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On Starting Procedures for IMEX General Linear Methods Based on Generalized Runge-Kutta Schemes

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ABSTRACT: General linear methods (GLMs) are an efficient class of time integration schemes to solve time-dependent differential equations, and GLMs need an initial input vector to start. Starting procedures are dedicated numerical schemes used to obtain the input vectors for executing GLMs. In this article, we present the construction of some starting procedures for implicit-explicit (IMEX) GLM solvers that are specially designed for partitioned differential systems. The construction of starting procedures is based on the generalized Runge-Kutta schemes in an IMEX environment, and we generate a sequence of starting procedures for computing the components of the initial input vector. We demonstrate examples of starting procedures of orders, $p = 2$ and $p = 3$. We provide two numerical illustrations of a time-dependent partitioned differential system, where IMEX-GLM is employed in conjunction with the proposed procedures. The reported numerical results confirm the desired convergence and do not report any order reduction.

Key Words: Starting procedure, IMEX schemes, general linear method, generalized Runge-Kutta schemes, partitioned problems.

Contents

1. Introduction

This paper deals with the partitioned system of differential equations that usually arises after the spatial discretization of partial differential equations [10]. This system of differential equations is then solved by suitable time integration schemes. The partitioned differential system as a system of first-order initial value problems (IVPs) is given as follows:

$$y' = f_1(t, y(t)) + f_2(t, y(t)), \qquad t_0 \le t \le T, \quad y(t_0) = y_0 \in \mathbb{R}^d. \tag{1.1}$$

Here, f_1 and f_2 correspond to the non-stiff and stiff components of the differential system, respectively. Assume that the functions y, f_1, and f_2 are sufficiently smooth, so that the problem given by (1.1) is well posed.

General linear methods (GLM) are a wide class of numerical solvers used for the time integration of systems of ordinary differential equations. However, GLMs are not self-starting schemes and need an input vector to start. This article aims to develop a dedicated initialization procedure for GLM to solve time-dependent partitioned differential systems (1.1). GLMs in the IMEX framework are of interest to

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various researchers [1, 2, 5, 12, 15, 25], and a lot of work is still being done to construct IMEX solvers that focus on optimizing the cost of applying numerical methods in practice to engineering problems.

The starting procedure is a very significant step in the implementation of GLMs, as the error introduced in computing the starting vectors carries through the numerical integration. In the work of Califano et al. [4], a systematic approach to the starting vector for solving unpartitioned problems using GLMs has been presented. Their work contains two approaches for starting vector generation: the first one is to compute the higher order derivative and approximate it to y at equally spaced time points and the second one uses a generalized Runge-Kutta scheme of the same order as GLM. More approaches based on Taylor series can be seen in [19, 20, 21]. In addition, an elaborate account of the starting procedure for two-step Runge-Kutta [7, 17, 18] and GLM with inherent stability of RK [22, 23, 24] is given in [13], and for more details, see [4, 9] and the references cited therein. To construct a dedicated numerical procedure for (1.1), we use the transformation given as follows [1, 6, 12, 25].

$$\begin{cases} y = x + z, \\ x' = f_1(x+z), \\ z' = f_2(x+z). \end{cases} \tag{1.2}$$

This transforms the additive partitioned system into a component-wise partitioned system of two differential equations where $x' = f_1, z' = f_2$ and $y' = x' + z'$.

We consider implicit-explicit (IMEX) GLMs characterized by the matrices $\mathbf{A}^{\{1\}} = [a_{ij}^{\{1\}}]_{s\times s}$, $\mathbf{B}^{\{1\}} = [b_{ij}^{\{1\}}]_{r\times s}$, and $\mathbf{A}^{\{2\}} = [a_{ij}^{\{2\}}]_{s\times s}$, $\mathbf{B}^{\{2\}} = [b_{ij}^{\{2\}}]_{r\times s}$ that correspond to the explicit and implicit bases, respectively. Also, the IMEX GLM adheres to some simplifying assumptions: the matrices associated with the input vector in the internal stage $\mathbf{U}^{\{1\}} = \mathbf{U}^{\{2\}} = \mathbf{U}$, and the external stage $\mathbf{V}^{\{1\}} = \mathbf{V}^{\{2\}} = \mathbf{V}$ along with the abscissa vector $\mathbf{c}^{\{1\}} = \mathbf{c}^{\{2\}} = \mathbf{c}$ to guarantee the internal consistency of the solution i.e., $y(t_n+h) = x(t_n+h) + z(t_n+h)$. Here, $\mathbf{U} = [u_{ij}]_{s\times r}$ and $\mathbf{V} = [v_{ij}]_{r\times r}$. The class was first introduced by [25] and one step ($t_n = t_{n-1}+h$) of the scheme is given as

$$\begin{aligned} Y_i^{[n]} &= h\left(\sum_{j=1}^{s} a_{ij}^{\{1\}} f_1(Y_j^{[n]}) + \sum_{j=1}^{s} a_{ij}^{\{2\}} f_2(Y_j^{[n]})\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \qquad i = 1,2,\dots,s, \\ y_i^{[n]} &= h\left(\sum_{j=1}^{s} b_{ij}^{\{1\}} f_1(Y_j^{[n]}) + \sum_{j=1}^{s} b_{ij}^{\{2\}} f_2(Y_j^{[n]})\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \qquad i = 1,2,\dots,r. \end{aligned} \tag{1.3}$$

