Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.78344

Refinement of Ostrowski Inequality and Functions of Bounded Variation on New f-divergence Measure with Applications

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ABSTRACT: Information Inequalities prove to be a powerful tool to quantify uncertainties and data dependencies. They provide useful insights into the relations between information-theoretic probability distributions. The entropy and divergence measure makes the data processing inequalities more effective for communication and informatic concepts. This study provides the constructive perspective of Ostrowski type inequality for functions of bounded variation. Explored approximation of new f divergence measure by applying principles of numerical integration theory. Discussed function f and its first derivatives exhibit bounded variation characteristics. Furthermore, by applying the bounded variation properties of the function f and its derivatives, accurate and efficient approximations can be achieved. Additionally, we have found applications of the obtained information inequalities related to the Relative Arithmetic-Geometric Divergence (AGD) that quantify the difference between probability distributions. Some identified means are also used to specify the results.

Key Words: New f-divergence, Ostrowski inequality, relative arithmetic-geometric divergence, variational distance, bounded variation.

Contents

1. Introduction

Inequalities in mathematics corresponds the correlation between two objects, mainly the non-equal relation between mathematical expressions. Information inequalities involve entropy, information measures and provide insights into the processing of information systems. They act as fundamentals for better understanding of relationships between various information-theoretic quantities, data compression, transmission and processing. The inequalities of information theory find utility in fields like data compression, machine learning, statistical inference, channel coding and many more.
An eminent information inequality in the field of information theory is Ostrowski Inequality [1] which calculate a differentiable function's deviation from its integral mean. The Ostrowski inequality has found applications in various fields, including Numerical integration, Probability theory, Information theory, Economics, Engineering. Many researchers have obtained generalizations of Ostrowski inequality in different fields. Previous studies such as [2, 3] discussed both discrete and continuous versions of Ostrowski inequality in the generalized form of different time scales. For functions of bounded first order derivatives [4] established q-Ostrowski inequality, significant in the field of quantum integration. The following is the general form of Ostrowski inequality

$$\left|\Gamma(\xi) - \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\Gamma(\alpha)\,d\alpha\right| \leq \sup_{\rho<\xi<\sigma}|\Gamma'(\xi)|\,(\rho-\sigma)\left[\frac{\left(\xi-\frac{\sigma+\rho}{2}\right)^2}{(\rho-\sigma)^2}+\frac{1}{4}\right]$$

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2010 *Mathematics Subject Classification:* 68P30, 94A15, 94A17.
Submitted August 11, 2025. Published October 16, 2025

The Divergence measures act as the building blocks of the various information inequalities. They can be understood as statistical distance useful in comparing two probability distributions. Interrelating these two leads to the development of new measures and their analogy has significant implications in fields like model selection and density estimation in statistics, anomaly detection and generative modelling in machine learning. These measures found numerous applications in information theoretic and statistical problems. Taneja [5, 6, 7], Dragomir [8, 9, 10] and many other researchers [11, 12, 13, 14] have done a significant amount of work as a contribution to the study of information inequalities and divergence measures. Their work focuses mainly on developing new divergence measures, establishing relationships between them, and deriving inequalities that bound these measures. Divergence measures are mainly derived from the generalization of relative entropy. The symmetry of the divergence measures depends on the probability distributions involved in it. Areas such as Fuzzy Inequalities [15] Pattern Recognition, Word Alignment, Approximation of probability distributions, medical diagnosis, etc. finds utility of divergence measures with wide applications. The extension of divergence measures in information theory also determines the value added to any event, i.e., to compare the usefulness of any event over the other. Let us consider a set of all complete finite discrete probability distributions: Then the New f-divergence introduced by Jain & Saraswat [11, 12] is defined as

$$S_f(P,Q) = \sum_{i=1}^{n} q_i\, f\left(\frac{p_i+q_i}{2q_i}\right), \tag{1.1}$$

where f is a convex function, and $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ are probability distributions. Convexity of the function f gives rise to the non-negativity property of the New f-divergence, i.e., the New f-divergence possesses the following properties:

$$S_f(P,Q) \geq f(1), \qquad S_f(P,Q) \geq 0.$$

The equality holds iff $P = Q$. Some other divergence measures are also used in this paper as particular instances of New f-divergence. By changing the convex function, different well known divergence measures can be obtained from new f-divergence. Few of them are discussed below:
Let $v > 0$, then we have the *Variational distance* [16] defined as

