$ is pairwise D-Lindelöf, there is a countable subcover $\{E_{\mathfrak{k}_n}\}_{n\in\mathbb{N}}$. Surjectivity of ϕ implies

$$\mathfrak{O} = \phi(\mathcal{L}) = \phi\left(\bigcup_{n\in\mathbb{N}} E_{\mathfrak{k}_n}\right) = \bigcup_{n\in\mathbb{N}} \phi(E_{\mathfrak{k}_n}) = \bigcup_{n\in\mathbb{N}} D_{\mathfrak{k}_n}.$$

Thus, $\{D_{\mathfrak{k}_n}\}_{n\in\mathbb{N}}$ is a countable D-subcover of $\mathfrak{O}$. □

Continuous functions preserve Lindelöf and countable compactness variants:

Theorem 2.11 *Consider* $\phi\colon (\mathcal{L}, \eta_1, \eta_2) \overset{onto}{\to} (\mathfrak{O}, \beta_1, \beta_2)$ *be pairwise continuous. Then:*

1. *If* $\mathcal{L}$ *is pairwise D-countably compact, then* $\mathfrak{O}$ *is pairwise D-countably compact.*
2. *If* $\mathcal{L}$ *is pairwise D-Lindelöf, then* $\mathfrak{O}$ *is pairwise D-Lindelöf.*

Proof: (1) Let $\mathcal{V} = \{V_n\}_{n=1}^{\infty}$ be a countable pairwise D-cover of $\mathfrak{O}$, where each $V_n = U'_n \setminus W'_n$ for some $U'_n \in \beta_1$ and $W'_n \in \beta_2$. Consider the family $\mathcal{U} = \{\phi^{-1}(V_n)\}_{n=1}^{\infty}$ in $\mathcal{L}$. Since ϕ is pairwise continuous, $\phi^{-1}(U'_n) \in \eta_1$ and $\phi^{-1}(W'_n) \in \eta_2$. Thus:

$$\phi^{-1}(V_n) = \phi^{-1}(U'_n \setminus W'_n) = \phi^{-1}(U'_n) \setminus \phi^{-1}(W'_n)$$

is a pairwise D-set in $\mathcal{L}$. Moreover, because $\mathcal{V}$ covers $\mathfrak{O}$ and ϕ is surjective, $\mathcal{U}$ covers $\mathcal{L}$. By the pairwise D-countable compactness of $\mathcal{L}$, there exists a finite subcover $\{\phi^{-1}(V_{n_k})\}_{k=1}^{m}$. Then $\{V_{n_k}\}_{k=1}^{m}$ covers $\mathfrak{O}$, proving $\mathfrak{O}$ is pairwise D-countably compact.

(2) By using the same technique, pairwise D-compactness follows by considering an arbitrary pairwise D-cover of $\mathfrak{O}$ and extracting a countable subcover via the pairwise D-compactness of $\mathcal{L}$. □

Particularized function classes allow for structure-preserving mappings between bitopological spaces:

Definition 2.12 *A function* $\phi\colon (\mathcal{L}, \eta_1, \eta_2) \to (\mathfrak{O}, \beta_1, \beta_2)$ *is pairwise D-irresolute if for every pairwise D-set* $D_{\mathfrak{O}} = U_{\mathfrak{O}} \setminus V_{\mathfrak{O}}$ *in* $\mathfrak{O}$ *(*$U_{\mathfrak{O}} \in \beta_1$*,* $V_{\mathfrak{O}} \in \beta_2$*),* $\phi^{-1}(D_{\mathfrak{O}})$ *is a pairwise D-set in* $\mathcal{L}$*.*

When domain spaces are effectively organized, pairwise D-irresolute functions preserve Lindel's properties:

Theorem 2.13 *Consider* $\phi\colon (\mathcal{L}, \eta_1, \eta_2) \overset{Onto}{\to} (\mathfrak{O}, \beta_1, \beta_2)$ *to be pairwise D-irresolute and surjective. If* $\mathcal{L}$ *is pairwise D-Lindelöf, then* $\mathfrak{O}$ *is pairwise D-Lindelöf.*

Proof: Let $\mathcal{D}_{\mathfrak{O}} = \{D_{\mathfrak{k}}\}_{\mathfrak{k} \in \mathsf{K}}$ be a pairwise D-cover of $\mathfrak{O}$. By pairwise D-irresoluteness, $\{\phi^{-1}(D_{\mathfrak{k}})\}_{\mathfrak{k} \in \mathsf{K}}$ is a pairwise D-cover of $\mathcal{L}$. Since $\mathcal{L}$ is pairwise D-Lindelöf, there exists a countable subcover $\{\phi^{-1}(D_{\mathfrak{k}_n})\}_{n=1}^{\infty}$. Then, By surjectivity $\{D_{\mathfrak{k}_n}\}_{n=1}^{\infty}$ covers $\mathfrak{O}$. □

Example 2.14 *Let* $\mathcal{L} = \mathbb{R}, \eta_1 = \eta_{std}, \eta_2 = \eta_{cof}$*,* $\mathfrak{O} = \{0, 1\}, \beta_1 = \beta_2 = discrete$*,*

$$\phi(l) = \begin{cases} 0 & l \in \mathbb{Q} \\ 1 & l \notin \mathbb{Q} \end{cases}.$$

Then $\phi^{-1}(\{0\}) = \mathbb{Q}$ *not pairwise D-set in* $\mathcal{L}$*, so* ϕ *not D-irresolute.*

The pairwise D-sets have closure properties under finite set operations:

Theorem 2.15 *The finite intersection of pairwise D-sets in any bitopological space* $(\mathcal{L}, \eta_1, \eta_2)$ *is a pairwise D-set.*

Proof: Let $D_k = U_k \setminus V_k$ $(k = 1, \dots, n)$ be pairwise D-sets with $U_k \in \eta_1$, $V_k \in \eta_2$, and $U_k \neq \mathcal{L}$. Then:

$$\bigcap_{k=1}^{n} D_k = \left(\bigcap_{k=1}^{n} U_k\right) \setminus \left(\bigcup_{k=1}^{n} V_k\right)$$

which is satisfies:

$\bigcap_{k=1}^{n} U_k \in \eta_1$, $\bigcup_{k=1}^{n} V_k \in \eta_2$, and $\bigcap_{k=1}^{n} U_k \neq \mathcal{L}$. Thus, $\bigcap_{k=1}^{n} D_k$ is a pairwise D-set. □

This intersection property is directly expressed in prevalent bitopological structures:

Example 2.16 *Consider* $(\mathbb{R}, \eta_{std}, \eta_{ll})$ *with:*

$$D_1 = (0,2) \setminus [1,3) = (0,1)$$
$$D_2 = (-1,1.5) \setminus [0.5,1) = (-1,0.5) \cup [1,1.5)$$

Then $D_1 \cap D_2 = (0,1) \cap [(-1,0.5) \cup [1,1.5)] = (0,0.5)$. *This expressed as a pairwise D-set:*

$$(0,0.5) = (0,1.5) \setminus [0.5,1.5)$$

where $(0,1.5) \in \eta_{std}$, $[0.5,1.5) \in \eta_{ll}$.

Although pairwise D-sets are closed under finite intersections, their structure under unions reveals inherent limitations.

Theorem 2.17 *The union of pairwise D-sets may not be a pairwise D-set, even in Lindelöf spaces.*

Proof: let $(\mathcal{L}, \eta_1, \eta_2)$ where $\mathcal{L} = \{a,b\}$ with topologies:

$$\eta_1 = \{\emptyset, \mathcal{L}, \{a\}\}, \quad \eta_2 = \{\emptyset, \mathcal{L}, \{b\}\}.$$

Define:

$$D_1 = \{a\} = \mathcal{L} \setminus \{b\} \quad (U_1 = \mathcal{L} \in \eta_1, V_1 = \{b\} \in \eta_2),$$

$$D_2 = \{b\} = \mathcal{L} \setminus \{a\} \quad (U_2 = \mathcal{L} \in \eta_1, V_2 = \{a\} \in \eta_2).$$

Both are pairwise D-sets since each is expressed as $U \setminus V$ where $U \in \eta_1$, $V \in \eta_2$, and $U \neq \emptyset$. Their union is:

$$D_1 \cup D_2 = \{a,b\} = \mathcal{L}.$$

To show $\mathcal{L}$ is not a pairwise D-set, assume there exist $U \in \eta_1$ and $V \in \eta_2$ such that $\mathcal{L} = U \setminus V$. The only open sets covering $\mathcal{L}$ are $U = \mathcal{L}$ (in η_1) or $U = \{a\}$ (in η_1). We examine all cases:

- If $U = \mathcal{L} \in \eta_1$, then $\mathcal{L} \setminus V = \mathcal{L}$ implies $V = \emptyset$. However, this violates the pairwise D-set condition $U \neq \mathcal{L}$.
- If $U = \{a\} \in \eta_1$, then $\{a\} \setminus V \subseteq \{a\} \neq \mathcal{L}$ for any $V \in \eta_2$.