The quantities $Y_i^{[n]}$ and $y_i^{[n]}$ are components of the internal and external stage vector. The method (1.3) in the matrix notation can also be written as

$$\begin{aligned} Y^{[n]} &= h(\mathbf{A}^{\{1\}} \otimes \mathbf{I}) f_1(Y^{[n]}) + h(\mathbf{A}^{\{2\}} \otimes \mathbf{I}) f_2(Y^{[n]}) + (\mathbf{U} \otimes \mathbf{I}) y^{[n-1]}, \\ y^{[n]} &= h(\mathbf{B}^{\{1\}} \otimes \mathbf{I}) f_1(Y^{[n]}) + h(\mathbf{B}^{\{2\}} \otimes \mathbf{I}) f_2(Y^{[n]}) + (\mathbf{V} \otimes \mathbf{I}) y^{[n-1]}. \end{aligned} \tag{1.4}$$

Here, $\mathbf{I}$ is the identity matrix of dimension d. An IMEX GLM is said to be of order p and stage order q, if and only if

$$\begin{aligned} e^{\mathbf{c}w} - w\mathbf{A}^{\{1\}} e^{\mathbf{c}w} + \mathbf{U}\mathbf{Q}^{\{1\}}(w) &= \mathcal{O}\left(w^{p+1}\right), \\ e^{\mathbf{c}w} - w\mathbf{A}^{\{2\}} e^{\mathbf{c}w} + \mathbf{U}\mathbf{Q}^{\{2\}}(w) &= \mathcal{O}\left(w^{p+1}\right), \\ e^{w}\mathbf{Q}^{\{1\}}(w) - w\mathbf{B}^{\{1\}} e^{\mathbf{c}w} + \mathbf{V}\mathbf{Q}^{\{1\}}(w) &= \mathcal{O}\left(w^{p+1}\right), \\ e^{w}\mathbf{Q}^{\{2\}}(w) - w\mathbf{B}^{\{2\}} e^{\mathbf{c}w} + \mathbf{V}\mathbf{Q}^{\{2\}}(w) &= \mathcal{O}\left(w^{p+1}\right), \end{aligned} \tag{1.5}$$

where $\mathbf{Q}^{\{1\}}(w)$ and $\mathbf{Q}^{\{2\}}(w)$ defined as $\mathbf{Q}^{\{1\}}(w) = \sum_{k=0}^{p} \ell_k^{\{1\}} w^k$, $\mathbf{Q}^{\{2\}}(w) = \sum_{k=0}^{p} \ell_k^{\{2\}} w^k$ and $e^{\mathbf{c}w}$ denote a vector with components given by $e^{(c_i w)}$, $i = 1, 2, \ldots, s$. Vectors $\ell_k^{\{1\}}$ and $\ell_k^{\{2\}}$ are given as

$$\ell_k^{\{1\}} = \begin{bmatrix} \ell_{1k}^{\{1\}} \\ \ell_{2k}^{\{1\}} \\ \vdots \\ \ell_{rk}^{\{1\}} \end{bmatrix}, \quad \ell_k^{\{2\}} = \begin{bmatrix} \ell_{1k}^{\{2\}} \\ \ell_{2k}^{\{2\}} \\ \vdots \\ \ell_{rk}^{\{2\}} \end{bmatrix}. \tag{1.6}$$

The real parameters $\ell_{ik}^{\{1\}}$ and $\ell_{ik}^{\{2\}}$, for $k = 1, 2, \ldots, p$, are the component-wise Taylor weights associated with the external stage approximations. Moreover, the preconsistency conditions of the IMEX GLM imply that $\ell_{i0}^{\{1\}} = \ell_{i0}^{\{2\}}$.
We aim to generate the input vector, and the components $y_i^{[0]}$, $i = 1, 2, \cdots, r$ of the input vector is given as

$$y_i^{[0]} = x_i^{[0]} + z_i^{[0]} + \mathcal{O}(h^{p+1}). \tag{1.7}$$

and $y_i^{[0]}$ satisfies the following order assumption

$$y_i^{[0]} = \ell_{i0} y(t_0) + \sum_{k=1}^{p} \ell_{ik}^{\{1\}} h^k x^{(k)}(t_0) + \sum_{k=1}^{p} l_{ik}^{\{2\}} h^k z^{(k)}(t_0) + \mathcal{O}(h^{p+1}), \quad i = 1, 2, \ldots, r. \tag{1.8}$$

