Bol. Soc. Paran. Mat. (3s.) **v. 2026 (44)** : 1–13.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.78341

Averaged Controllability of the Klein-Gordon Equation with a Parametric Electromagnetic Potential

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ABSTRACT: In this paper, we consider the average null controllability problem for a Klein-Gordon equation with an electromagnetic potential, which depends on a parameter, that represents the electromagnetic field properties. These properties are affected by many different factors related to the behavior of particles. Thus, to achieve the desired result, we apply the Hilbert Uniqueness Method, which provides direct and inverse averaged inequalities. These inequalities assume the continuity and coercivity of a constructed operator, this entails establishing a parameter independent control that brings the average (with respect to the parameter) of the system to zero.

Key Words: averaged controllability, HUM, observability inequality, the Klein-Gordon equation, the electromagnetic potential depends on a parameter.

Contents

1. Introduction

This paper addresses the following averaged control problem of the wave equation with potential depending on a parameter:

$$\left\{\begin{array}{cc} y_{tt}-\Delta y+p(x,\sigma)y=0 & \text{in } Q,\\ y=u & \text{on } \Sigma_0,\\ y=0 & \text{on}\Sigma\backslash\Sigma_0,\\ y(x,0)=y_0\left(x\right),y_t\left(x,0\right)=y_1\left(x\right) & \text{in } \Omega. \end{array}\right. \tag{1.1}$$

With localized Dirichlet control $u(x,t)\in L^2(\Sigma_0)$,$Q=(0,T)\times\Omega$, $\Omega\subset\mathbb{R}^n$ is an open bounded domain with a regular boundary $\Gamma=\partial\Omega$, $\Sigma=(0,T)\times\Gamma$ and Σ_0 is a nonempty part of Σ. The potential $p\left(x,\sigma\right)$, dependent on the space variable and the real unknown parameter σ in $[0,1]$, is considered a random variable that follows the uniform probability law. The potential is assumed to be in $L^{\infty}\left(\Omega\right)$ for all $\sigma\in[0,1]$, and (y_0,y_1) are the initial datum independent of σ in $H_0^1\left(\Omega\right)\times L^2\left(\Omega\right)$. For each value of $\sigma\in[0,1]$, $y\left(x,t,\sigma\right)$ is the unique solution of (1.1) in $C\left(0,T;H_0^1\left(\Omega\right)\right)\cap C^1\left(0,T;L^2\left(\Omega\right)\right)$, (see [26]). One of the most widely used models in quantum field theory is the Klein-Gordon equation, a fundamental wave equation describing the behavior of spinless scalar particles; according to [22], this equation is likewise regarded as a more practical form of the Schrodinger equation. We refined the Klein-Gordon equation with an electromagnetic potential to explain the interaction of a charged particle with the electromagnetic field. This is related to the electric and magnetic fields ([16],[18]). Its exact controllability problem has been studied in many works, such as [5], and [12]. In addition, the Klein-Gordon equation which has electromagnetic potential is applicable in many fields, such as geophysics, medical imaging, hydrology, and earth sciences (see [4], [3], [21], [24], [25], [17]). Its exact controllability in one dimension has been

2020 *Mathematics Subject Classification:* 93B05, 93C20.
Submitted August 10, 2025. Published January 20, 2026

studied in [15], [7], while the case of several dimensions has been shown in [19]. The electromagnetic potential is a practical and powerful tool in the Klein-Gordon equations. It combines the electronic field and the magnetic field into a single quantity that can clearly and accurately describe the interaction of charged particles with electromagnetic fields at the quantum level. This interaction depends on the properties of the electromagnetic field represented by the parameter defined in (0,1). To this effect, the classical controllability notion in [14] could not lead us to the desired result more precisely and is not enough to determine the exact controllability concering the potential coefficient. Hence, the concept of averaged controllability is a suitable notion for obtaining the main result. The average controllability significantly increased, just as it appeared in 2014 by E. Zuazua [6] due to its application in numerous fields, such as microbiology, electromagnetics, and economics ([1], [2], [3]), where their dynamic is represented by parametric systems (PDEs, ODEs). In those systems, the parameter plays an important role. The average controllability notion consists of manipulating the state average to find a parameter-independent control, which differs from the classical notion. In this context, we find studies such as ([2], [8], [9], [19], and [26]) for parameter-dependent wave equations and for randomly evolving PDEs in [23]. In addition, other studies have reported the mean controllability in finite dimensions, such as [10]. Thus, one of the user methods for solving controllability problems is the Hilbert uniqueness method proposed by J. L. Lions [13], [11] for classical controllability problems, which consist of providing a uniqueness theorem, a direct inequality, and an inverse inequality [1]. This article aims to study the controllability of the wave equation concerning potential depending on the parameter used to determine the properties of the mixture of media.

This paper is organized as follows: First, we introduce our problem with some key results to use in the second part, where the HUM method is applied to demonstrate the average null controllability by giving an averaged direct and inverse inequalities.