Thus, $\mathcal{L}$ cannot be expressed as a pairwise D-set. Since $\mathcal{L}$ is finite, it is Lindelöf. □

Even in finite Lindelöf spaces, there are minimal counterexamples demonstrating union failure:

Example 2.18 *Let* $(\mathcal{L} = \{a,b\}, \eta_1, \eta_2)$ *where:*

$$\eta_1 = \{\emptyset, \mathcal{L}, \{a\}\}, \quad \eta_2 = \{\emptyset, \mathcal{L}, \{b\}\}.$$

Define:

$$D_1 = \mathcal{L} \setminus \{b\} = \{a\} \quad (U_1 = \mathcal{L} \in \eta_1, V_1 = \{b\} \in \eta_2),$$

$$D_2 = \mathcal{L} \setminus \{a\} = \{b\} \quad (U_2 = \mathcal{L} \in \eta_1, V_2 = \{a\} \in \eta_2).$$

Both satisfy the pairwise D-set definition $U_k \in \eta_1$, $V_k \in \eta_2$, *and* $U_k = \mathcal{L} \neq \emptyset$.

Their union $D_1 \cup D_2 = \mathcal{L}$ *is not a pairwise D-set because* $\mathcal{L} = \mathcal{L} \setminus \emptyset$ *which is violates* $U \neq \mathcal{L}$ *Despite:* $\mathcal{L}$ *being finite, hence Lindelöf, and both* D_1 *and* D_2 *being pairwise D-sets.*

The condition remains in larger spaces with asymmetric topologies:

Example 2.19 *Let* $(\mathcal{L} = \{a, b, c\}, \eta_1, \eta_2)$ *where:*

$$\eta_1 = \{\emptyset, \mathcal{L}, \{a\}, \{c\}, \{a, c\}\}, \quad \eta_2 = \{\emptyset, \mathcal{L}, \{a\}, \{b\}, \{c\}, \{b, c\}\}$$

Define:

$$D_1 = \{a\} = \{a\} \setminus \emptyset \quad (U_1 = \{a\} \in \eta_1, \ V_1 = \emptyset \in \eta_2)$$

$$D_2 = \{b, c\} = \mathcal{L} \setminus \{a\} \quad (U_2 = \mathcal{L} \in \eta_1, \ V_2 = \{a\} \in \eta_2)$$

Both are valid pairwise D-sets. Their union:

$$D_1 \cup D_2 = \{a\} \cup \{b, c\} = \mathcal{L}$$

is not a pairwise D-set.

Pairwise D-Lindelöfness is preserved under countable sums, but unions of pairwise D-sets can invalidate the property, even in well-behaved spaces.

Proposition 2.20 *There exist pairwise D-Lindelöf spaces where unions of pairwise D-sets fail to be pairwise D-Lindelöf.*

Proof: Let $X = [0, \omega_1]$ with:

$$\eta_1 = \text{order topology}, \quad \eta_2 = \eta_{\text{dis}}.$$

This space is pairwise D-Lindelöf: η_1 is compact (hence Lindelöf), any pairwise D-cover has a η_1-open refinement $\{U_\mathfrak{k}\}$, compactness $\Rightarrow$ finite subcover $\{U_{\mathfrak{k}_i}\}_{i=1}^n$, and corresponding $D_{\mathfrak{k}_i}$ form a finite D-subcover.

Each singleton $\{\mathfrak{k}\}$ $(\mathfrak{k} < \omega_1)$ is a pairwise D-set:

$$\{\mathfrak{k}\} = U_\mathfrak{k} \setminus V_\mathfrak{k} \quad \text{with} \quad U_\mathfrak{k} = [0, \mathfrak{k} + 1) \in \eta_1, \quad V_\mathfrak{k} = [0, \mathfrak{k} + 1) \setminus \{\mathfrak{k}\} \in \eta_2.$$

Consider $A = [0, \omega_1) = \bigcup_{\mathfrak{k}<\omega_1} \{\mathfrak{k}\}$. In $(A, \eta_1|_A, \eta_2|_A)$: Each $\{\mathfrak{k}\}$ remains a pairwise D-set: $U'_\mathfrak{k} = [0, \mathfrak{k} + 1) \cap A \in \eta_1|_A$, $V'_\mathfrak{k} = ([0, \mathfrak{k} + 1) \setminus \{\mathfrak{k}\}) \cap A \in \eta_2|_A$, $\mathcal{D} = \{\{\mathfrak{k}\} : \mathfrak{k} < \omega_1\}$ is a pairwise D-cover of A, $\mathcal{D}$ has no countable subcover: Any countable subset covers $\{\mathfrak{k}_1, \mathfrak{k}_2, \ldots\}$. Let $\beta = \sup\{\mathfrak{k}_i\} < \omega_1 \Rightarrow \beta \notin \bigcup_i \{\mathfrak{k}_i\}$

Thus, A is not pairwise D-Lindelöf. □

3. Inheritance of Pairwise D-Lindelöfness

This section examines the inheritance of pairwise D-Lindelöfness among subspaces in bitopological spaces. It establishes that tau_1-closed subspaces preserve the property, illustrates failure in open subspaces using counterexamples, and proves preservation under countable unions of closed subspaces. In addition, D-set lemmas and constructive proofs establish inheritance patterns.

Definition 3.1 *A subspace* $\mathfrak{O} \subseteq \mathcal{L}$ *is termed as pairwise D-Lindelöf if every pairwise D-cover of* $\mathfrak{O}$ *admits a countable subcover.*

Closed subspaces preserve pairwise D-Lindelöfness under appropriate topological conditions:

Theorem 3.2 *Consider* $(\mathcal{L}, \eta_1, \eta_2)$ *pairwise D-Lindelöf and* $A \subseteq \mathcal{L}$ *be* η_1*-closed. Then the subspace* $(A, \eta_{1_A}, \eta_{2_A})$ *is pairwise D-Lindelöf.*

Proof: Assume $A \neq \emptyset$. Let $\mathcal{E} = \{E_{\mathfrak{k}}\}_{\mathfrak{k}\in I}$ be a pairwise D-cover of A by pairwise D-sets in the subspace topology. Thus for each $\mathfrak{k}$,

$$E_{\mathfrak{k}} = U'_{\mathfrak{k}} \setminus V'_{\mathfrak{k}}$$

where $U'_{\mathfrak{k}} \in \eta_{1_A}$, $V'_{\mathfrak{k}} \in \eta_{2_A}$, and $U'_{\mathfrak{k}} \neq A$. By definition, there exist $U_{\mathfrak{k}} \in \eta_1$ and $V_{\mathfrak{k}} \in \eta_2$ such that:

$$U'_{\mathfrak{k}} = U_{\mathfrak{k}} \cap A \quad \text{and} \quad V'_{\mathfrak{k}} = V_{\mathfrak{k}} \cap A.$$

Note that $U_{\mathfrak{k}} \neq \mathcal{L}$ because if $U_{\mathfrak{k}} = \mathcal{L}$, then $U'_{\mathfrak{k}} = A$, contradicting $U'_{\mathfrak{k}} \neq A$. Therefore, $D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}$ is a pairwise D-set in $\mathcal{L}$, and

$$E_{\mathfrak{k}} = (U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}) \cap A = D_{\mathfrak{k}} \cap A.$$

The collection $\mathcal{D} = \{D_{\mathfrak{k}} : \mathfrak{k} \in I\}$ covers A since $\mathcal{E}$ covers A and $E_{\mathfrak{k}} \subseteq D_{\mathfrak{k}}$ for each $\mathfrak{k}$.

As A is η_1-closed, $\mathcal{L} \setminus A$ is η_1-open. Define:

$$D_0 = (\mathcal{L} \setminus A) \setminus \emptyset,$$

which is a pairwise D-set in $\mathcal{L}$ (as $\mathcal{L} \setminus A \in \eta_1$, $\emptyset \in \eta_2$, and $\mathcal{L} \setminus A \neq \mathcal{L}$ since $A \neq \emptyset$). Then $\mathcal{D}^* = \mathcal{D} \cup \{D_0\}$ covers $\mathcal{L}$: If $l \in A$, then $l \in D_{\mathfrak{k}}$ for some $\mathfrak{k}$. If $l \in \mathcal{L} \setminus A$, then $l \in D_0$.