It is obvious that $y(t_0)$, $x'(t_0)$ and $z'(t_0)$ is known from the problem (1.1). Therefore, the generation of $y^{[0]}$ reduces to computing $x^{(k)}(t_0)$ and $z^{(k)}(t_0)$ for $k = 2, \cdots, p$. In problems modeled for practical engineering problems, it is not always easy to calculate derivatives up to order p; hence, alternatives to derivative computation are required.
In most of the literature [1, 12, 14, 16, 25], the computation of higher-order derivatives is done by using a Taylor series transform that calculates the approximation of y at equally spaced time points. This particular approach is the classical method for computing the input vector and is used in most existing GLM algorithms; however, it requires the use of an external methodology, such as IMEX RK or Runge-Kutta schemes of suitable order, to obtain an approximation to y. The starting procedure that we are proposing in this work will not use any derivative approximation. These procedures are fully customized for particular IMEX GLMs, as they depend on the corresponding Taylor weights $\ell_{ik}^{[m]}$ of the IMEX GLM method and are designed specifically to generate the input vectors.

The remainder of the article is organized as follows. Section 2 outlines the construction of a starting procedure for IMEX GLMs using the GIMEX-RK approach and derives the associated order conditions. Section 3 presents two example procedures of orders 2 and 3, with starting coefficients expressed via Taylor weights. Numerical results for two test problems are reported in Section 4, followed by conclusions in Section 5.

2. Starting procedures for IMEX general linear methods

To construct the starting procedure, we combine two Runge–Kutta methods of order p with s stages each: a diagonally implicit Runge–Kutta (DIRK) scheme and an explicit Runge–Kutta (ERK) scheme. This combination is feasible due to the DIRK structure of the matrix $\mathbf{A}^2$, which enables the formulation of GIMEX-RK methods with $s = p$. Moreover, this construction also ensures that the internal consistency condition is satisfied.
The components of the input vector $y_i^{[0]}$, $i = 1, 2, \cdots, r$, are calculated with the help of a sequence of generalized IMEX RK (GIMEX RK) methods, i.e., each y_i is calculated by a different GIMEX RK scheme of the same order, similar to the approach given in [4] for unpartitioned problems.
For partitioned problem (1.1), the GIMEX RK method $(\mathbf{c}, \mathbf{A}, \mathbf{b})$ of order p with s stages can be formulated

as

$$\begin{aligned} Y_i^{[0]} &= y_0 + h\left(\sum_{j=1}^{i-1} a_{ij}^{\{1\}} f_1(Y_j^{[0]})\right) + h\left(\sum_{j=1}^{i} a_{ij}^{\{2\}} f_2(Y_j^{[0]})\right), \qquad i = 1,2,\ldots,s, \\ y_1 &= b_0 y_0 + h\left(\sum_{j=1}^{s} b_j^{\{1\}} f_1(Y_j^{[0]})\right) + h\left(\sum_{j=1}^{s} b_j^{\{2\}} f_2(Y_j^{[0]})\right), \qquad i = 1,2,\ldots,r. \end{aligned} \tag{2.1}$$

On collecting the coefficients in the expressions in (2.1), we get the matrix triad $(\mathbf{c}^{\{1\}}, \mathbf{A}^{\{1\}}, \mathbf{b}^{\{1\}})$ for the explicit counterpart and $(\mathbf{c}^{\{2\}}, \mathbf{A}^{\{2\}}, \mathbf{b}^{\{2\}})$ as the coefficient matrices for the implicit base, with the assumption that the abscissa vector is same for both the implicit base and the explicit counterpart i.e., $\mathbf{c}^{\{1\}} = \mathbf{c}^{\{2\}}$ to maintain the internal consistency.
The method coefficients in a compact form can be represented by the Butcher tableau

$$\begin{array}{c|c|c} \mathbf{c} & \mathbf{A}^{\{1\}} & \mathbf{A}^{\{2\}} \\ \hline \mathbf{b}_0 & (\mathbf{b}^{\{1\}})^T & (\mathbf{b}^{\{2\}})^T \end{array}. \tag{2.2}$$

Note that the matrices $\mathbf{A}^{\{m\}}$ and $\mathbf{c}^{\{m\}}$ for $m = 1, 2$ are associated with the starting procedures (2.1) and are different from the matrices given in (1.4).