2. Averaged Null Controllability (Hilbert Uniqueness Method)

The average null controllability of the problem (1.1) is defined as follows:

Definition 2.1 *[6] We say that the system (1.1) is average null controllable if there exists a control $u(x,t)$ in $L^2(\Sigma)$ independent of the parameter σ such that*

$$\left(\int_0^1 y\left(x,T,\sigma\right)d\sigma, \int_0^1 y_t\left(x,T,\sigma\right)d\sigma\right) = (0,0)\,. \tag{2.1}$$

.

To achieve average null controllability, we apply the HUM method, which consists of proving a uniqueness theorem, and direct and inverse inequalities [12]. Let us begin by establishing the precise definition of the operator Λ to introduce the main theorem (the uniqueness theorem).

Now, we introduce the backward equation and the homogenous equation:

$$\begin{cases} \psi_{tt} = \Delta\psi - p(x,\sigma)\psi & \text{in } Q, \\ \psi = \int_0^1 \frac{\partial\phi}{\partial\eta} d\sigma & \text{on } \Sigma_0, \\ \psi = 0 & \text{on } \Sigma\diagdown\Sigma_0, \\ \psi\left(x,T\right) = 0\ ; \psi_t\left(x,T\right) = 0 & \text{in } \Omega. \end{cases} . \tag{2.2}$$

$$\begin{cases} \phi_{tt} - \Delta\phi + p\left(x,\sigma\right)\phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi\left(x,0\right) = \phi_0\left(x\right); \phi_t\left(x,0\right) = \phi_1\left(x\right) & \text{in } \Omega. \end{cases} \tag{2.3}$$

For all (ϕ_0,ϕ_1) in the Hilbertian space of the initial data X is independent of the parameter σ.

We multiply the backward equation by ϕ and integrate on Q concerning σ using the integration by parts to obtain:

$$\int_0^1\int_Q \psi_{tt}\phi dxdtd\sigma - \int_0^1\int_Q \Delta\psi\phi dxdtd\sigma = -\int_0^1\int_Q p(x,\sigma)\psi\phi dxdtd\sigma.$$

$$\int_0^1 \int_Q \psi_{tt}\phi dxdtd\sigma = -\int_0^1 \int_\Omega \psi_t (x,0)\, \phi (x,0)\, dxd\sigma + \int_0^1 \int_\Omega \psi (x,0)\, \phi_t (x,0)\, dxd\sigma + \int_0^1 \int_Q \psi\phi_{tt} dxdtd\sigma.$$

We use Green's formula,

$$\int_0^1 \int_Q \Delta\psi\phi dxdtd\sigma = \int_0^1 \int_Q \psi\Delta\phi dxdtd\sigma + \int_0^1 \int_{\Sigma_0} \psi\frac{\partial\phi}{\partial\eta} d\Gamma dtd\sigma.$$

Where η is the unit normal vector directed outward from Q and η_i is η's i^{Th} component.

Implies that:

$$-\left(\int_0^1 \psi_1 (x)\, d\sigma \;, \phi_0 (x)\right)_{L^2(\Omega)} + \left(\int_0^1 \psi_0 (x)\, d\sigma, \phi_1 (x)\right)_{L^2(\Omega)} + \int_0^1 \int_Q (\phi_{tt} - \Delta\phi + p(x,\sigma)\,\phi)\, \psi dxdtd\sigma + \int_0^1 \int_{\Sigma_0} \psi\frac{\partial\phi}{\partial\eta} d\Gamma dtd\sigma = 0.$$

Where $\psi_0 (x) = \psi (.,0)$ and $\psi_1 (x) = \psi_t (.,0)$

$$\int_0^1 \int_{\Sigma_0} \psi\frac{\partial\phi}{\partial\eta} d\Gamma dtd\sigma = \left(\int_0^1 \psi_1 (x)\, d\sigma ; \phi_0 (x)\right)_{L^2(\Omega)} - \left(\int_0^1 \psi_0 (x)\, d\sigma ; \phi_1 (x)\right)_{L^2(\Omega)}.$$

$$\int_{\Sigma_0} \left|\int_0^1 \frac{\partial\phi}{\partial\eta} d\sigma\right|^2 d\Gamma dt = \left(\int_0^1 \psi_1 (x)\, d\sigma ; \phi_0 (x)\right)_{L^2(\Omega)} - \left(\int_0^1 \psi_0 (x)\, d\sigma ; \phi_1 (x)\right)_{L^2(\Omega)}.$$

Moreover, we define the operator Λ as follows:

$$\begin{aligned} &\Lambda : X \longrightarrow X^* \\ &\Lambda \{\phi_0, \phi_1\} = \left\{\textstyle\int_0^1 \psi_1 (\sigma)\, d\sigma, -\int_0^1 \psi_0 (\sigma)\, d\sigma\right\}. \end{aligned} \tag{2.4}$$

Where X^* the adjoint space of X, and we define a semi norm on X as follows:

$$\|\phi_0, \phi_1\|_X^2 = \int_\Sigma \left|\int_0^1 \frac{\partial\phi}{\partial\eta} d\sigma\right|^2 d\Gamma dt. \tag{2.5}$$

Now, we prove the coercivity and the continuity of the operator Λ by introducing direct and inverse inequalities.