Since $\mathcal{L}$ is pairwise D-Lindelöf, there exists a countable subcover $\mathcal{S} \subseteq \mathcal{D}^*$. $\mathcal{S} = \{D_{\mathfrak{k}_n} : n \in \mathbb{N}\} \cup \{D_0\}$. Removing D_0, the collection $\{D_{\mathfrak{k}_n} : n \in \mathbb{N}\}$ covers A. Thus:

$$\{E_{\mathfrak{k}_n} = D_{\mathfrak{k}_n} \cap A : n \in \mathbb{N}\}$$

covers A, and each $E_{\mathfrak{k}_n}$ is in $\mathcal{E}$ (hence is a pairwise D-set in A). Therefore, $\{E_{\mathfrak{k}_n} : n \in \mathbb{N}\}$ is a countable subcover of $\mathcal{E}$. □

The complement of closed sets naturally forms pairwise D-sets in this framework:

Lemma 3.3 *Let $(\mathcal{L}, \eta_1, \eta_2)$ be a bitopological space and $A \subseteq \mathcal{L}$ be nonempty and η_1-closed. Then $\mathcal{L} \setminus A$ is a pairwise D-set.*

Proof: Since A is η_1-closed, $\mathcal{L} \setminus A$ is η_1-open.

$$\mathcal{L} \setminus A = (\mathcal{L} \setminus A) \setminus \emptyset$$

where $\mathcal{L} \setminus A \in \eta_1$ and $\emptyset \in \eta_2$. Moreover, because $A \neq \emptyset$, we have $\mathcal{L} \setminus A \neq \mathcal{L}$. Therefore, $\mathcal{L} \setminus A$ satisfies the definition of a pairwise D-set. □

The closed interval subspace demonstrates successful inheritance:

Example 3.4 *Let $(\mathbb{N}, \eta_{cof}, \eta_{dis})$ is pairwise D-Lindelöf since $\mathbb{N}$ is countable. And $A = \{0, 1\} \subseteq \mathbb{N}$. The set A is η_{cof}-closed because it is finite.*

Consider the following pairwise D-cover of A in the subspace topology:

$$\mathcal{E} = \{E_1, E_2\} \quad \textit{where} \quad E_1 = \{0\}, \quad E_2 = \{1\}.$$

Every E_i is a pairwise D-set in $(A, \eta_{cof_A}, \eta_{dis_A})$ because:

$$E_1 = \{0\} \setminus \emptyset \quad \textit{with} \quad \{0\} \in \eta_{cof_A},\ \emptyset \in \eta_{dis_A},\ \textit{and } \{0\} \neq A$$
$$E_2 = \{1\} \setminus \emptyset \quad \textit{with} \quad \{1\} \in \eta_{cof_A},\ \emptyset \in \eta_{dis_A},\ \textit{and } \{1\} \neq A$$

The cover $\mathcal{E}$ has a finite subcover, confirming that A is pairwise D-Lindelöf.

Proposition 3.5 *In pairwise D-Lindelöf spaces:*

1. *Pairwise D-Lindelöfness is hereditary for closed subspaces.*

2. *It is not hereditary for open subspaces.*

Proof: (1) Follows from Theorem 3.2 (closed subspaces preserve pairwise D-Lindelöfness).

(2) Consider the bitopological space $(\mathcal{L}, \eta_1, \eta_2)$ where $\mathcal{L} = [0, 1]$, with both η_1 and η_2 being the standard topology. This space is pairwise D-Lindelöf since $[0, 1]$ is compact. Let $A = (0, 1)$, which is open in $\mathcal{L}$ (as $\mathcal{L} \setminus A = \{0\}$ is closed).

Define:

$$\mathcal{E} = \left\{ \left(0, \tfrac{1}{2}\right) \right\} \cup \left\{ (l, 1) \mid l \in \left[\tfrac{1}{2}, 1\right) \right\}.$$

Each set in $\mathcal{E}$ is a pairwise D-set in the subspace $(A, \eta_{1_A}, \eta_{2_A})$:

- $\left(0, \frac{1}{2}\right) = U_1 \setminus \emptyset$ where $U_1 = \left(0, \frac{1}{2}\right) \in \eta_{1_A}$, $\emptyset \in \eta_{2_A}$, and $U_1 \neq A$
- $(l, 1) = U_l \setminus \emptyset$ where $U_l = (l, 1) \in \eta_{1_A}$, $\emptyset \in \eta_{2_A}$, and $U_l \neq A$

$\mathcal{E}$ covers A: For any $o \in (0, 1)$, if $o < \frac{1}{2}$, then $o \in \left(0, \frac{1}{2}\right)$, if $o \geq \frac{1}{2}$, choose $l \in \left[\frac{1}{2}, o\right)$ to get $o \in (l, 1)$
However, $\mathcal{E}$ has no countable subcover. Let $\mathcal{S} \subseteq \mathcal{E}$ be countable. Define:

$$l_0 = \inf \left\{ l \mid (l, 1) \in \mathcal{S} \cap \{(l, 1)\}_{l \geq 1/2} \right\}$$

- If $l_0 > \frac{1}{2}$, choose $o \in \left[\frac{1}{2}, l_0\right)$. Then $o \notin \left(0, \frac{1}{2}\right)$ and $o \notin (l, 1)$ for any $(l, 1) \in \mathcal{S}$ (since $l \geq l_0 > o$).
- If $l_0 = \frac{1}{2}$, then $o = \frac{1}{2}$ satisfies: $o \notin \left(0, \frac{1}{2}\right)$, $o \notin (l, 1)$ for any $l \geq \frac{1}{2}$ (since $l > o$ or $l = o$ both exclude o)

In both cases, $\mathcal{S}$ misses some $o \in A$, so A is not pairwise D-Lindelöf. □

Special subspace classes inherit pairwise D-Lindelöfness unconditionally:

Corollary 3.6 *If $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf and $A \subseteq \mathcal{L}$ is countable, then $(A, \eta_{1_A}, \eta_{2_A})$ is pairwise D-Lindelöf without requiring $U \cap A \neq A$.*

Closed subspaces naturally satisfy the complement condition:

Corollary 3.7 *If $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf and $A \subseteq \mathcal{L}$ is both η_1-closed and η_2-closed, then the subspace $(A, \eta_{1_A}, \eta_{2_A})$ is pairwise D-Lindelöf.*

Countable unions of closed pairwise D-Lindelöf subspaces preserve the property:

Proposition 3.8 *Consider a bitopological space $(\mathcal{L}, \eta_1, \eta_2)$ and $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \subseteq \mathcal{L}$ is η_1-closed and the subspace $(A_n, \eta_{1_{A_n}}, \eta_{2_{A_n}})$ is pairwise D-Lindelöf. Then the subspace $(A, \eta_{1_A}, \eta_{2_A})$ is pairwise D-Lindelöf.*

Proof: Let $\mathcal{D} = \{D_{\mathfrak{k}}\}_{\mathfrak{k} \in \mathtt{K}}$ be a pairwise D-cover of A in the subspace bitopology. Thus, each $D_{\mathfrak{k}} \subseteq A$ is a pairwise D-set in A, so:

$$D_{\mathfrak{k}} = U'_{\mathfrak{k}} \setminus V'_{\mathfrak{k}} \quad \text{for some} \quad U'_{\mathfrak{k}} \in \eta_{1_A}, V'_{\mathfrak{k}} \in \eta_{2_A}, U'_{\mathfrak{k}} \neq A.$$

For all $n \in \mathbb{N}$, consider the restriction to A_n:

$$\mathcal{D}_n = \{D_{\mathfrak{k}} \cap A_n : \mathfrak{k} \in \mathtt{K}\}.$$

This is a pairwise D-cover of A_n in its subspace bitopology because: For any $l \in A_n \subseteq A$, there exists $\mathfrak{k}$ such that $l \in D_{\mathfrak{k}}$, so $l \in D_{\mathfrak{k}} \cap A_n$.
Each $D_{\mathfrak{k}} \cap A_n = (U'_{\mathfrak{k}} \cap A_n) \setminus (V'_{\mathfrak{k}} \cap A_n)$, where:

- $U'_{\mathfrak{k}} \cap A_n \in \eta_{1_{A_n}}$ and $V'_{\mathfrak{k}} \cap A_n \in \eta_{2_{A_n}}$.
- $U'_{\mathfrak{k}} \cap A_n \neq A_n$: If $U'_{\mathfrak{k}} \cap A_n = A_n$, then $A_n \subseteq U'_{\mathfrak{k}}$. Since $U'_{\mathfrak{k}} \neq A$, there exists $o \in A \setminus U'_{\mathfrak{k}}$. But $o \notin A_n$ (as $A_n \subseteq U'_{\mathfrak{k}}$), contradicting $A = \bigcup_n A_n$.

Thus, $D_{\mathfrak{k}} \cap A_n$ is a pairwise D-set in A_n.

By pairwise D-Lindelöfness of A_n, there exists a countable subcover $\mathcal{S}_n \subseteq \mathcal{D}_n$ of A_n. Let $\mathcal{S}_n = \{D_{\mathfrak{k}_{n,k}} \cap A_n\}_{k=1}^{\infty}$, where each $\mathfrak{k}_{n,k} \in \mathtt{K}$.