The scheme described by (2.1), used to generates the components of the input vector $y_i^{[0]}$ under the assumption that it approximates y_1 up to the order p, i.e.,

$$y_i^{[0]} = y_1 + \mathcal{O}(h^{p+1}) = x_1 + z_1 + \mathcal{O}(h^{p+1}). \tag{2.3}$$

We derive the order conditions for the GIMEX RK scheme, keeping (2.3) in consideration. The most important property of IMEX schemes is that they do not require any coupling order conditions. Hence, we first start with the generalized ERK method and derive order conditions for it. The generalized ERK scheme is given as

$$\begin{aligned} X_i^{[0]} &= x_0 + h\sum_{j=1}^{i-1} a_{ij}^{\{1\}} f_1(Y_j^{[0]}), \quad i = 1, \cdots, s, \\ x_1 &= b_0 x_0 + h\sum_{j=1}^{s} b_j^{\{1\}} f_1(Y_j^{[0]}). \end{aligned} \tag{2.4}$$

Now, consider the continuous extension of (2.4), given by

$$\begin{aligned} X_i^{[0]}(t) &= x_0 + (t - t_0)\sum_{j=1}^{i-1} a_{ij}^{\{1\}} f_1(Y_j^{[0]}), \quad i = 1, \cdots, s, \\ \tilde{x}_1(t) &= b_0 x_0 + (t - t_0)\sum_{j=1}^{s} b_j^{\{1\}} f_1(Y_j^{[0]}), \end{aligned} \tag{2.5}$$

$t \in [t_0, t_0 + h]$. Expanding the numerical solution $\tilde{x}_1(t_0 + h)$ into Taylor series around t_0, keeping in account that $\tilde{x}_1(t_0) = b_0 x(t_0)$,

$$\tilde{x}_1(t_0 + h) = b_0 x(t_0) + h\tilde{x}_1'(t_0) + \frac{h^2}{2!}\tilde{x}_1''(t_0) + \cdots + \frac{h^p}{p!}\tilde{x}_1^{(p)}(t_0) + \mathcal{O}(h^{p+1}). \tag{2.6}$$

Now, we consider the generalized DIRK scheme as

$$\begin{aligned} Z_i^{[0]} &= z_0 + h\sum_{j=1}^{i} a_{ij}^{\{2\}} f_2(Y_j^{[0]}), \quad i = 1, \cdots, s, \\ z_1 &= b_0 z_0 + h\sum_{j=1}^{s} b_j^{\{2\}} f_2(Y_j^{[0]}), \end{aligned} \tag{2.7}$$

and, the continuous extension of (2.7) is given by

$$\begin{aligned} Z_i^{[0]}(t) &= z_0 + (t-t_0)\sum_{j=1}^{i} a_{ij}^{\{2\}} f_2(Y_j^{[0]}) \quad i=1,\cdots,s, \\ \tilde{z}_1(t) &= b_0 z_0 + (t-t_0)\sum_{j=1}^{s} b_j^{\{2\}} f_2(Y_j^{[0]}), \end{aligned} \tag{2.8}$$

$t \in [t_0, t_0+h]$. Expanding the numerical solution $\tilde{z}_1(t_0+h)$ into a Taylor series around t_0 and $\tilde{z}_1(t_0) = b_0 z(t_0)$,

$$\tilde{z}_1(t_0+h) = b_0 z(t_0) + h\tilde{z}_1'(t_0) + \frac{h^2}{2!}\tilde{z}_1''(t_0) + \cdots + \frac{h^p}{p!}\tilde{z}_1^{(p)}(t_0) + \mathcal{O}(h^{p+1}). \tag{2.9}$$

Using the transformation (1.2), (2.3) becomes

$$\begin{aligned} x_i^{[0]} &= x_1 + \mathcal{O}(h^{p+1}), \\ z_i^{[0]} &= z_1 + \mathcal{O}(h^{p+1}). \end{aligned}$$

Adding (2.6) and (2.9),

$$\begin{aligned} \tilde{x}_1(t_0+h) + \tilde{z}_1(t_0+h) = b_0\left(x(t_0)+z(t_0)\right) + h\left(\tilde{x}_1'(t_0)+\tilde{z}_1'(t_0)\right) + \frac{h^2}{2!}\left(\tilde{x}_1''(t_0)+\tilde{z}_1''(t_0)\right) + \cdots \\ + \frac{h^p}{p!}\left(\tilde{x}_1^{(p)}(t_0)+\tilde{z}_1^{(p)}(t_0)\right) + \mathcal{O}(h^{p+1}), \end{aligned}$$

$$\tilde{y}_1(t_0+h) = b_0 y(t_0) + h\tilde{y}_1'(t_0) + \frac{h}{2!}\tilde{y}_1''(t_0) + \cdots + \frac{h^p}{p!}\tilde{y}_1^{(p)}(t_0) + \mathcal{O}(h^{p+1}). \tag{2.10}$$

In order to obtain the order conditions for GIMEX RK (2.1), we compare (2.10) and (1.7), i.e., $y_i^{[0]}$ and $\tilde{y}_1(t_0+h)$, we get

$$x_i^{[0]} = \tilde{x}_1(t_0+h) + \mathcal{O}(h^{p+1}) \text{ and } z_i[0] = \tilde{z}_1(t_0+h) + \mathcal{O}(h^{p+1}).$$