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2}V(P,Q)$$

Let $v > 0$, then we have the *Relative Arithmetic–Geometric divergence* [16] defined as

$$S_f(P,Q) = \sum_{i=1}^{n} \frac{p_i+q_i}{2}\log\left(\frac{p_i+q_i}{2q_i}\right) = G(P,Q). \tag{1.2}$$

Furthermore, to resolve the problems discussed, we introduce constants K and k with

$$0 < \zeta < 1 < \xi < \infty \quad \text{and} \quad \zeta \leq \frac{p_i+q_i}{2q_i} \leq \xi, \quad i = 1, 2, \ldots, n. \tag{1.3}$$

The following results based on Ostrowski type inequality were discussed in [16].

Theorem 1.1 *Let us consider a function $f : [\zeta, \xi] \to \mathbb{R}$ which is absolutely continuous, and $f' : [\zeta, \xi] \to \mathbb{R}$ is essentially bounded, that is $f' \in L_\infty[\zeta, \xi]$. Hence, given condition (1.3), we have*

$$\left| S_f(P,Q) - \frac{1}{\xi-\zeta}\int_{\zeta}^{\xi} f(\omega)\, d\omega \right| \leq \left[\tfrac{1}{4} + \frac{1}{(\xi-\zeta)^2}\left(\tfrac{1}{4}\chi^2(Q,P) + \left(\tfrac{\xi+\zeta}{2} - 1\right)^2\right)\right] \quad \times (\xi-\zeta)\,\|f'\|_\infty$$
$$\leq \tfrac{1}{2}(\xi-\zeta)\,\|f'\|_\infty.$$

Furthermore, we define extensions of f in the form of its first derivative as [17, 18]

$$f^{\wedge}(z) = f(1) + (z-1)f\left(\tfrac{1+z}{2}\right),$$
$$f^{\Gamma}(z) = f(1) + \tfrac{z-1}{2}f'(z). \tag{1.4}$$

For the convex functions provided above, the following results can be obtained. [18]

Theorem 1.2 *Let $f : [\zeta, \xi] \to \mathbb{R}$ be a function with absolutely continuous derivative on $[\zeta, \xi]$ and $f' \in L_\infty[\zeta, \xi]$. Hence, given condition (1.3), we have*

$$\begin{aligned} |S_f(P,Q) - S_{f^{\wedge}}(P,Q)| &\le \frac{1}{16}\|f'\|_\infty \chi^2(P,Q) \\ &\le \frac{1}{16}\|f'\|_\infty (\xi-1)(1-\zeta) \\ &\le \frac{1}{64}\|f'\|_\infty (\xi-\zeta)^2. \end{aligned}$$

Similarly, for the second function in (1.4), the following result holds true.

Theorem 1.3 *Let $f : [\zeta, \xi] \to \mathbb{R}$ be a function with absolutely continuous derivative on $[\zeta, \xi]$ and $f' \in L_\infty[\zeta, \xi]$. Hence, given condition (1.3), we have*

$$\begin{aligned} \left|S_f(P,Q) - S_{f^{\Gamma}}(P,Q)\right| &\le \frac{1}{16}\|f'\|_\infty \chi^2(P,Q) - \frac{1}{4\|f''\|_\infty} S_{f^{\circ}}(P,Q) \\ &\le \frac{1}{16}\|f'\|_\infty (\xi-1)(1-\zeta) \\ &\le \frac{1}{64}\|f'\|_\infty (\xi-\zeta)^2. \end{aligned}$$

These results in Theorem 1.2 and Theorem 1.3 assume the absolute continuity of the first and essential boundedness of the second or higher order derivatives. The following important Lemma discussed by Pearce and Dragomir [19] has a significant use in proving inequalities discussed further in this paper.

Lemma 1.1 *Given condition (1.3)*

$$V(P,Q) \le \frac{2(\xi-1)(1-\zeta)}{\xi-\zeta} \le \frac{\xi-\zeta}{2},$$
$$V(P,Q) \le \sum_{i=1}^{n} |p_i - q_i|,$$

with probability distributions $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$.

2. Preliminaries

This section contains basic Ostrowski type inequality discussed in [16, 19]. Such type of inequalities is studied for functions of bounded variation.