Therefore, we define the average energy for later use.

We multiply the homogenous equation by ϕ_t, and we integrate on Ω with respect to the parameter σ. The average energy is defined in (2.6), $\forall t \in (0,T)$.

$$E_a (t) = \frac{1}{2}\left(\int_0^1 \int_\Omega |\phi_t|^2 \, dx\, d\sigma + \int_0^1 \int_\Omega |\nabla\phi|^2 \, dx\, d\sigma + \int_0^1 \int_\Omega p(x,\sigma)\, |\phi|^2 \, dx\, d\sigma\right). \tag{2.6}$$

Hence, in the following lemma, we prove the conservation of the average energy in (2.6).

Lemma 2.1 *We take $\phi = \phi (x,t,\sigma)$ as the solution of the homogenous equation, then the average energy (2.6) is conserved for all $t \in (0,T)$.*

$$E_a (0) = E_a (t) = \frac{1}{2}\left(\int_0^1 \int_\Omega |\phi_t|^2 dxd\sigma + \int_0^1 \int_\Omega |\nabla\phi|^2 dxd\sigma + \int_0^1 \int_\Omega p(x,\sigma)\, |\phi|^2 dxd\sigma\right).$$

Proof: By multiplying the homogenous equation with ϕ_t, integrate on $(0,1)\times\Omega$, using the Green formula and the Fubini theorem, and taking boundary conditions into account, we find that for all $t\in(0,T)$:

$$\begin{aligned}\int_0^1\int_\Omega(\phi_{tt}-\Delta\phi+p(x,\sigma)\phi_t)\,dx\,d\sigma &= \frac{1}{2}\frac{d}{dt}\int_0^1\int_\Omega|\phi_t|^2\,dx\,d\sigma+\int_0^1\int_\Omega\nabla\phi_t\nabla\phi\,dx\,d\sigma\\ &- \int_0^1\int_{\partial\Omega}\phi_t\nabla\phi\cdot\eta\,d\partial\Omega\,d\sigma+\frac{1}{2}\frac{d}{dt}\int_0^1\int_\Omega p(x,\sigma)\,|\phi|^2\,dx\,d\sigma\\ &= \frac{1}{2}\frac{d}{dt}\left(\int_0^1\int_\Omega|\phi_t|^2\,dx\,d\sigma+\int_0^1\int_\Omega|\nabla\phi|^2\,dx\,d\sigma\right.\\ &+ \left.\int_0^1\int_\Omega p(x,\sigma)\,|\phi|^2\,dx\,d\sigma\right)\\ &= \frac{d}{dt}E_a(t).\end{aligned}$$

Then,

$$\frac{d}{dt}E_a(t)=0.$$

This is precisely the assertion of the lemma. □

3. Averaged Inverse and Direct Inequalities

We start by proving a lemma that we will use later.

Lemma 3.1 *Let $q=(q_k)$ be a vector field in $\left[C^1\left(\bar{\Omega}\right)\right]^n$ independent of the parameter σ then, for all weak solutions for the homogenous equation (2.3), we have:*

$$\begin{aligned}\frac{1}{2}\int_0^1\int_\Sigma q_i\,|\nabla\phi|^2.\eta_i d\Gamma dt d\sigma &= \int_0^1\int_\Omega\phi_t\left(q_i\frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma\\ &+ \frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}\left(|\phi_t|^2-|\nabla\phi|^2-p(x,\sigma)\,|\phi|^2\right)dxdtd\sigma\\ &+ \int_0^1\int_Q\left(\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right)dxdtd\sigma\\ &- \frac{1}{2}\int_0^1\int_Q\frac{\partial p(x,\sigma)}{\partial x_i}q_i\;|\phi|^2\,dxdtd\sigma.\end{aligned}\tag{3.1}$$

Proof: First, for simplicity of notation, the convention of repeated indices will be applied as follows: $q_i\frac{\partial\phi}{\partial x_i}=\sum_{i=1}^n q_i\frac{\partial\phi}{\partial x_i}$. Then, we make the convention with the homogeneous equation, and $q_i\frac{\partial\phi}{\partial x_i}$ and integrate on $Q\times[0,1]$.