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \{D_{\mathfrak{k}_{n,k}} : k \in \mathbb{N}\}$$

is countable and covers A:

$$A = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (D_{\mathfrak{k}_{n,k}} \cap A_n) \subseteq \bigcup_{D \in \mathcal{S}} D.$$

Hence, $\mathcal{S}$ is a countable pairwise D-subcover of $\mathcal{D}$ for A. □

4. Subspace Inheritance and Representation Constraints for Pairwise D-Lindelöfness

This section establishes conditions that dictate that pairwise D-Lindelöfness is preserved in subspaces of bitopological spaces. The rule of thumb is that subspace complements be pairwise D. It presents pathological counterexamples violating this condition, analyzes limitations of pairwise D-set representations in subspaces, and expands to preservation under perfect maps and projections. Theorems and constructive counterexamples formalize the necessity criteria for inheritance.

Theorem 4.1 *Consider $(\mathcal{L}, \eta_1, \eta_2)$ pairwise D-Lindelöf and $A \subseteq \mathcal{L}$. If $\mathcal{L} \setminus A$ is a pairwise D-set, then the subspace A with induced bitopologies is pairwise D-Lindelöf.*

Proof: Let $\mathcal{D}_A = \{D_\beta \cap A\}$ be a pairwise D-cover of A, where each $D_\beta = U_\beta \setminus V_\beta$ is a pairwise D-set in $\mathcal{L}$. Since $\mathcal{L} \setminus A$ is a pairwise D-set, $\mathcal{D}^* = \{D_\beta : \beta\} \cup \{\mathcal{L} \setminus A\}$ is a pairwise D-cover of $\mathcal{L}$. By pairwise D-Lindelöfness of $\mathcal{L}$, there exists a countable subcover $\{D_{\beta_n}\}_{n\in\mathbb{N}} \cup \{\mathcal{L} \setminus A\}$. Then:

$$A \subseteq \bigcup_{n\in\mathbb{N}} D_{\beta_n} \quad \text{(since } \mathcal{L} \setminus A \text{ covers no points in } A\text{)},$$

implying $\{D_{\beta_n} \cap A\}_{n\in\mathbb{N}}$ is a countable D-subcover of A. □

Closed intervals satisfy the subspace inheritance condition in appropriate topologies:

Example 4.2 *Let $(\mathbb{Q}, \eta_{std}, \eta_{cof})$ where $\mathcal{L} = \mathbb{Q}$ is pairwise D-Lindelöf (by Theorem 2.6), $A = \mathbb{Q} \cap [0, 1]$, $\mathcal{L} \setminus A = \mathbb{Q} \cap ((-\infty, 0) \cup (1, \infty)) = \underbrace{(\mathbb{Q} \cap ((-\infty, 0) \cup (1, \infty)))}_{\in \eta_1} \setminus \emptyset$ (pairwise D-set).*

By Theorem 4.1, the subspace A is pairwise D-Lindelöf

Subspaces that don't succeed the complement condition exhibit pathological behavior.

Example 4.3 *Let $\mathcal{L} = \mathbb{R}$ with $\eta_1 = \eta_{cof}$, $\eta_2 = \eta_{cof}$, and $A = \mathbb{R} \setminus \mathbb{Z}$. Then $\mathcal{L}$ is pairwise D-Lindelöf. $\mathcal{L} \setminus A = \mathbb{Z}$ is not a pairwise D-set:*

$$\nexists U \in \eta_1, V \in \eta_2 \text{ such that } \mathbb{Z} = U \setminus V$$

(No cofinite set equals $\mathbb{Z}$ since $\mathbb{R} \setminus \mathbb{Z}$ is not finite).
Consider

$$\mathcal{D} = \{(\mathbb{R} \setminus \{n\}) \cap A : n \in \mathbb{Z}\}$$

where each $\mathbb{R} \setminus \{n\}$ is cofinite open. This is a pairwise D-cover of A since:

$$\bigcup_{n\in\mathbb{Z}} ((\mathbb{R} \setminus \{n\}) \cap A) = A$$

$\mathcal{D}$ has no countable subcover: Each $n \in \mathbb{Z}$ is excluded from its corresponding set, and $\mathbb{Z}$ is countable but $\mathcal{D}$ requires all sets to cover A.
Thus, $(A, \eta_{1_A}, \eta_{2_A})$ is not pairwise D-Lindelöf.

The following counterexamples demonstrate the necessity of key conditions in subspace inheritance:

Example 4.4 *Let $\mathcal{L} = [0, \omega_1]$, where ω_1 is the first uncountable ordinal. with: η_1: Order topology (compact, hence pairwise D-Lindelöf), η_2:$\{\emptyset, \mathcal{L}\}$, $A = [0, \omega_1)$, $\mathcal{L} \setminus A = \{\omega_1\}$.*
Then, Pairwise D-cover of A:

$$\mathcal{D} = \{[0, \mathfrak{k}) : \mathfrak{k} < \omega_1\}$$

Each $[0, \mathfrak{k}) = \underbrace{[0, \mathfrak{k})}_{\eta_1\text{-open}} \setminus \emptyset$ is a pairwise D-set. No countable subcover: For any countable $\{\mathfrak{k}_n\} \subset \omega_1$, $\sup \mathfrak{k}_n < \omega_1$, so $\beta = \sup \mathfrak{k}_n + 1$ is uncovered. Thus, A is not pairwise D-Lindelöf despite $\mathcal{L}$ being pairwise D-Lindelöf.

The subspace condition $U \cap A \neq A$ is essential for proper pairwise D-set formation in subspaces:

Lemma 4.5 *For a pairwise D-set $D = U \setminus V$ in $(\mathcal{L}, \eta_1, \eta_2)$ to induce a pairwise D-set in the subspace A, the condition $U \cap A \neq A$ is not inherently necessary. However, if $U \cap A = A$, the representation $D \cap A = A \setminus (V \cap A)$ may violate the requirement that pairwise D-sets in A cannot equal A itself.*

The interval subspace demonstrates successful inheritance of pairwise D-Lindelöfness:

Example 4.6 *Let $(\mathbb{Q}, \eta_{std}, \eta_{cof})$ (countable, hence pairwise D-Lindelöf), $A = \mathbb{Q} \cap [0,1]$, $D = (\mathbb{Q} \cap (0,2)) \setminus \{\sqrt{2}\}$ (where $\sqrt{2} \notin \mathbb{Q}$), satisfies $U \cap A = \mathbb{Q} \cap (0,1] \neq A$ (since $0 \in A$ but $0 \notin U$), $D \cap A = \mathbb{Q} \cap (0,1)$. The pairwise D-cover of A:*

$$\mathcal{D}_A = \left\{ \left(\mathbb{Q} \cap \left(\frac{1}{n+1}, 1 \right) \right) \setminus \emptyset \mid n \in \mathbb{N} \right\} \cup \left\{ \mathbb{Q} \cap \left(0, \tfrac{1}{2}\right) \setminus \emptyset \right\}$$

admits a finite subcover: $\left\{ \mathbb{Q} \cap \left(0, \frac{1}{2}\right) \setminus \emptyset \right\} \cup \left\{ \mathbb{Q} \cap \left(\frac{1}{k+1}, 1 \right) \setminus \emptyset \right\}$ for any $k \geq 1$, illustrating Theorem 4.1.

Example 4.7 *Let $\mathcal{L} = [0, \omega_1)$ (order topology $\eta_1 = \eta_2$), $A = [0, \omega)$ (countable ordinals), $\mathcal{L}$ not pairwise D-Lindelöf (cover $\{[0, \mathfrak{k}) : \mathfrak{k} < \omega_1\}$ uncountable), A pairwise D-Lindelöf (countable), A not η_1-closed (ω is limit point). Shows ambient pairwise D-Lindelöfness is necessary for Theorem 4.1.*

Example 4.8 *Consider: $\mathcal{L} = [0, \omega_1)$ (first uncountable ordinal, order topology $\eta_1 = \eta_2$), $A = [0, \omega)$ (countable ordinals), $\mathcal{L}$ is not pairwise D-Lindelöf (cover $\{[0, \mathfrak{k}) : \mathfrak{k} < \omega_1\}$ has no countable subcover), A is pairwise D-Lindelöf (countable space). This shows Theorem 4.1 requires ambient pairwise D-Lindelöfness.*

Pathological cases demonstrate the necessity of subspace conditions:

Example 4.9 *Let $\mathcal{L} = \mathbb{Q}$ with $\eta_1 = \eta_{std}$, $\eta_2 = \eta_{cof}$. By Theorem 2.6 we found it is pairwise D-Lindelöf . $A = \mathbb{Q} \cap (0,1)$, $D = (\mathbb{Q} \cap (0,2)) \setminus \emptyset$ where:*

$$U = \mathbb{Q} \cap (0,2) \in \eta_1, \quad V = \emptyset \in \eta_2$$

$U \cap A = \mathbb{Q} \cap (0,1) = A$
Then $D \cap A = A$. This cannot be expressed as $W \setminus B$ in subspace A where $W \in \eta_{1_A}$ and $W \neq A$, $B \in \eta_{2_A}$ because any η_{1_A}-open $W \subsetneq A$ is proper subset of $\mathbb{Q} \cap (0,1)$, $W \setminus B \subsetneq A$ for any B, while $D \cap A = A$. The definition requires $W \neq A$ for pairwise D-sets in A.