Proposition 2.1 *Let $g : [\zeta, \xi] \to \mathbb{R}$ be a function of bounded variation on $[\zeta, \xi]$, and let $\bigvee_h^{\hbar}(g)$ denote the total variation of g on $[h, \hbar]$. Then, $\forall x \in [h, \hbar]$,*

$$\left| \int_h^{\hbar} g(u)\, du - g(x)(\hbar - h) \right| \leq \left[\frac{\hbar - h}{2} + \left| x - \frac{\hbar - h}{2} \right| \right] \bigvee_h^{\hbar}(g) \leq (\hbar - h) \bigvee_h^{\hbar}(g). \tag{2.1}$$

By redefining and summing over i in Proposition 2.1, the following form of Ostrowski inequality can be obtained.

Proposition 2.2 *Consider a function $g : [h, \hbar] \to \mathbb{R}$ of bounded variation. Then, $\forall x_1, x_2 \in [h, \hbar]$,*

$$\left| \int_h^{\hbar} g(u)\, du - \frac{\hbar - h}{2} \sum_{i=1}^{2} g(x_i) \right| \leq \left[\frac{\hbar - h}{2} + \frac{1}{2} \sum_{i=1}^{2} \left| x_i - \frac{\hbar - h}{2} \right| \right] \bigvee_h^{\hbar}(g). \tag{2.2}$$

Here, the key tool for deriving this result is the triangle inequality.

Reevaluating (2.2) by putting $g = f'$, $x_1 = h = 1$, and $x_2 = \hbar = 1$, we get the following important lemma:

Lemma 2.1 *Consider a differentiable function $f : [\zeta, \xi] \to \mathbb{R}$ whose first derivative f' has bounded variation on $[\zeta, \xi]$. Then, for $\zeta < 1 < \xi$ and $x \in [\zeta, \xi]$,*

$$\left| f(x) - f(1) - \frac{x - 1}{2} \left[f'(1) + f'(x) \right] \right| \leq (x - 1) \bigvee_{\zeta}^{\xi}(f'). \tag{2.3}$$

The above result holds for $x \geq 1$. For $x < 1$, (2.3) takes the form:

$$\left| f(x) - f(1) - \frac{1 - x}{2} \left[f'(1) + f'(x) \right] \right| \leq (1 - x) \bigvee_{\zeta}^{\xi}(f'). \tag{2.4}$$

3. Main Results

This section establishes some useful inequalities for distinct convex functions in new f-divergence measure and interrelate it with existing divergence measure. One such measure is Variational distance defined as:

$$V(P, Q) = \sum_{i=1}^{n} |p_i - q_i|,$$

where $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ being the probability distributions. These relations open up wide scopes in the literature of information theory and inequalities.

Theorem 3.1 *Let us consider a function $f : [\zeta, \xi] \to \mathbb{R}$ of bounded variation. Then, given condition (1.3), we have*

$$\left|S_f(P,Q)-\frac{1}{\xi-\zeta}\int_\zeta^\xi f(\omega)\,d\omega\right| \le \left[\frac{1}{2}+\frac{1}{\xi-\zeta}\left\{\frac{1}{2}V(P,Q)-\left(\frac{\xi+\zeta}{2}-1\right)\right\}\right]\bigvee_\zeta^\xi(f)$$
$$\le \frac{1}{2}(\xi-\zeta)\,V(P,Q)\bigvee_\zeta^\xi(f) \tag{3.1}$$
$$\le \frac{1}{4}\bigvee_\zeta^\xi(f).$$

Proof: Redefining (2.1) with $g=f$, $x=\frac{p_i+q_i}{2q_i}$, $i=\{1,2,\dots,n\}$, $h=k$, $\hbar=K$, we get

$$\int_\zeta^\xi f(\omega)\,d\omega - f\Big(\frac{p_i+q_i}{2q_i}\Big)(\xi-\zeta) \le \left[\frac{\xi-\zeta}{2}+\left|\frac{p_i+q_i}{2q_i}-\frac{\xi-\zeta}{2}\right|\right]\bigvee_\zeta^\xi(f).$$

We multiply by q_i and take summation over $i=\{1,2,\dots,n\}$, then, by using the generalized triangle inequality, we obtain

$$\left|S_f(P,Q)-\frac{1}{\xi-\zeta}\int_\zeta^\xi f(\omega)\,d\omega\right| \le \left[\frac{1}{2}+\sum_{i=1}^n q_i\frac{1}{\xi-\zeta}\left|\frac{p_i+q_i}{2q_i}-\frac{\xi-\zeta}{2}\right|\right]\bigvee_\zeta^\xi(f).$$