$$\begin{aligned}\int_0^1\int_Q(\phi_{tt}-\Delta\phi+p(x,\sigma)\,\phi)\,q_i\frac{\partial\phi}{\partial x_i}\eta_i dxdtd\sigma &= \int_0^1\int_Q\phi_{tt}q_i\frac{\partial\phi}{\partial x_i}dxdtd\sigma\\ &- \int_0^1\int_Q\Delta\phi q_i\frac{\partial\phi}{\partial x_i}dxdtd\sigma\\ &+\int_0^1\int_Q p(x,\sigma)\,\phi q_i\frac{\partial\phi}{\partial x_i}dxdtd\sigma.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^1\int_Q \phi_{tt} q_i \frac{\partial\phi}{\partial x_i} dxdtd\sigma &= \int_0^1\int_\Omega \left(\phi_t q_i \frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma - \int_0^1\int_Q \phi_t q_i \frac{\partial\phi_t}{\partial x_i} dxdtd\sigma \\ &= \int_0^1\int_\Omega \left(\phi_t q_i \frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma - \frac{1}{2}\int_0^1\int_Q q_i \frac{\partial}{\partial x_i}\left|\phi_t\right|^2 dxdtd\sigma \\ &= \int_0^1\int_\Omega \left(\phi_t q_i \frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma - \frac{1}{2}\int_0^1\int_\Sigma q_i \left|\phi_t\right|^2 d\Gamma dtd\sigma \\ &+ \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left|\phi_t\right|^2 dxdtd\sigma \\ &= \int_0^1\int_\Omega \left(\phi_t q_i \frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left|\phi_t\right|^2 dxdtd\sigma.\end{aligned}$$

For the second integral term, we obtain:

$$\int_0^1\int_Q \Delta\phi q_i \frac{\partial\phi}{\partial x_i} dxdtd\sigma = \int_0^1\int_\Sigma \nabla\phi\left(q_i \frac{\partial\phi}{\partial x_i}\right)\eta_i d\Gamma dtd\sigma - \int_0^1\int_Q \nabla\phi\nabla\left(q_i \frac{\partial\phi}{\partial x_i}\right) dxdtd\sigma.$$

Then,

$$\nabla\phi\nabla\left(q_i \frac{\partial\phi}{\partial x_i}\right) = \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i} + \frac{1}{2}q_i \frac{\partial}{\partial x_i}\left|\nabla\phi\right|^2 \quad j=1,\ ...,\ n$$

Consequently,

$$\int_0^1\int_Q \Delta\phi q_i \frac{\partial\phi}{\partial x_i} dxdtd\sigma = \frac{1}{2}\int_0^1\int_\Sigma q_i\left|\nabla\phi\right|^2\eta_i d\Gamma dtd\sigma - \int_0^1\int_Q\left(\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right)dxdtd\sigma$$
$$-\frac{1}{2}\int_0^1\int_Q q_i \frac{\partial}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\sigma.$$

We thus obtain:

$$-\frac{1}{2}\int_0^1\int_Q q_i \frac{\partial}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\sigma = -\frac{1}{2}\int_0^1\int_\Sigma q_i\left|\nabla\phi\right|^2\eta_i d\Gamma dtd\sigma + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\sigma.$$

Where,

$$\int_0^1\int_\Sigma \left|\nabla\phi\right|^2\eta_i d\Gamma dtd\sigma = \int_0^1\int_\Sigma \nabla\phi\frac{\partial\phi}{\partial x_i} d\Gamma dtd\sigma.$$

This gives:

$$\begin{aligned}\int_0^1\int_Q \Delta\phi q_i \frac{\partial\phi}{\partial x_i} dxdtd\sigma &= \frac{1}{2}\int_0^1\int_\Sigma q_i\left|\nabla\phi\right|^2 d\Gamma dtd\sigma + \frac{1}{2}\int_0^1\int_Q \frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\sigma \\ &\quad -\int_0^1\int_Q\left(\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right)dxdtd\sigma.\end{aligned}$$

The last integral is rewritten as follows:

$$\begin{aligned}\int_0^1\int_Q p(x,\sigma)\phi q_i \frac{\partial\phi}{\partial x_i} dxdtd\sigma &= -\frac{1}{2}\int_0^1\int_Q \frac{\partial p(x,\sigma)}{\partial x_i} q_i\ \left|\phi\right|^2 dxdtd\sigma \\ &\quad -\frac{1}{2}\int_0^1\int_Q \frac{\partial q_i(x,\sigma)}{\partial x_i}\ p(x,\sigma)\left|\phi\right|^2 dxdtd\sigma.\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{1}{2}\int_0^1\int_\Sigma q_i\left|\nabla\phi\right|^2\eta_i d\Gamma dt d\sigma &= \int_0^1\int_\Omega \phi_t\left(q_i\frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma+\frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}\left|\phi_t\right|^2 dxdtd\sigma\\
&\quad -\frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}\left|\nabla\phi\right|^2 dxdtd\sigma+\int_0^1\int_Q\left(\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right)dxdtd\sigma\\
&\quad -\frac{1}{2}\int_0^1\int_Q\frac{\partial p(x,\sigma)}{\partial x_i}q_i\ \left|\phi\right|^2 dxdtd\sigma\\
&\quad -\ \frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}p(x,\sigma)\left|\phi\right|^2 dxdtd\sigma\\
&= \int_0^1\int_\Omega\phi_t\left(q_i\frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma\\
&\quad +\frac{1}{2}\int_0^1\int_Q\frac{\partial q_i}{\partial x_i}\left(\left|\phi_t\right|^2-\left|\nabla\phi\right|^2-p(x,\sigma)\left|\phi\right|^2\right)dxdtd\sigma\\
&\quad +\int_0^1\int_Q\left(\frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}\right)dxdtd\sigma\\
&\quad -\ \frac{1}{2}\int_0^1\int_Q\frac{\partial p(x,\sigma)}{\partial x_i}q_i\ \left|\phi\right|^2 dxdtd\sigma.
\end{aligned}$$