Thus, $D \cap A$ fails to be a pairwise D-set in A, illustrating Lemma 4.5.

Example 4.10 *Let $(\mathcal{L}, \eta_1, \eta_2)$ where $\mathcal{L} = [0,1]$, with both η_1 and η_2 being the standard topology. This space is pairwise D-Lindelöf since $[0,1]$ is compact. Let $A = (0,1)$, which is open in $\mathcal{L}$ (as $\mathcal{L} \setminus A = \{0\}$ is closed).*

Define:

$$\mathcal{E} = \left\{ \left(0, \tfrac{1}{2}\right) \right\} \cup \left\{ (l, 1) \mid l \in \left[\tfrac{1}{2}, 1\right) \right\}.$$

Each set in $\mathcal{E}$ is a pairwise D-set in the subspace $(A, \eta_{1_A}, \eta_{2_A})$ because:

- $\left(0, \frac{1}{2}\right) = U_1 \setminus \emptyset$ *where* $U_1 = \left(0, \frac{1}{2}\right) \in \eta_{1_A}$, $\emptyset \in \eta_{2_A}$, *and* $U_1 \neq A$
- $(l, 1) = U_l \setminus \emptyset$ *where* $U_l = (l, 1) \in \eta_{1_A}$, $\emptyset \in \eta_{2_A}$, *and* $U_l \neq A$

$\mathcal{E}$ covers A: For any $o \in (0,1)$, if $o < \frac{1}{2}$, then $o \in \left(0, \frac{1}{2}\right)$. If $o \geq \frac{1}{2}$, choose $l \in \left[\frac{1}{2}, o\right)$ to get $o \in (l, 1)$. However, $\mathcal{E}$ has no countable subcover. Let $\mathcal{S} \subseteq \mathcal{E}$ be countable. Define:

$$l_0 = \inf \left\{ l \mid (l,1) \in \mathcal{S} \cap \{(l,1)\}_{l \geq 1/2} \right\}$$

- *If $l_0 > \frac{1}{2}$, choose $o \in \left(\frac{1}{2}, l_0\right)$. Then $o \notin \left(0, \frac{1}{2}\right)$ and $o \notin (l,1)$ for any $(l,1) \in \mathcal{S}$ (since $l \geq l_0 > o$).*
- *If $l_0 = \frac{1}{2}$, then $o = \frac{1}{2}$ satisfies $o \notin \left(0, \frac{1}{2}\right)$, $o \notin (l,1)$ for any $l \geq \frac{1}{2}$ (as $l > o$ or $l = o$ both exclude o).*

Thus, A is not pairwise D-Lindelöf, demonstrating that open subspaces may not inherit pairwise D-Lindelöfness.

Perfect functions characterize equivalence of pairwise D-Lindelöfness under local indiscreteness:

Theorem 4.11 *Consider $\phi\colon (\mathcal{L}, \eta_1, \eta_2) \to (\mathfrak{O}, \beta_1, \beta_2)$ be pairwise continuous, pairwise closed, and such that $\phi^{-1}(o)$ is pairwise D-Lindelöf. If $\mathcal{L}$ is pairwise locally indiscrete, then:*

$$\mathcal{L} \text{ is pairwise } D\text{-Lindelöf} \iff \mathfrak{O} \text{ is pairwise } D\text{-Lindelöf.}$$

Proof: (⇒): Follows from Theorem 2.13 since pairwise continuous functions are D-irresolute.

(⇐): Let $\mathcal{D} = \{D_{\mathfrak{k}}\}_{\mathfrak{k}\in\mathbb{K}}$ be a pairwise D-cover of $\mathcal{L}$. For each $o \in \mathfrak{O}$, $A_o = \phi^{-1}(o)$ is pairwise D-Lindelöf, so there exists countable subcover $\mathcal{D}_o = \{D_{\mathfrak{k}_n^o}\}_{n=1}^{\infty}$. Define:

$$Y_o = \mathfrak{O} \setminus \phi\left(\mathcal{L} \setminus \bigcup_{n=1}^{\infty} D_{\mathfrak{k}_n^o}\right)$$

which is β_1-open (since ϕ is closed). Then $\{Y_o\}_{o\in\mathfrak{O}}$ covers $\mathfrak{O}$. By pairwise D-Lindelöfness of $\mathfrak{O}$, there exists countable subcover $\{Y_{o_m}\}_{m=1}^{\infty}$. Thus:

$$\mathcal{L} = \bigcup_{m=1}^{\infty} \phi^{-1}(Y_{o_m}) \subseteq \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} D_{\mathfrak{k}_n^{o_m}}$$

is a countable pairwise D-subcover. □

Locally indiscrete spaces induce strong clopen structures that facilitate preservation proofs:

Lemma 4.12 *In pairwise locally indiscrete spaces:*

1. *Every pairwise D-set is η_1-clopen and η_2-clopen*
2. *Countable unions of pairwise D-sets are pairwise D-sets*
3. *Continuous preimages preserve clopen properties*

Proof: For (1): Let $D = U \setminus V$ be pairwise D. By local indiscreteness:

- U is η_1-clopen $\Rightarrow D = U \setminus V$ is η_1-clopen.
- V is η_2-clopen $\Rightarrow D$ is η_2-clopen

(2) and (3) follow from (1) and topology axioms. □

Corollary 4.13 *If $\phi\colon \mathcal{L} \to \mathfrak{O}$ is pairwise continuous, pairwise closed, finite-to-one, and $\mathcal{L}$ pairwise locally indiscrete, then:*

$$\mathcal{L} \text{ pairwise } D\text{-Lindelöf} \iff \mathfrak{O} \text{ pairwise } D\text{-Lindelöf.}$$

Perfect functions preserve Lindelöfness when combined with pairwise D-Lindelöfness:

Theorem 4.14 *Let $\phi\colon (\mathcal{L}, \eta_1, \eta_2) \overset{Onto}{\to} (\mathfrak{O}, \beta_1, \beta_2)$ be a pairwise continuous, closed function with Lindelöf fibers. If $\mathfrak{O}$ is pairwise D-Lindelöf, then $\mathcal{L}$ is pairwise Lindelöf.*

Proof: Let $\mathcal{U}$ be a pairwise open cover of $\mathcal{L}$. For each $o \in \mathfrak{O}$, the fiber $\phi^{-1}(o)$ is Lindelöf, so there exists a finite subcollection $\mathcal{U}_o \subseteq \mathcal{U}$ covering $\phi^{-1}(o)$. Define:

$$Y_o = \mathfrak{O} \setminus \phi\left(\mathcal{L} \setminus \bigcup \mathcal{U}_o\right)$$

Since ϕ is closed, Y_o is β_1-open. The collection $\{Y_o\}_{o\in\mathfrak{O}}$ covers $\mathfrak{O}$ (as $o \in Y_o$) and consists of β_1-open sets. By pairwise D-Lindelöfness of $\mathfrak{O}$, there exists a countable subcover $\{Y_{o_n}\}_{n\in\mathbb{N}}$. Then:

$$\bigcup_{n\in\mathbb{N}} \mathcal{U}_{o_n}$$

is a countable pairwise open subcover of $\mathcal{L}$. □

Projection functions exhibit perfectness properties under Lindelöf conditions:

Theorem 4.15 *Let $(\mathcal{L}, \eta_1, \eta_2)$ be pairwise Lindelöf and $(\mathfrak{O}, \beta_1, \beta_2)$ pairwise Hausdorff. Then the projection $\xi_{\mathfrak{O}} : \mathcal{L} \times \mathfrak{O} \to \mathfrak{O}$ is pairwise continuous with Lindelöf fibers. $\xi_{\mathfrak{O}}$ is closed if both spaces are pairwise D_2*

Proof: Preimages of β_k-open sets are ξ_k-open by definition. $\xi_{\mathfrak{O}}^{-1}(o) = \mathcal{L} \times \{o\} \cong \mathcal{L}$ is Lindelöf.
Let $C \subseteq \mathcal{L} \times \mathfrak{O}$ be ξ_1-closed. For $o \notin \xi_{\mathfrak{O}}(C)$, for each $l \in \mathcal{L}$, choose pairwise disjoint $\eta_1 \times \beta_1$-open $U_{lo} \ni (l, o)$ and $V_{lo} \supseteq C$ with $U_{lo} \cap V_{lo} = \emptyset$. By Lindelöfness of $\mathcal{L}$, cover $\{U_{lo}\}_{l \in \mathcal{L}}$ has finite subcover. Intersect corresponding $\mathfrak{O}$-neighborhoods to obtain β_1-open $W_o \ni o$ disjoint from $\xi_{\mathfrak{O}}(C)$. □