Since

$$\sum_{i=1}^n q_i\frac{1}{\xi-\zeta}\left|\frac{p_i+q_i}{2q_i}-\frac{\xi-\zeta}{2}\right| = \frac{1}{\xi-\zeta}\left[\sum_{i=1}^n q_i\left(\frac{p_i+q_i}{2q_i}-\left(\frac{\xi+\zeta}{2}-1\right)\right)\right]$$
$$\le \frac{1}{\xi-\zeta}\left[\frac{1}{2}V(P,Q)-\left(\frac{\xi+\zeta}{2}-1\right)\right].$$

yielding the first part of inequality in (3.1).
For the second inequality, we have, for any positive variables ξ,ζ,

$$\left|\frac{\xi+\zeta}{2}-1\right| \le \frac{\xi-\zeta}{2},$$

and the third inequality follows from Lemma (i), i.e.,

$$V(P,Q)\le\frac{\xi-\zeta}{2},$$

yielding the result of the theorem. □

Corollary 3.1 *Suppose $f:[\zeta,\xi]\to\mathbb{R}$ satisfies (3.1), and let there exist $\epsilon>0$ such that*

$$0<\frac{1}{2\epsilon/\bigvee_\zeta^\xi(f)}<(\xi-\zeta).$$

Then, we have

$$\left|S_f(P,Q)-\frac{1}{\xi-\zeta}\int_\zeta^\xi f(t)\,dt\right| < \epsilon\,V(P,Q).$$

Corollary 3.2 *Consider an absolutely continuous function $f : [0,2] \to \mathbb{R}$ of bounded variation. Let $\Upsilon \in (0,1)$, and let $p_i(\Upsilon), q_i(\Upsilon)$ be probability distributions satisfying*

$$\left|\frac{p_i(\Upsilon)+q_i(\Upsilon)}{2q_i(\Upsilon)} - 1\right| \leq \Upsilon$$

for all $i = 1, 2, \ldots, n$.

Then, we have

$$S_f\big(P(\Upsilon), Q(\Upsilon)\big) = \frac{1}{2\Upsilon}\int_{1-\Upsilon}^{1+\Upsilon} f(t)\,dt + R_f\big(P, Q, \Upsilon\big),$$

where $R_f(P, Q, \Upsilon)$ is the remainder term satisfying

$$R_f\big(P, Q, \Upsilon\big) = \frac{1}{2}\left[1 + \frac{1}{2}V\big(P(\Upsilon), Q(\Upsilon)\big) - (\Upsilon - 1)\right]\bigvee_{\Upsilon-1}^{\Upsilon+1}(f).$$

This follows from (3.1) by taking $\zeta = 1 - \Upsilon$ and $\xi = 1 + \Upsilon$.

Theorem 3.2 *(Comparison Theorem of Theorems 1.2 and 1.3) Consider a differentiable function where the first derivative f' is of bounded variation on $[\zeta, \xi]$, then for given condition (1.3), we have*

$$|S_f(P,Q) - S_{f^\wedge}(P,Q)| \leq \frac{1}{4}\,V(P,Q)\bigvee_{\zeta}^{\xi}(f') \leq \frac{\xi-\zeta}{8}\bigvee_{\zeta}^{\xi}(f') \tag{3.2}$$

$$\big|S_f(P,Q) - S_{f^\Gamma}(P,Q)\big| \leq \frac{1}{2}\,V(P,Q)\bigvee_{\zeta}^{\xi}(f') \leq \frac{\xi-\zeta}{4}\bigvee_{\zeta}^{\xi}(f') \tag{3.3}$$

Proof: Redefining (2.1) by putting $x = (h+\hbar)/2$, we get the inequality

$$\left|\int_h^{\hbar} g(u)\,du - g\left(\frac{h+\hbar}{2}\right)(\hbar - h)\right| \leq \frac{\hbar - h}{2}\bigvee_{h}^{\hbar}(g).$$

Taking $g = f'$, $m = 1$, $M = x$, where $x \in [\zeta, \xi]$, we have

$$\left|f(x) - f(1) - f'\left(\frac{1+x}{2}\right)(x-1)\right| \leq \frac{|x-1|}{2}\bigvee_{1}^{x}(f') \leq \frac{|x-1|}{2}\bigvee_{\zeta}^{\xi}(f').$$

From (1.4), it follows that

$$|f(x) - f^\wedge(x)| \leq \frac{|x-1|}{2}\bigvee_{1}^{x}(f') \leq \frac{|x-1|}{2}\bigvee_{\zeta}^{\xi}(f').$$

Now, using $x = \frac{p_i+q_i}{2q_i}$, multiplying by q_i and summing over $i = 1, 2, \ldots, n$, then applying the generalized triangle inequality, the first inequality of (3.2) is obtained.
The second inequality follows from Lemma (i).