As a result, we obtain (3.1). □

3.1. The average direct inequality

In this section, we prove the following theorem:

Theorem 3.1 *(**The direct inequality**): Let ϕ, the solution of (2.3), verify the following inequality:*

$$\int_\Sigma\left|\int_0^1\frac{\partial\phi}{\partial\eta}d\sigma\right|^2 d\Gamma dt\leq C\int_0^1\int_\Omega\left[\left|\phi_1\right|^2+\left|\nabla\phi_0\right|^2+p(x,\sigma)\left|\phi_0\right|^2\right]dxd\sigma. \tag{3.2}$$

C is a positive constant. Under the above result, we assume that the regularity property for the solution of (2.3) holds.

$$\frac{\partial\phi}{\partial\eta}(x,t,\sigma)\in L^2(\Sigma\times(0,1)). \tag{3.3}$$

η is the outward unit normal vector.

Proof: Now, according to Lemma 1 and [12], we take $q=h$, $h\cdot\eta=1$ on Γ:

$$\begin{aligned}
\frac{1}{2}\int_0^1\int_\Sigma\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma &= \int_0^1\int_\Omega h\phi_t\frac{\partial\phi}{\partial x_i}{}_0^T dxd\sigma\\
&\quad +\ \frac{1}{2}\int_0^1\int_Q\nabla h\left(\left|\phi_t\right|^2-\left|\nabla\phi\right|^2-p(x;\sigma)\left|\phi\right|^2\right)dxdtd\sigma\\
&\quad +\ \int_0^1\int_Q\nabla h\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_i}dxdtd\sigma\\
&\quad -\ \frac{1}{2}\int_0^1\int_Q h\frac{\partial p(x,\sigma)}{\partial x_i}\left|\phi\right|^2 dxdtd\sigma.
\end{aligned}$$

We estimate each integral then,

$$\int_0^1\int_\Sigma\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma\leq CE_a(0).$$

According to Cauchy-Schwarz, we obtain:

$$\int_{\Sigma}\left|\int_{0}^{1}\frac{\partial\phi}{\partial\eta}d\sigma\right|^{2}d\Gamma dt\leq K\int_{0}^{1}\int_{\Sigma}\left|\frac{\partial\phi}{\partial\eta}\right|^{2}d\Gamma dtd\sigma \tag{3.4}$$

Where $k>0$.

□

We obtain the average direct inequality defined in (3.2), which implies (3.3).

3.2. The average inverse inequality

In the following section, we will demonstrate a theorem to ensure the inverse inequality.

Theorem 3.2 *(The inverse inequality) Assume* $R(x^0)=\|m(x)\|_{L^\infty(\Omega)}$ *and* $T\left(x^0\right)=2\alpha R(x^0)\max\left\{1,\frac{1}{\alpha}\right\}$ *where* $\alpha>0$ *then, for every* $T>T\left(x^0\right)$ *and every weak solution for* (2.3)*, the next averaged observability inequality holds:*

$$\left(T-T\left(x^0\right)\right)E_a\left(0\right)\leq\frac{R\left(x^0\right)}{2}\int_{\Sigma(x^0)}\left|\int_{0}^{1}\frac{\partial\phi}{\partial\eta}d\sigma\right|^{2}d\Gamma dt. \tag{3.5}$$

To simplify the calculation, we use the following notations: following:

Let $x\in\mathbb{R}^n$ and x^0 fixed in $\mathbb{R}^n$, $m\left(x\right)=x-x^0$ a partition of the boundary where $\Gamma_0=\{x\in\Gamma;m\left(x\right).\eta\left(x\right)>0\}$where $\eta\left(x\right)$ is a unit normal vector directed outward Ω [12].

$$\begin{aligned}m(x)&=x-x^0,\\ m_i(x)&=x_i-x_i^0,1\leq i\leq n,\end{aligned}$$

$$\Gamma\left(x^0\right)=\left\{x\in\Gamma\ m\left(x\right).\eta\left(x\right)>0\right\},$$

$$\Sigma\left(x^0\right)=\Gamma\left(x^0\right)\times\left]0,T\right[.$$

$$R\left(x^0\right)=\|m(x)\|_{L^\infty(\Omega)}=\sup_{x\in\overline{\Omega}}\left\|x-x^0\right\|,$$

$$T\left(x^0\right)=2R\ \left(\ x^0\right),$$

We put $q_i\left(x\right)=m_i\left(x\right)=x_i-x_i^0$, $1\leq i\leq n$

$$\begin{aligned}\frac{\partial q_i}{\partial x_j}&=\delta_{ij},\text{ where }\delta_{ij}=\left\{\begin{array}{ll}1 & si\ i=j\\ 0 & si\ i\neq j\end{array}\right.\\ \sum_{i=1}^{n}\frac{\partial q_i}{\partial x_i}&=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(x_i-x_i^0\right)=n\\ \frac{\partial\phi}{\partial x_j}\frac{\partial q_i}{\partial x_j}\frac{\partial\phi}{\partial x_i}&=\frac{\partial\phi}{\partial x_i}\frac{\partial q_i}{\partial x_i}\frac{\partial\phi}{\partial x_i}=n\left|\nabla\phi\right|^{2}.\end{aligned}$$