Projections preserve Lindelöfness properties in D-Lindelöf domains:

Theorem 4.16 *Let $(\mathcal{L}, \eta_1, \eta_2)$ be pairwise D-Lindelöf and $(\mathfrak{O}, \beta_1, \beta_2)$ pairwise Hausdorff. Then projection $\xi_{\mathfrak{O}} : \mathcal{L} \times \mathfrak{O} \to \mathfrak{O}$ is pairwise continuous with Lindelöf fibers.*

Proof: Preimages of β_k-open sets are ξ_k-open. $\xi_{\mathfrak{O}}^{-1}(o) = \mathcal{L} \times \{o\} \cong \mathcal{L}$ is pairwise D-Lindelöf, hence Lindelöf. □

Compact factors simplify product preservation results:

Corollary 4.17 *Let $(\mathcal{L}, \kappa_1, \kappa_2)$ be compact Hausdorff and pairwise locally indiscrete, and $(\mathfrak{O}, \beta_1, \beta_2)$ pairwise locally indiscrete D-Lindelöf. Then $\mathcal{L} \times \mathfrak{O}$ is pairwise D-Lindelöf.*

Example 4.18 *Let $\mathcal{L} = \mathbb{R}$ with $\eta_1 = \eta_{cof}$ and $\eta_2 = \eta_{dis}$. For each $l \in \mathbb{R}$, fix a distinct point $o_l \neq l$. Define:*

$$U_l = \mathbb{R} \setminus \{o_l\} \in \eta_1, \quad V_l = \mathbb{R} \setminus \{l, o_l\} \in \eta_2.$$

Then $D_l = U_l \setminus V_l = \{l\}$ is a pairwise D-set. The collection $\mathcal{D} = \{D_l\}_{l \in \mathbb{R}}$ is a pairwise D-cover of $\mathbb{R}$ since each $l \in \mathbb{R}$ is covered by D_l, U_l is cofinite (hence η_1-open) and $U_l \neq \mathbb{R}$, also V_l is η_2-open.

Suppose $\mathcal{D}' = \{D_{l_n}\}_{n=1}^{\infty}$ is a countable subcover. Then $\mathcal{D}'$ covers only the countable set $\{l_n : n \in \mathbb{N}\}$, leaving uncountably many points uncovered. Thus, $(\mathbb{R}, \eta_{cof}, \eta_{dis})$ is not pairwise D-Lindelöf.

Counterexamples further illuminate the limitations of pairwise D-Lindelöfness:

Example 4.19 *Let $\mathcal{L} = \mathbb{N}$ with $\eta_1 = \eta_{dis}$ and $\eta_2 = \eta_{ind}$ is pairwise D-Lindelöf: Since $\mathcal{L}$ is countable, any pairwise D-cover has a countable subcover.*

Consider the pairwise D-cover $\mathcal{D} = \{\{n\} \mid n \in \mathbb{N}\}$, where each singleton is constructed as:

$$\{n\} = U_n \setminus V_n, \quad U_n = \{n\} \in \eta_1, \quad V_n = \emptyset \in \eta_2.$$

Each $\{n\}$ is a pairwise D-set (U_n is discrete-open, V_n is indiscrete-open, $U_n \neq \mathcal{L}$).

$\mathcal{D}$ covers $\mathbb{N}$ but has no finite subcover: For any finite $\{\{n_1\}, \ldots, \{n_k\}\}$, the point $\max\{n_i\} + 1$ *remains uncovered. Thus, $(\mathcal{L}, \eta_1, \eta_2)$ is pairwise D-Lindelöf but not pairwise D-compact.*

The following examples demonstrate how pairwise D-Lindelöfness fails in common bitopological structures, even when individual topologies satisfy separation axioms:

Example 4.20 *Consider$(\mathbb{R}, \eta_{std}, \eta_{cof})$. For each $l \in \mathbb{R} \setminus \mathbb{Q}$, define:*

$$D_l = (l - 1, l + 1) \setminus \{l\}.$$

This is a pairwise D-set because: $U_l = (l - 1, l + 1) \in \eta_{std}$, $V_l = \{l\} \in \eta_{cof}$, $U_l \neq \mathbb{R}$. The collection $\mathcal{D} = \{D_l \mid l \in \mathbb{R} \setminus \mathbb{Q}\}$ is a pairwise D-cover of $\mathbb{R}$: For rational $q \in \mathbb{Q}$, choose $l_q = q + \sqrt{2}/k$ for k large *enough that* $q \in (l_q - 1, l_q + 1)$*. Since* $q \neq l_q$ *(rational vs. irrational),* $q \in D_{l_q}$*. For irrational* $o \in \mathbb{R} \setminus \mathbb{Q}$*, choose* $l_o = o + \sqrt{2}/k$ $(k \geq 2)$*. Then* $o \in (l_o - 1, l_o + 1)$ *and* $o \neq l_o$*, so* $o \in D_{l_o}$*. Suppose* $\mathcal{D}$ *has a countable subcover* $\{D_{l_1}, D_{l_2}, \ldots\}$ *with* $l_n \in \mathbb{R} \setminus \mathbb{Q}$*. Let*

$$M = \sup\{|l_n| : n \in \mathbb{N}\} + 2$$

Then $o = M + 1$ *satisfies:*

$$|o| > |l_n| + 1 \quad \forall n \implies o \notin (l_n - 1, l_n + 1) \quad \forall n \implies y \notin \bigcup_n D_{l_n}.$$

Thus $\mathcal{D}$ *has no countable subcover, proving* $(\mathbb{R}, \eta_{std}, \eta_{cof})$ *is not pairwise D-Lindelöf.*

Example 4.21 *Consider the bitopological space* $(\mathbb{R}^2, \eta_{std} \times \eta_{std}, \eta_{cof} \times \eta_{cof})$*. For each* $p = (l_p, o_p) \in \mathbb{R}^2$*, define:*

$$D_p = \{p\} = B(p, 1) \setminus V_p$$

where:

$$U_p = B(p, 1) \in \eta_{std} \times \eta_{std} \quad \text{(open disk of radius 1 centered at } p\text{)}$$
$$V_p = [(\mathbb{R} \setminus \{l_p\}) \times \mathbb{R}] \cup [\mathbb{R} \times (\mathbb{R} \setminus \{o_p\})] \in \eta_{cof} \times \eta_{cof}$$

This is a pairwise D-set because U_p *is standard-open and* $U_p \neq \mathbb{R}^2$*,* V_p *is cofinite* $\times$ *cofinite-open,* $D_p = U_p \setminus V_p = \{p\}$ *(since* $V_p^c = \{(l_p, o_p)\}$*). The collection* $\mathcal{D} = \{D_p \mid p \in \mathbb{R}^2\}$ *is a pairwise D-cover of* $\mathbb{R}^2$ *because every point* $q \in \mathbb{R}^2$ *is covered by* $D_q = \{q\}$*.*

Since $|\mathbb{R}^2| = \mathfrak{c} > \aleph_0$*,* $\mathcal{D}$ *has no countable subcover: For any countable subcollection* $\{D_{p_n}\}_{n \in \mathbb{N}}$*, the set* $\{p_n : n \in \mathbb{N}\}$ *is countable, so there exists* $q \in \mathbb{R}^2 \setminus \{p_n : n \in \mathbb{N}\}$ *that remains uncovered. Thus, the space is not pairwise D-Lindelöf.*

The subsequent counterexamples highlight critical distinctions between pairwise D-Lindelöfness and compactness, and its sensitivity to product topologies:

Example 4.22 *Let* $\mathcal{L} = \mathbb{R}$ *with* $\eta_1 = \eta_{dis}$*,* $\eta_2 = \{\emptyset, \mathcal{L}\}$*. For each* $l \in \mathbb{R}$*, define:*

$$D_l = \{l\} = \{l\} \setminus \emptyset \quad (\eta_1\text{-open} \setminus \eta_2\text{-open}).$$

Then $\mathcal{D} = \{D_l\}_{l \in \mathbb{R}}$ *is a pairwise D-cover. It admits a countable subcover iff the continuum is countable, which is false. Moreover, it has no finite subcover, showing pairwise D-Lindelöfness does not imply pairwise D-compactness.*

The following theorem establishes fundamental properties of pairwise D-sets in pairwise D-Lindelöf spaces and demonstrates the necessity of its topological constraints:

Theorem 4.23 *Let* $(\mathcal{L}, \eta_1, \eta_2)$ *be pairwise D-Lindelöf.*

1. *If* $(\mathcal{L}, \eta_1, \eta_2)$ *is pairwise Hausdorff and either* $\eta_1 \supseteq \eta_2$ *or* $\eta_2 \subseteq \eta_1$*, then every pairwise D-set is* η_k*-closed for the finer topology* η_k*.*
2. *The conditions are essential:*
   - *(a) If pairwise Hausdorff is omitted, there exists a pairwise D-Lindelöf space with a pairwise D-set that is not closed in either topology .*
   - *(b) If no topology contains the other, there exists a pairwise D-Lindelöf space with a pairwise D-set that is not Lindelöf in either topology.*

Proof: Part (1): Assume $\eta_1 \subseteq \eta_2$. Let $A = U \setminus V$ be a pairwise D-set with $U \in \eta_1$, $V \in \eta_2$, $U \neq \mathcal{L}$.