Using the transformation (1.2) again, we have,

$$y_i^{[0]} = \tilde{y}_1(t_0+h) + \mathcal{O}(h^{p+1}) = \tilde{x}_1(t_0+h) + \tilde{z}_1(t_0+h) + \mathcal{O}(h^{p+1}), \tag{2.11}$$

if and only if

$$\begin{cases} b_0 = \ell_{i0}, \\ \tilde{x}^{(k)}(t_0) = k!\ell_{ik}^{\{1\}} x^{(k)}(t_0), \\ \tilde{z}^{(k)}(t_0) = k!\ell_{ik}^{\{2\}} z^{(k)}(t_0), \end{cases} \tag{2.12}$$

for $k = 1, 2, \cdots, p$. Using the theory of rooted trees, the order conditions and elementary differentials for the starting procedure of GLM have been explained in [4, 11]. Now, we proceed to find the order conditions for GIMEX RK using the theory of rooted trees and elementary differentials analogous to the approach given in [3, 4, 11].

$$x^{(k)}(t_0) = \sum_{t\in T_k} \alpha(t) F_1[t](x_0) \text{ and } z^{(k)}(t_0) = \sum_{t\in T_k} \alpha(t) F_2[t](z_0), \tag{2.13}$$

also,

$$\tilde{x}^{(p)}(t_0) = \sum_{t\in T_k} \alpha(t)\gamma(t)\overline{\Phi(t)} F_1[t](x_0) \text{ and } \tilde{z}^{(p)}(t_0) = \sum_{t\in T_k} \alpha(t)\gamma(t)\Phi(t) F_2[t](z_0), \tag{2.14}$$

for $k = 1, 2, \cdots, p$. Here, $\overline{\Phi(t)}$ and $\Phi(t)$ are the elementary weights and $F_1[t]$ and $F_2[t]$ are the elementary differentials corresponding to the non-stiff and stiff component of the system (1.1) respectively with $\gamma(t)$ as the density of the tree t. Using (2.13) and (2.14) in (2.12) can be written as

$$\sum_{t\in T_k} \alpha(t)\gamma(t)\overline{\Phi(t)}F_1[t](x_0) = k!\ell_{ik}^{\{1\}} \sum_{t\in T_k} \alpha(t)F_1[t](x_0) \text{ and,}$$
$$\sum_{t\in T_k} \alpha(t)\gamma(t)\Phi(t)F_2[t](z_0) = k!\ell_{ik}^{\{2\}} \sum_{t\in T_k} \alpha(t)F_2[t](z_0).$$

Now we obtain the order conditions for the starting procedures to compute the components $y_i^{[0]}, i = 1, 2\cdots, r$, of the starting vector.

$$\begin{aligned} \overline{\Phi(t)} &= \frac{k!\ell_{ik}^{\{1\}}}{\gamma(t)}, \\ \Phi(t) &= \frac{k!\ell_{ik}^{\{2\}}}{\gamma(t)}. \end{aligned} \tag{2.15}$$

The elementary weights $\overline{\Phi(t)}$ and $\Phi(t)$ are similar to those defined for unpartitioned problems and are reported in [4, 11], and that is why we refrain from stating them here explicitly. Another result that we have derived in our earlier works is the error estimate at the start of integration due to the order of the starting procedure chosen in the Taylor series approach of input vector generation. This work is currently under consideration for publication, and hence we omit the proof of that theorem here and only provide the derived estimate.
If the starting vector is correct up to $\mathcal{O}(h^{p+2})$, then the error in one step of an IMEX GLM ($p = q = r = s$) scheme (1.3) is given by

$$\begin{aligned} y(t_1) - y_1 = h^{p+1}x^{(p+1)}(t_0)\left(\frac{1}{(p+1)!} - \sum_{j=1}^{s} a_{sj}^{\{1\}}\frac{c_j^p}{p!}\right) + \\ h^{p+1}z^{(p+1)}(t_0)\left(\frac{1}{(p+1)!} - \sum_{j=1}^{s} a_{sj}^{\{2\}}\frac{c_j^p}{p!}\right) + \mathcal{O}(h^{p+2}). \end{aligned} \tag{2.16}$$

3. Examples of starting procedures based on generalized IMEX RK

We present two examples of the starting procedure of order 2 and order 3 constructed using the order conditions given in Section 2. Since the structure of matrices $\mathbf{A}^{\{m\}}$, for $m = 1, 2$ in (2.2) doesn't allow the construction of starting procedure order 1, we start the construction with order 2.