Proof: We rewrite Lemma 2 as follows:

$$\begin{aligned}
\frac{1}{2}\int_0^1\int_\Sigma m_i(x)\left|\frac{\partial\phi}{\partial\eta}\right|^2 .\eta_i(x)\,d\Gamma dt d\sigma &= \int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}\right)_0^T dx d\sigma \\
&+ \frac{1}{2}\int_0^1\int_Q n\left(|\phi_t|^2-|\nabla\phi|^2-p(x,\sigma)|\phi|^2\right)dxdtd\sigma \\
&+ \int_0^1\int_Q |\nabla\phi|^2\,dxdtd\sigma \\
&- \frac{1}{2}\int_0^1\int_Q \frac{\partial p(x,\sigma)}{\partial x_i}m_i(x)\ \ |\phi|^2\,dxdtd\sigma \\
&= \int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}\right)_0^T dx d\sigma + \frac{n}{2}\int_0^1\int_Q|\phi_t|^2\,dxdtd\sigma \\
&+ \frac{2-n}{2}\int_0^1\int_Q|\nabla\phi|^2\,dxdtd\sigma \\
&- \frac{n}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma \\
&- \frac{1}{2}\int_0^1\int_Q \frac{\partial p(x,\sigma)}{\partial x_i}m_i(x)\ |\phi|^2\,dxdtd\sigma \\
&\leq \int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}\right)_0^T dx d\sigma + \frac{n}{2}\int_0^1\int_Q|\phi_t|^2\,dxdtd\sigma \\
&+ \frac{2-n}{2}\int_0^1\int_Q|\nabla\phi|^2\,dxdtd\sigma \\
&- \frac{n}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma.
\end{aligned}$$

In the next steps, we estimate each integral, and we use the Cauchy-Schwarz inequality on $\Sigma(x^0)$ as follows:

$$0 \quad < \quad m(x).\eta(x) \leq \|m(x)\|\,\|\eta_i(x)\| \leq R(x^0).$$

We estimate the first member,

$$\begin{aligned}
\frac{1}{2}\int_0^1\int_\Sigma m_i(x).\eta_i(x)\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma &\leq \frac{1}{2}\int_0^1\int_\Sigma\sum_{i=1}^n m_i(x).\eta_i(x)\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma \\
&\leq \|m(x)\|\,\|\eta_i(x)\|\frac{1}{2}\int_0^1\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma \\
&\leq \frac{R(x^0)}{2}\int_0^1\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma.
\end{aligned}$$

We take,

$$F=\int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}\right)_0^T dxd\sigma-\frac{1}{2}\int_0^1\int_Q\frac{\partial p(x,\sigma)}{\partial x_i}m_i(x)|\phi|^2\,dxdtd\sigma.$$

We thus obtain:

$$\begin{aligned}
&F+\frac{n}{2}\int_0^1\int_Q|\phi_t|^2\,dxdtd\sigma+\frac{2-n}{2}\int_0^1\int_Q|\nabla\phi|^2\,dxdtd\sigma-\frac{n}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma \\
\leq\ & \frac{R(x^0)}{2}\int_0^1\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma.
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
& F+\frac{n}{2}\int_0^1\int_Q |\phi_t|^2\,dxdtd\sigma+\frac{2-n}{2}\int_0^1\int_Q |\nabla\phi|^2\,dxdtd\sigma \\
& -\frac{n}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma+\frac{1}{2}\int_0^1\int_Q |\phi_t|^2\,dxdtd\sigma \\
& -\frac{1}{2}\int_0^1\int_Q |\phi_t|^2\,dxdtd\sigma+\frac{1}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma \\
& -\frac{1}{2}\int_0^1\int_Q p(x,\sigma)|\phi|^2\,dxdtd\sigma \\
= \; & F+\frac{1}{2}\int_0^1\int_Q \left[|\phi_t|^2+|\nabla\phi|^2+p(x,\sigma)|\phi|^2\right]dxdtd\sigma \\
& +\frac{n-1}{2}\int_0^1\int_Q \left[|\phi_t|^2-|\nabla\phi|^2-p(x,\sigma)|\phi|^2\right]dxdtd\sigma.
\end{aligned}$$

Implies that:

$$\begin{aligned}
& F+\frac{n-1}{2}\int_0^1\int_Q \left[|\phi_t|^2-|\nabla\phi|^2-p(x,\sigma)|\phi|^2\right]dxdtd\sigma+TE_a(0) \\
\leq \; & \frac{R(x^0)}{2}\int_0^1\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma.
\end{aligned}$$