□

In a similar manner, the proof of (3.3) follows from Lemma (ii).

Corollary 3.3 *Suppose $f : [\zeta, \xi] \to \mathbb{R}$ satisfies (3.2) and (3.3), and let there exist $\epsilon > 0$ such that*

$$\xi - \zeta < \frac{8\epsilon}{\bigvee_{\zeta}^{\xi}(f')}.$$

Then, we have

$$|S_f(P,Q) - S_{f^\wedge}(P,Q)| < \epsilon,$$

$$\left|S_f(P,Q) - S_{f^\Gamma}(P,Q)\right| < \frac{\epsilon}{2}.$$

Corollary 3.4 *Consider an absolutely continuous function $f : [0,2] \to \mathbb{R}$ of bounded variation. Let $\Upsilon \in (0,1)$, and let $p_i(\Upsilon), q_i(\Upsilon)$ be probability distributions satisfying*

$$\left|\frac{p_i(\Upsilon) + q_i(\Upsilon)}{2q_i(\Upsilon)} - 1\right| \leq \Upsilon$$

for all $i = 1, 2, \dots, n$. Then, we have

$$S_f\big(P(\Upsilon), Q(\Upsilon)\big) = S_{f^\wedge}\big(P(\Upsilon), Q(\Upsilon)\big) + R_f\big(P, Q, \Upsilon\big),$$

where the remainder term $R_f(P,Q,\Upsilon)$ satisfies

$$R_f(P,Q,\Upsilon) \leq \frac{1}{4} V\big(P(\Upsilon), Q(\Upsilon)\big) \bigvee_{\Upsilon-1}^{\Upsilon+1}(f) \leq \frac{1}{4} \bigvee_{\Upsilon-1}^{\Upsilon+1}(f).$$

This follows from (3.2) by taking $\zeta = 1 - \Upsilon$ and $\xi = 1 + \Upsilon$.

4. Applications to Special Mean

The aforementioned inequalities are obtained by using a certain convex function, which results in different inequalities depending on the divergence measures. One such divergence measure is discussed above (1.2). Given condition (1.3) We have

$$S_f(P,Q) = G(P,Q) \tag{4.1}$$

Also, when $f(u) = u \ln u$, we get

$$\int_{\zeta}^{\xi} f(\omega)\, d\omega = \int_{\zeta}^{\xi} \omega \ln \omega \, d\omega = \frac{1}{4}\Big[\xi^2 \ln \xi^2 - \zeta^2 \ln \zeta^2 - (\xi^2 - \zeta^2)\Big] = \frac{\xi^2 - \zeta^2}{4} \ln \Big[I(\zeta^2, \xi^2)\Big],$$

where $I(\gamma, \delta)$ is the **identric mean** of two numbers γ, δ defined as

$$I(\gamma,\delta) = \begin{cases} \gamma & \text{if } \gamma = \delta, \\ \frac{1}{e}\left(\frac{\delta^\delta}{\gamma^\gamma}\right)^{1/(\delta-\gamma)} & \text{otherwise.} \end{cases}$$

Moreover,

$$\bigvee_{\zeta}^{\xi}(f) = \int_{\zeta}^{\xi} |f'(t)|\, dt = \int_{\zeta}^{\xi} |\ln t|\, dt = \vartheta(\xi, \zeta) \tag{4.3}$$

Using (3.1), (4.1), (4.2), and (4.3), we obtain

$$\left|G(Q,P) - \frac{1}{\xi-\zeta}\frac{\xi^2-\zeta^2}{4}\ln[I(\zeta^2,\xi^2)]\right| \le \left[\frac{1}{2} + \frac{1}{\xi-\zeta}\left(\frac{1}{2}V(P,Q) - \frac{\xi+\zeta}{2} + 1\right)\right]\vartheta(\xi,\zeta)$$
$$\le \frac{1}{2}(\xi-\zeta)V(P,Q)\,\vartheta(\xi,\zeta)$$
$$\le \frac{1}{4}\vartheta(\xi,\zeta).$$