(A is η_2-closed): For any $l \notin A$:

- If $l \notin U$, since $U \in \eta_1 \subseteq \eta_2$, $\mathcal{L} \setminus U$ is η_2-closed and contains l.
- If $l \in U \cap V$, then $V \in \eta_2$ contains l and $V \cap A = \emptyset$.

Thus $\mathcal{L} \setminus A$ is η_2-open.

Let $\{O_i\}_{i \in I}$ be a η_1-open cover of A. For each i, define $D_i = O_i \setminus \emptyset$. Then:

$$\mathcal{D} = \{D_i : i \in I\} \cup \{(\mathcal{L} \setminus A) \setminus \emptyset\}$$

is a pairwise D-cover of $\mathcal{L}$ only if $\mathcal{L} \setminus A$ is η_1-open, which holds if $\eta_2 \subseteq \eta_1$ but not necessarily under $\eta_1 \subseteq \eta_2$. This shows the condition $\eta_2 \subseteq \eta_1$ is needed for the Lindelöf claim.

Part (2): Let $\mathcal{L} = \mathbb{N} \times \{0, 1\}$ with:

$$\eta_1 = \eta_{\text{dis}}, \quad \eta_2 = \{\emptyset, \mathcal{L}, \mathbb{N} \times \{0\}, \mathbb{N} \times \{1\}\}.$$

The set $A = \mathbb{N} \times \{0\}$ is a pairwise D-set ($A = A \setminus \emptyset$), but not η_2-closed. $\mathcal{L}$ is pairwise D-Lindelöf.
Let $\mathcal{L} = \mathbb{R}$ with:

$$\eta_1 = \eta_{\text{coc}}, \quad \eta_2 = \eta_{\text{std}}.$$

The set $A = (0, 1) = (0, 1) \setminus \emptyset$ is a pairwise D-set. It is not η_1-Lindelöf and not η_2-closed. Here $\eta_1 \not\supseteq \eta_2$ and $\eta_2 \not\supseteq \eta_1$.

□

The next example illustrates a space satisfying all conditions of Theorem 4.23(1):

Example 4.24 *Consider $(\mathbb{R}, \eta_{std}, \eta_{dis})$*

This space satisfies Theorem 4.23(1) because for $l \neq o$, use $Y_l = \mathbb{R} \setminus \{o\} \in \eta_{std}$, $W_o = \{o\} \in \eta_{dis}$. $\eta_{std} \subseteq \eta_{dis}$. Pairwise D-Lindelöf: Follows from $\mathbb{R}$ being η_{std}-Lindelöf. For pairwise D-set $A = (0, 1) = U \setminus V$ with: $U = (0, 1) \cup (2, 3) \in \eta_{std}$, $V = (2, 3) \in \eta_{dis}$

Theorem 4.23 implies A is η_{dis}-closed. A is η_{std}-Lindelöf .

The necessity of Theorem 4.23's conditions is demonstrated through pathological examples:

Example 4.25 *Consider $(\mathcal{L}, \eta_1, \eta_2)$ where $\mathcal{L} = [0, \omega_1]$, $\eta_1 =$ order topology (compact Hausdorff)*

$\eta_2 =$ topology where open sets are $\emptyset$, $\mathcal{L}$, and all $[0, \mathfrak{k})$ for $\mathfrak{k} < \omega_1$. This space is pairwise D-Lindelöf: Any pairwise D-cover $\mathcal{D}$ induces a η_1-open cover $\{U_{\mathfrak{k}}\}$. By η_1-compactness, a finite subcover exists, yielding finite D-subcover.

The pairwise D-set:

$$A = [0, \omega_1) = U \setminus V \quad \textit{with} \quad U = [0, \omega_1) \in \eta_2, \quad V = \emptyset \in \eta_1, \quad U \neq \mathcal{L}$$

satisfies: not η_1-closed, also η_2-Lindelöf.

Pairwise Hausdorff fails: ω_1 and any $\mathfrak{k} > \omega_0$ lack disjoint η_1/η_2-neighborhoods.

Example 4.26 *Consider $(\mathbb{R}, \eta_1, \eta_2)$ where $\eta_1 = \eta_{coc}$*

$\eta_2 = \eta_{std}$. This space is pairwise D-Lindelöf. Let $\mathcal{D} = \{D_{\mathfrak{k}} = U_{\mathfrak{k}} \setminus V_{\mathfrak{k}}\}$ be a pairwise D-cover. The sets $\{U_{\mathfrak{k}}\}$ form a η_1-open cover of $\mathbb{R}$. Since η_1 is not Lindelöf, but let $S = \bigcup_{\mathfrak{k}} V_{\mathfrak{k}}$. Then S is η_2-Lindelöf. Cover S by countable η_2-subcover, and $\mathbb{R} \setminus S$ by countable η_1-subcover.

The pairwise D-set:

$$A = (0, 1) = U \setminus V \quad \textit{with} \quad U = (0, 1) \in \eta_2, \quad V = \emptyset \in \eta_1, \quad U \neq \mathbb{R}$$

satisfies: Not η_1-Lindelöf, η_2-closed.

Here $\eta_1 \not\subseteq \eta_2$ since $\{0.5\} \in \eta_2$ but $\{0.5\} \notin \eta_1$.

Remark 4.27 *In any bitopological space $(\mathcal{L}, \eta_1, \eta_2)$, every singleton is Lindelöf under both η_1 and η_2, as finite sets are compact and hence Lindelöf.*

5. Applications to Statistical Spaces and Data Analysis

The mathematical structure of pairwise D-Lindelöf bitopological spaces, provides new perspectives and potential applications in statistics. While topology has long been associated with statistical concepts such as estimator convergence and continuity, the introduction of bitopological forms and D-sets provides a more comprehensive mathematical language for dealing with complex data structures and probabilistic objects. This section investigates several potential applications for these concepts, focusing on data classification, convergence analysis, and measuring compactness in probability spaces.

In the fields of statistics and data classifying, the objective frequently involves to separate a data space into distinct regions corresponding to various classes. Conventional approaches use metrics or comparable measure to define boundaries. The pairwise D-set ($A = U \setminus V$, where $U \in \eta_1$ and $V \in \eta_2$) is an adaptable mechanism for defining complex regions within a data space $\mathcal{L}$.

Consider a dataset where every point of data $x \in \mathcal{L}$ has two distinct types of features, obviously leading to a different topological structure, η_1 and η_2. For example, η_1 represents a topology based on continuous numerical attributes (e.g., Euclidean distance), whereas η_2 represents a topology generated by classified or normal features (e.g., a discrete or cofinite topology). A pairwise D-set may represent a class or cluster defined by a combination of these feature types: points that are 'open' with respect to one set of features ($U \in \eta_1$) but closed or excluded with respect to another ($V \in \eta_2$).

This framework allows for:

- Many data sets in reality are already mixed, which with continuous and discrete variables. Bitopological spaces are a suitable setting for modeling such data, as η_1 and η_2 can be determined to represent the basic characteristics of different feature subsets. A pairwise D-set can then define a decision region that takes advantage of both types of information.
- The condition $U \neq \mathcal{L}$ for D-sets indicates that the defining open set U does not cover the entire space. This might be determined as an approach to avoid trivial or overly simplistic classification rules while ensuring that the defined regions are truly discriminative. The ability to construct finite intersections of pairwise D-sets implies that complex, differing classification boundaries can be created by combining simpler D-set regions.
- Pairwise D-sets could lead to new feature engineering methods. Instead of dependent entirely on geometric properties, features could be constructed using the topological connections defined by D-sets. For example, a new feature for a data point could be its membership in various pairwise D-sets, or the D-Lindelöf degree of its surroundings. This could be especially useful for high-dimensional data, where traditional geometric concepts may break down.

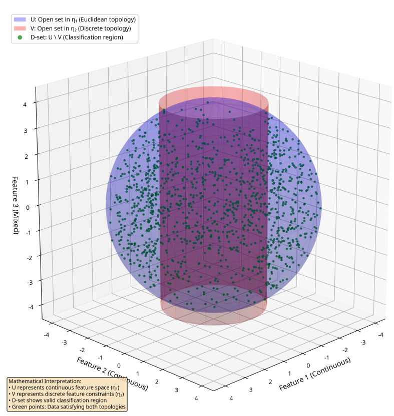

Figure 1: Visualization of a Pairwise D-set in Statistical Data Classification

The convergence of random variables is an important aspect of probability theory and statistics. Concepts such as probability convergence, close to certain convergence, and distribution convergence are commonly defined in metric or measure spaces. The manuscript's exploration of Lindelöf properties in bitopological spaces provides a new perspective on these convergence concepts.