3.1. Starting procedure of order $p = 2$ (GIMEX-RK2)

Taking a two-stage, second-order combination of DIRK and ERK schemes, the starting procedure for the method of order p with stages s, and $p = s$ can be seen as

$$\left[\begin{array}{c|cc|cc} c_1 & 0 & 0 & a_{11}^{\{2\}} & 0 \\ c_2 & a_{21}^{\{1\}} & 0 & a_{21}^{\{2\}} & a_{22}^{\{2\}} \\ \hline b_0 & b_1^{\{1\}} & b_2^{\{1\}} & b_1^{\{2\}} & b_2^{\{2\}} \end{array}\right].$$

Here, we have considered a 2-stage explicit method of order 2 and a diagonally implicit method of the same order. The method can now be written as

$$\begin{aligned} Y_1 &= y_0 + h(a_{11}^{\{2\}}f_2(Y_1)) \\ Y_2 &= y_0 + h(a_{21}^{\{1\}}f_1(Y_1)) + h(a_{21}^{\{2\}}f_2(Y_1) + a_{22}^{\{2\}}f_2(Y_2)) \\ y_1 &= b_0y_0 + h(b_1^{\{1\}}f_1(Y_1) + b_2^{\{1\}}f_1(Y_2)) + h(b_1^{\{2\}}f_2(Y_1) + b_2^{\{2\}}f_2(Y_2)). \end{aligned}$$

To determine the coefficients of these methods, we use the order conditions provided in Section 2 for $p=2$, with $b_0=\ell_{i0}$ and $s=2$. $\sum_{j=1}^{s} a_{ij}^{\{m\}}=c_i$, for $i=1,2$, and $m=1,2$, $\sum_{j=1}^{s} a_{ij}^{\{m\}} c_j=\frac{c_i^2}{2}$, $i=1,2,$ and $m=2$

$$\begin{aligned}
\Phi(\tau) &= b_1^{\{m\}}+b_2^{\{m\}}=\ell_{i1}^{\{m\}}, \quad m=1,2\\
\Phi([\tau]) &= b_2^{\{m\}} c_2=\ell_{i2}^{\{m\}}, \quad m=1,2.
\end{aligned} \tag{3.1}$$

Solving these conditions, we arrive at the sequence of methods with $\mathbf{c}=[c_1=0, c_2]$ with c_2 taken as λ, and the explicit counterpart as obtained in [4],

$$\begin{aligned}
&a_{21}^{\{1\}}=\lambda, \quad b_1^{\{1\}}=\frac{1}{\lambda}(-\ell_{i1}^{\{1\}}\lambda+\ell_{i2}^{\{1\}}), \quad b_2^{\{1\}}=\frac{1}{\lambda}(\ell_{i2}^{\{1\}}),\\
&a_{11}^{\{2\}}=0, \quad a_{21}^{\{2\}}=\frac{\lambda}{2}, \quad a_{22}^{\{2\}}=\frac{\lambda}{2},\\
&b_1^{\{2\}}=\frac{1}{\lambda}(-\ell_{i1}^{\{2\}}\lambda+\ell_{i2}^{\{2\}}), \quad b_2^{\{2\}}=\frac{1}{\lambda}(\ell_{i2}^{\{2\}}).
\end{aligned} \tag{3.2}$$

3.2. Starting procedure of order $p=3$ (GIMEX-RK3)

The GIMEX RK for the starting procedure of order $p=3$ with the number of stages $s=3$ in the Butcher tableau representation is written as

$$\left[\begin{array}{c|ccc|ccc}
c_1 & 0 & 0 & 0 & a_{11}^{\{2\}} & 0 & 0\\
c_2 & a_{21}^{\{1\}} & 0 & 0 & a_{21}^{\{2\}} & a_{22}^{\{2\}} & 0\\
c_3 & a_{31}^{\{1\}} & a_{32}^{\{1\}} & 0 & a_{31}^{\{2\}} & a_{32}^{\{2\}} & a_{33}^{\{2\}}\\
\hline
b_0 & b_1^{\{1\}} & b_2^{\{1\}} & b_3^{\{1\}} & b_1^{\{2\}} & b_2^{\{2\}} & b_3^{\{2\}}
\end{array}\right].$$

The method can be written as follows :

$$\begin{aligned}
Y_1 &= y_0+h\left(a_{11}^{\{2\}} f_2(Y_1)\right),\\
Y_2 &= y_0+h\left(a_{21}^{\{1\}} f_1(Y_1)\right)+h\left(a_{21}^{\{2\}} f_2(Y_1)+a_{22}^{\{2\}} f_2(Y_2)\right),\\
Y_3 &= y_0+h\left(a_{31}^{\{1\}} f_1(Y_1)+a_{32}^{\{1\}} f_1(Y_2)\right)+h\left(a_{31}^{\{2\}} f_2(Y_1)+a_{32}^{\{2\}} f_2(Y_2)+a_{33}^{\{2\}} f_2(Y_3)\right),\\
y_1 &= b_0 y_0+h\left(b_1^{\{1\}} f_1(Y_1)+b_2^{\{1\}} f_1(Y_2)+b_3^{\{1\}} f_1(Y_3)\right)+h\left(b_1^{\{2\}} f_2(Y_1)+b_2^{\{2\}} f_2(Y_2)+b_3^{\{2\}} f_2(Y_3)\right).
\end{aligned}$$