We take,

$$G=\int_0^1\int_Q \left[|\phi_t|^2-|\nabla\phi|^2-p(x,\sigma)|\phi|^2\right]dxdtd\sigma.$$

$$F+\frac{n-1}{2}G+TE_a(0)\leq\frac{R(x^0)}{2}\int_0^1\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\eta}\right|^2 d\Gamma dtd\sigma. \tag{3.6}$$

Now, we return to the homogenous equation, multiply it by ϕ, and integrate it into $(0,1)\times Q$, we obtain:

$$\int_0^1\int_Q \phi\left(\phi_{tt}-\Delta\phi+p(x,\sigma)\phi\right)dxdtd\sigma=0.$$

Hence,

$$\begin{aligned}
\int_0^1\int_Q \phi\left(\phi_{tt}-\Delta\phi+p(x,\sigma)\phi\right)dxdtd\sigma &= \int_0^1\int_\Omega (\phi_t\phi)_0^T\,dx\;d\sigma-\int_0^1\int_Q |\phi_t|^2\;dxdtd\sigma \\
&+ \int_0^1\int_Q |\nabla\phi|^2\,dxdtd\sigma+\int_0^1\int_Q p(x,\sigma)\phi^2 dxdtd\sigma \\
&= 0.
\end{aligned}$$

Implies that:

$$\int_0^1\int_\Omega (\phi_t\phi)_0^T\,dxd\sigma=Y.$$

We have,

$$\begin{aligned} F+\frac{n-1}{2}G=\int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}\right)dx_0^T d\sigma &+ \frac{n-1}{2}\int_0^1\int_\Omega (\phi_t\phi)_0^T\,dxd\sigma \\ &- \frac{1}{2}\int_0^1\int_Q \frac{\partial p(x,\sigma)}{\partial x_i}m_i(x)\,|\phi|^2\,dxdtd\sigma \\ &= \int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right)_0^T dxd\sigma \\ &- \frac{1}{2}\int_0^1\int_Q \frac{\partial p(x,\sigma)}{\partial x_i}m_i(x)\,|\phi|^2\,dxdtd\sigma. \end{aligned}$$

Moreover, we use Cauchy $\varepsilon-$inequality to find that:

$$\begin{aligned} \int_0^1\int_\Omega \phi_t\left(m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right)dxd\sigma &\le \frac{\varepsilon}{2}\int_0^1\int_\Omega |\phi_t|^2\,dxd\sigma \\ &+ \frac{1}{2\varepsilon}\int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right|^2 dxd\sigma. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right|^2 dxd\sigma &= \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}\right|^2 dxd\sigma \\ &+ \left(\frac{n-1}{2}\right)^2\int_0^1\int_\Omega |\phi|^2\,dxd\sigma \\ &+ (n-1)\int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}\phi\right| dxd\sigma. \end{aligned}$$

Additionally, we have that:

$$\int_0^1\int_\Omega m_i(x)\frac{\partial\phi}{\partial x_i}\phi dxd\sigma=\frac{1}{2}\int_0^1\int_\Omega m_i(x)\frac{\partial}{\partial x_i}|\phi|^2\,dxd\sigma.$$

We use the Green formula:

$$\frac{1}{2}\int_0^1\int_\Omega m_i(x)\frac{\partial}{\partial x_i}|\phi|^2\,dxd\sigma=-\frac{n}{2}\int_0^1\int_\Omega |\phi|^2\,dxd\sigma.$$

We rewrite (3.8) as follows:

$$\begin{aligned} \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right|^2 dxd\sigma &= \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}\right|^2 dxd\sigma \\ &+ \left(\frac{n-1}{2}\right)^2\int_0^1\int_\Omega |\phi|^2\,dxd\sigma \\ &-\frac{n(n-1)}{2}\int_0^1\int_\Omega |\phi|^2\,dxd\sigma \\ &= \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}\right|^2 dxd\sigma-\frac{n^2-1}{4}\int_0^1\int_\Omega |\phi|^2\,dxd\sigma. \end{aligned}$$

Implies that:

$$\begin{aligned} \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}+\frac{n-1}{2}\phi\right|^2 dxd\sigma &\le \int_0^1\int_\Omega \left|m_i(x)\frac{\partial\phi}{\partial x_i}\right|^2 dxd\sigma \\ &\le \|m_i(x)\|^2\int_0^1\int_\Omega |\nabla\phi|^2\,dxd\sigma \\ &\le R\left(x^0\right)^2\int_0^1\int_\Omega |\nabla\phi|^2\,dxd\sigma. \end{aligned}$$

Now, by taking $\varepsilon = R\left(x^0\right)$ in (3.7), we obtain:

$$\int_0^1 \int_\Omega \phi_t \left(m_i\left(x\right) \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dxd\sigma \quad \leq \quad \frac{R\left(x^0\right)}{2} \int_0^1 \int_\Omega \left|\phi_t\right|^2 dxd\sigma + \frac{R\left(x^0\right)}{2} \int_0^1 \int_\Omega \left|\nabla \phi\right|^2 dxd\sigma.$$