Consider a sequence of random variables (X_n) defined on a probability space $((\Omega, \mathcal{F}, P))$. The space of random variables can be given various topologies that reflect different modes of convergence. For example, convergence in probability can be metrized, resulting in a topology. When looking at two different concept of convergence, say convergence in probability η_1) and almost sure converge (η_2), the space of random variables becomes a bitopological space $(\mathcal{L}, \eta_1, \eta_2)$.

- The Lindelöf property, which ensures the existence of a countable subcover for any open cover, is completely connected with separability and the existence of countable dense subsets. In terms of probability spaces, this could imply that if a space of random variables is pairwise D-Lindelöf, then expected complex convergence conduct can be defined as countable collections of D-events or D-regions. This might make it easier to analyze convergence for complex random procedures.

- A D-set $U \setminus V$ may represent a specific type of 'event' in the probability space, where U is an event evident under one topological lens (e.g., related to the probability measure) and V is an event evident under another (e.g., related to the sample space structure). The pairwise D-Lindelöf property would then imply that any probability space cover consisting of such D-events can be reduced to a countable subcover. This could have implications for building minimal sigma-algebras or assessing the size of event spaces required to capture certain probabilistic events.

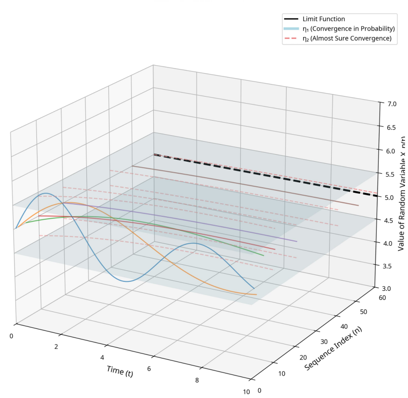

Figure 2: visualization for Convergence in Probability Spaces

Compactness is a significant idea in analysis, as it often ensures the existence of limits or the boundedness of specific functions. In probability theory, there are various concepts of compactness, such as tightness of probability measures, which is a type of compactness in the space of probability measures. The bitopological structures enables a more complex understanding of compactness in these spaces.

Consider the probability measures $\mathcal{P}$ on a measurable space $(X, \mathcal{B})$. $\mathcal{P}$ can be filled with various topologies, including the weak topology (η_1) and the stronger total variation topology $(\eta_2$. A bitopological space $(\mathcal{P}, \eta_1, \eta_2)$ would allow the study of compactness from these two perspectives.

- The concept of D-sets may lead to the definition of pairwise D-compactness. This would require finite subcovers of pairwise D-covers. Such a concept could be useful for identifying subsets of probability measures that are compact under specific weak and strong convergence requirements. For example, a set of evaluates could be pairwise D-compact if it is compact enough in the weakening topology while also being well-behaved in the overall variation topology, possibly by excluding certain pathological measures.

- The tightness of a collection of probability measures is an important concept in demonstrating distribution convergence. A collection of measures is tight if for every $\epsilon > 0$, there exists a compact set K such that the measure of K is greater than $1 - \epsilon$. In a bitopological setting, one could investigate pairwise D-tightness, where the compact set K is replaced by a pairwise D-set, or where the notion of compactness itself is defined in terms of D-sets and bitopologies. This could result in new tightness conditions that are more appropriate for complex, multidimensional probability spaces.

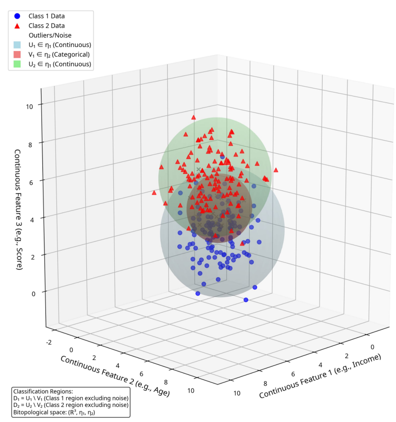

Figure 3: Visualization of Data Classification

The models of statistics frequently use parameter spaces that are topological. These parameter spaces' properties, such as connectedness, compactness, or Lindelöfness, can have significant implications for the inference of statistics, including the existence and properties of estimators, hypothesis testing, and confidence intervals. The bitopological structure permits you to model situations in which the parameter space may contain multiple relevant topological structures.

- In some statistical models, parameters may be estimated using various methods, each resulting in a different concept of closeness or neighborhood in the space of parameters. A frequentist conduct may induce one topology (η_1) based on the probability function, whereas a Bayesian approach may induce another (η_2) based on the prior distribution. A bitopological space on the parameter space would allow for a more comprehensive study of these dual structures.

- A pairwise D-set in a parameter space may represent a region of identifiable parameters (where U is an open set of identifiable parameters) from which non-identifiable or problematic parameters are excluded (represented by V). In this context, the Lindelöf property could indicate that, even if the parameter space is incredibly large, a countable collection of such D-sets can cover the effective or relevant regions for conclusion.

- The non-preservation of pairwise D-Lindelöfness under arbitrary unions, as demonstrated in the manuscript, illustrates a potential problem when combining statistical models or parameter regions. This could imply that, while individual model components may have desirable topological properties, their arbitrary combination may result in an absence of these properties, reducing the reliability or interpretability of the combined model. This knowledge could inform model selection strategies, promoting combinations that preserve these topological properties.

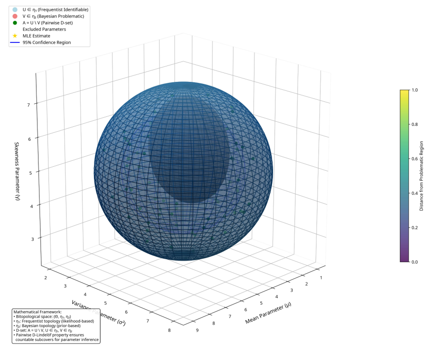

Figure 4: visualization for the D-sets in Statistical Parameter Space

In conclusion, theoretical advances in pairwise D-Lindelöf bitopological spaces provide a solid mathematical foundation for solving current statistical challenges. This structure provides an improved language for describing complex data structures, convergence behaviors, compactness properties, and model structures, opening up novel possibilities for research and application in a variety of statistical domains.

6. Conclusion

This study thoroughly developed an extensive theoretical structure for pairwise (D-Lindelöf spaces and their associated functional properties within bitopological configurations. This study fills a gap in understanding covering properties in asymmetric topological environments by introducing the concept of D-sets, which are defined as differences between open sets from different topologies. The core contribution is the formalization and rigorous analysis of pairwise D-Lindelöfness, a property that synthesizes interactions between two distinct topologies and provides a substantial generalization of existing Lindelöf concepts.

Our investigation provided several fundamental conclusions. We found that countable spaces have pairwise D-Lindelöfness under any bitopology, which is a fundamental characteristic. Pairwise D-Lindelöfness implies η_1-Lindelöfness, but it does not always guarantee full pairwise Lindelöfness. We analyzed the mathematical properties of pairwise D-sets, proving their closure under finite intersections but exposing their failure to preserve this property under arbitrary unions, even in well-behaved spaces. This understanding of D-set behavior is essential for future topological constructions.

In terms of preservation and inheritance, the study confirms that pairwise D-Lindelö is effectively retained under pairwise continuous surjections, pairwise D-irresolute functions, and perfect functions, particularly under conditions of local indiscreteness. We derived accurate inheritance conditions for η_1-closed subspaces and subspaces whose complements are pairwise D-sets, defining the boundaries of this property's adaptability. The importance of these conditions was rigorously emphasized through a series of counterexamples, which highlighted the intricate interplay between bitopological structure and the preservation of covering properties. Bitopological analysis revealed that pairwise D-Lindelö is not hereditary for open subspaces, despite being hereditary for closed subspaces.

Beyond theoretical advances, this work has significant application potential in data and statistical research. Our structure of pairwise D-Lindelöf bitopological spaces enables novel data classification processes through pairwise D-sets that define complex decision boundaries in mixed feature spaces (combining continuous/discrete variables), convergence analysis for random variables via bitopological interpretations of different convergence modes, and these applications provide pairwise D-Lindelöf spaces an efficient language for contemporary statistical challenges.

Future research directions include extending this framework to other covering properties such as paracompactness and metacompactness, exploring interactions with weaker separation axioms, and generalizing the concepts to tri-topological or fuzzy bitopological spaces. Further research could look into additional variations of these functionalities, such as define and study soft D-Lindelöf functions, examine fuzzy D-Lindelöf . This study firmly positions pairwise D-Lindelöfness as an indispensable tool for advancing both theoretical and applied topology, offering robust new machinery for analyzing complex topological systems in asymmetric environments.

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