Using the order conditions, (2.15) for $p=3$ with $b_0=\ell_{i0}$ and $s=3$,$\sum_{j=1}^{s} a_{ij}^{\{m\}}=c_i$, and $\sum_{j=1}^{s} a_{ij}^{\{m\}} c_j=\frac{c_i^2}{2}$, for $i=1,2,3,$ and $m=2$

$$\begin{aligned}
\Phi(\tau) &= b_1^{\{m\}}+b_2^{\{m\}}+b_3^{\{m\}}=\ell_{i1}^{\{m\}}, \quad m=1,2,\\
\Phi([\tau]) &= b_2^{\{m\}} c_2+b_3^{\{m\}} c_3=\ell_{i2}^{\{m\}}, \quad m=1,2,\\
\Phi([\tau^2]) &= b_2^{\{m\}} c_2^2+b_3^{\{m\}} c_3^2=2\ell_{i3}^{\{m\}}, \quad m=1,2,\\
\Phi([_2\tau]_2) &= b_3^{\{m\}} a_{32}^{\{m\}} c_2=\ell_{i3}^{\{m\}}, \quad m=1,2.
\end{aligned} \tag{3.3}$$

On solving these order conditions, we arrive at the GIMEX-RK scheme for each component of the input vector, for $y_i^{[0]}$, $i=1,2,3$, and $[c_1,c_2,c_3]=[0,\lambda,1]$ is given as

$$\begin{aligned}
&a_{21}^{\{1\}}=\lambda, \quad a_{31}^{\{1\}}=1+\frac{\ell_{i3}-\lambda\ell_{i3}}{\lambda^2\ell_{i2}-2\lambda\ell_{i3}}, \quad a_{32}^{\{1\}}=-\frac{(-1+\lambda)\ell_{i3}}{\lambda(\lambda\ell_{i2}-2\ell_{i3})},\\
&b_1^{\{1\}}=\frac{\lambda\ell_{i1}^{\{1\}}-\ell_{i2}^{\{1\}}-\lambda\ell_{i2}^{\{1\}}+2\ell_{i3}^{\{1\}}}{\lambda}, \quad b_2^{\{1\}}=\frac{-\ell_{i2}^{\{1\}}+2\ell_{i3}^{\{1\}}}{(\lambda-1)\lambda}, \quad b_3^{\{1\}}=\frac{\lambda\ell_{i2}^{\{1\}}-2\ell_{i3}^{\{1\}}}{\lambda-1},\\
&a_{11}^{\{2\}}=0, \quad a_{21}^{\{2\}}=\lambda/2, \quad a_{22}^{\{2\}}=\lambda/2, \quad a_{31}^{\{2\}}=\frac{1}{2}(1-2a_{32}^{\{2\}}+2a_{32}^{\{2\}}\lambda), \quad a_{33}^{\{2\}}=\frac{1}{2}(1-2a_{32}^{\{2\}}\lambda),\\
&b_1^{\{2\}}=\frac{\lambda\ell_{i1}^{\{2\}}-\ell_{i2}^{\{2\}}-\lambda\ell_{i2}^{\{2\}}+2\ell_{i3}^{\{2\}}}{\lambda}, \quad b_2^{\{2\}}=\frac{-\ell_{i2}^{\{2\}}+2\ell_{i3}^{\{1\}}}{(\lambda-1)\lambda}, \quad b_3^{\{2\}}=\frac{\lambda\ell_{i2}^{\{2\}}-2\ell_{i3}^{\{2\}}}{\lambda-1}.
\end{aligned} \tag{3.4}$$

4. Numerical illustration

To illustrate the applicability of the proposed starting procedure, we demonstrate two test problems, namely the Prothero-Robinson (PR) problem and the van der problem. The IMEX GLM to apply these starting procedures has been taken from [25] for both the orders 2 and 3. The error norms have been reported in the form of Tables. The error with N number of steps is denoted by $\|e_N(T)\|$ and taking $h = (T - t_0)/N$ the error norm becomes $\|e_h(T)\|$. The estimated order is calculated using the formula

$$p = \frac{\log\left(\frac{\|e_{N_1}(T)\|}{\|e_{N_2}(T)\|}\right)}{\log 2}, \tag{4.1}$$

where $\|e_{N_1}(T)\|$, $\|e_{N_2}(T)\|$ denote the error norms at time T when using two consecutive time steps N_1, N_2 respectively and $N_2 = 2N_1$.

4.1. Prothero-Robinson problem

We consider the PR-problem as introduced by Prothero and Robinson in 1974 [8]

$$y' = \lambda(y - \psi(t)) + \psi'(t), \quad y(t_0) = \psi(t_0), \quad t \in [0, T], \tag{4.2}$$

where, λ is the stiffness parameter with $Re(\lambda) < 0$. For the experimental purpose, we choose $\psi(t) = \sin t$, $\lambda = -10^5$, and $T = 50$. The numerical results obtained using both the starting procedures have been presented in the form of Table 1, and the error versus step size decay is shown in Fig. 1. The results obtained show convergence of the numerical solution with the expected order, same as the imposed order. Here, the error is obtained from the exact solution $\psi(t) = \sin t$.