Then,

$$\begin{aligned} \left| F + \frac{n-1}{2} G \right| &= \left| \int_0^1 \int_\Omega \phi_t \left(m_i\left(x\right) \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right)_0^T dxd\sigma - \frac{1}{2} \int_0^1 \int_Q \frac{\partial p\left(x,\sigma\right)}{\partial x_i} m_i\left(x\right) \left|\phi\right|^2 dxdtd\sigma \right| \\ &\leq \frac{R\left(x^0\right)}{2} \int_0^1 \int_\Omega \left|\phi_t\right|^2 dxd\sigma + \frac{R\left(x^0\right)}{2} \int_0^1 \int_\Omega \left|\nabla \phi\right|^2 dxd\sigma. \end{aligned}$$

Conversely, we have:

$$\left| \int_0^1 \int_\Omega \phi_t \left(m_i\left(x\right) \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right)_0^T dxd\sigma \right| \leq 2 \left\| \int_0^1 \int_\Omega \phi_t \left(m_i\left(x\right) \frac{\partial \phi}{\partial x_i} + \frac{n-1}{2}\phi \right) dxd\sigma \right\|_{L^\infty(0,T)}.$$

This implies that:

$$\left| F + \frac{n-1}{2} G \right| \leq 2R\left(x^0\right) E_a\left(0\right).$$

We take $2R\left(x^0\right) = T\left(x^0\right)$,

$$\left| F + \frac{n-1}{2} G \right| \leq T\left(x^0\right) E_a\left(0\right).$$

Consequently,

$$F + \frac{n-1}{2} G + TE_a\left(0\right) \leq \frac{R\left(x^0\right)}{2} \int_0^1 \int_{\Sigma(x^0)} \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Gamma dtd\sigma.$$

and,

$$\begin{aligned} TE_a\left(0\right) - T\left(x^0\right) E_a\left(0\right) &\leq TE_a\left(0\right) - \left(F + \frac{n-1}{2} G \right) \\ &\leq F + \frac{n-1}{2} G + TE_a\left(0\right) \\ &\leq \frac{R\left(x^0\right)}{2} \int_0^1 \int_{\Sigma(x^0)} \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\Gamma dtd\sigma. \end{aligned}$$

Implies that:

$$\left(T - T\left(x^0\right)\right) E_a\left(0\right) \leq \frac{R\left(x^0\right)}{2} \int_{\Sigma(x^0)} \int_0^1 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\sigma d\Gamma dt.$$

Moreover,

$$\int_{\Sigma(x^0)} \int_0^1 \left| \frac{\partial \phi}{\partial \eta} \right|^2 d\sigma d\Gamma dt \leq \int_{\Sigma(x^0)} \left(\int_0^1 \frac{\partial \phi}{\partial \eta} d\sigma \right)^2 d\Gamma dt.$$

Which yields:

$$\left(T - T\left(x^0\right)\right) E_a\left(0\right) \leq \frac{R\left(x^0\right)}{2} \int_{\Sigma(x^0)} \left| \int_0^1 \frac{\partial \phi}{\partial \eta} d\sigma \right|^2 d\Gamma dt.$$

Which is the averaged inverse inequality. □

Theorem 3.3 (Uniqueness theorem) *ϕ is a solution for the homogeneous equation, if $\frac{\partial \phi}{\partial \eta}\left(x,t,\sigma\right) = 0$ on $\Sigma \times (0,1)$ then, $\phi = 0$ in $Q \times (0,1)$.*

Consequently, $\|\phi_0, \phi_1\|_X^2 = \int_\Sigma \left| \int_0^1 \frac{\partial \phi}{\partial \eta} d\sigma \right|^2 d\Gamma dt$ is a norm on X, (see [11]).

Proof: In the context of the HUM method, the uniqueness theorem depends on proving direct and inverse effects. In detail, the averaged inequalities show that the seminorm in (2.5) is a norm on X(the Hilbert space of the initial data), which is identified in $H_0^1(\Omega) \times L^2(\Omega)$:

$$\begin{aligned} \frac{2}{R(x^0)}\left(T - T(x^0)\right) E_a(0) &\leq \int_{\Sigma(x^0)} \left| \int_0^1 \frac{\partial \phi}{\partial \eta} d\sigma \right|^2 d\Gamma dt \\ &\leq C \int_0^1 \int_\Omega \left[|\phi_1|^2 + |\nabla \phi_0|^2 + p(x,\sigma)|\phi_0|^2 \right] dx d\sigma. \end{aligned} \tag{3.8}$$

Importantly, Λ is an isomorphism on $H_0^1(\Omega) \times L^2(\Omega)$ to $H_0^{-1}(\Omega) \times L^2(\Omega)$. More specifically (2.4) has a unique solution given by

$$\{\phi_0, \phi_1\} = \Lambda^{-1}\left(\int_0^1 \psi_1(\sigma)\, d\sigma, -\int_0^1 \psi_0(\sigma)\, d\sigma \right)_{H_0^{-1} \times L^2(\Omega)}. \tag{3.9}$$

This approach aims to design parameter-independent control to achieve the averaged null controllability of the problem in (1.1) mentioned in (2.1).

□

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