Table 1: Computation of error norm and estimated order for the PR-problem using both the starting procedures.

	GIMEX-RK2		GIMEX-RK3	
N	$\|e_N(T)\|$	p	$\|e_N(T)\|$	p
2^9	3.41329×10^{-7}	–	4.72784×10^{-9}	–
2^{10}	8.80690×10^{-8}	1.9545	4.57862×10^{-10}	3.3682
2^{11}	2.22632×10^{-8}	1.9840	4.86067×10^{-11}	3.2357
2^{12}	5.57310×10^{-9}	1.9981	5.48722×10^{-12}	3.1470
2^{13}	1.38422×10^{-9}	2.0094	6.35492×10^{-13}	3.1101
2^{14}	3.40184×10^{-10}	2.0247	6.60583×10^{-14}	3.2661

4.2. van der Pol problem

The van der Pol oscillator is a nonlinear second-order differential equation originally introduced to model electrical circuits involving vacuum tubes [8]. For numerical purposes, we consider its equivalent formulation as a system of two first-order differential equations, expressed in partitioned form as follows:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\left(1 - u^2\right) v - u\right)/\epsilon \end{bmatrix}. \tag{4.3}$$

The system is integrated in the time interval $t \in [0, 3/4]$, with the final time point as $T = 3/4$, where ϵ is taken as 10^{-3}. The initial conditions are given by $u(0) = 2$ and $v(0) = 0$. The reference solution is obtained using `MATLAB` ode15s. The error norm and expected order are reported in the Table 2, and the convergence of the IMEX GLM schemes using the proposed starting procedures is shown in the Fig. 2.

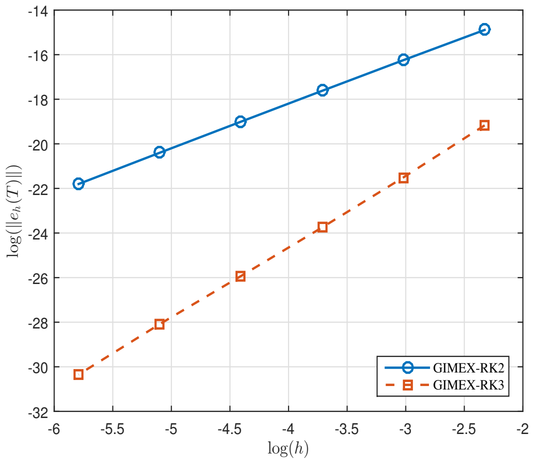

Figure 1: $log(\|e_h(T)\|)$ vs $log(h)$ convergence plots of IMEX GLM for PR problem

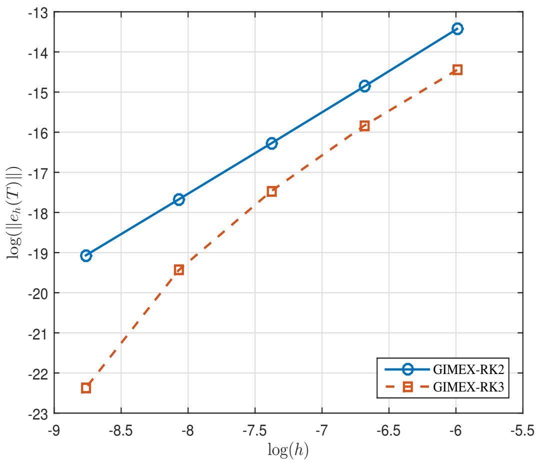

Figure 2: $log(\|e_h(T)\|)$ vs $log(h)$ convergence plots of IMEX GLM for van der Pol problem

Table 2: Computation of error norm and estimated order for the van der Pol problem using both the starting procedures.

	GIMEX-RK2		GIMEX-RK3	
N	$\|e_N(T)\|$	p	$\|e_n(T)\|$	p
200	1.47680×10^{-6}	–	5.30399×10^{-7}	–
400	3.51593×10^{-7}	2.0705	1.32531×10^{-7}	2.0070
800	8.54780×10^{-8}	2.0403	2.59873×10^{-8}	2.3505
1600	2.10507×10^{-8}	2.0217	3.60424×10^{-9}	2.8500
3200	5.22243×10^{-9}	2.0111	1.91917×10^{-10}	4.2311

5. Conclusion

In this article, we have successfully proposed a starting procedure for the computation of the input vector of an IMEX GLM. We propose the starting procedure based on the GIMEX RK method to com-

pute the components of the input vector. The procedures are capable of generating the input vector, and further these input vectors are used to execute IMEX GLM for two partitioned problems. For both the PR problem and the van der Pol problem, we have reported the results of IMEX GLMs along with the expected order via both the example procedures. The reported results are very conclusive to infer a significantly small error norm, as well as the convergence of the IMEX GLM is not hindered.
For our future work, we are interested in studying the optimal step size management for targeted GLM with an imposed step size for the initial procedure. We are also focused on developing more dedicated and class-oriented starting procedures.

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