Next, from (1.1) and (1.4), we have

$$S_{f^\wedge}(P,Q) = \sum_{i=1}^{n}\frac{1}{2}\frac{p_i+q_i}{q_i}\ln\frac{p_i+3q_i}{4q_i} \tag{4.4}$$

Also,

$$\bigvee_{\zeta}^{\xi}(f') = \int_{\zeta}^{\xi}|f''(t)|\,dt = \int_{\zeta}^{\xi}\frac{dt}{t} = \ln\frac{\xi}{\zeta} \tag{4.5}$$

Using (3.2), (4.4), and (4.5):

$$\left|G(Q,P) - \sum_{i=1}^{n}\frac{1}{2}\frac{p_i+q_i}{q_i}\ln\frac{p_i+3q_i}{4q_i}\right| \le \frac{1}{4}V(P,Q)\ln\frac{\xi}{\zeta} \le \frac{\xi-\zeta}{8}\ln\frac{\xi}{\zeta}.$$

Since

$$\frac{\xi-\zeta}{8}\ln\frac{\xi}{\zeta} = \frac{(\xi-\zeta)^2}{8L(\zeta,\xi)},$$

where $L(\alpha,\beta)$ is the **logarithmic mean** defined as

$$L(\alpha,\beta) = \begin{cases}\alpha & \text{if } \alpha=\beta,\\ \frac{\beta-\alpha}{\ln\beta-\ln\alpha} & \text{otherwise.}\end{cases}$$

Hence, we have the result

$$\left|G(Q,P) - \sum_{i=1}^{n}\frac{1}{2}\frac{p_i+q_i}{q_i}\ln\frac{p_i+3q_i}{4q_i}\right| \le \frac{1}{4}V(P,Q)\ln\frac{\xi}{\zeta} \le \frac{(\xi-\zeta)^2}{8L(\zeta,\xi)}.$$

Finally, for $t>0$, $S_f(P,Q)$ takes the form of the **variational distance**:

$$S_f(P,Q) = \frac{1}{2}\sum_{i=1}^{n}|p_i-q_i| = V(P,Q),$$

and

$$\frac{1}{\xi-\zeta}\int_{\zeta}^{\xi}f(\omega)\,d\omega = \frac{1}{\xi-\zeta}\left[\frac{(\zeta-1)^2}{2} + \frac{(\xi-1)^2}{2}\right] = \frac{1}{\xi-\zeta}\left[\left(\frac{\xi-\zeta}{2}\right)^2 + \left(\frac{\xi+\zeta}{2}-1\right)^2\right].$$

Also,

$$\bigvee_{\zeta}^{\xi}(f) = \bigvee_{\zeta}^{1}(f) + \bigvee_{1}^{\xi}(f) = 1-\zeta+\xi-1 = \xi-\zeta.$$

Using these in (3.1), we obtain

$$\begin{aligned}&\left|\frac{1}{2}V(P,Q)-\frac{1}{\xi-\zeta}\Big[\left(\frac{\xi-\zeta}{2}\right)^2+\left(\frac{\xi+\zeta}{2}-1\right)^2\Big]\right|\\&\leq\left[\frac{1}{2}+\frac{1}{\xi-\zeta}\left(\frac{1}{2}V(P,Q)-\frac{\xi+\zeta}{2}+1\right)\right](\xi-\zeta)\\&\leq\frac{1}{2}V(P,Q)\\&\leq\frac{1}{4}(\xi-\zeta).\end{aligned}$$

5. Conclusion

The new f-divergence measure expands the possibilities for comparing probability distributions and offers wider area for research involving divergence measures. The results obtained successfully demonstrate the feasibility of approximating new f-divergence measure using numerical integration techniques. By leveraging the bounded variation properties of the function f and its derivatives, accurate and efficient approximations can be achieved. Furthermore, the derived information inequalities have found practical applications in quantifying the difference between probability distributions, specifically in relation to the Relative Arithmetic-Geometric Divergence (AGD). Advancements in new f-divergence measures and their associated inequalities not only enhance theoretical understanding, but also offer practical methodologies for effectively analyzing and comparing probability distributions.

Funding: Not applicable.
Availability of data and material: The work did not use data from any external repository.
Declarations Conflict of interest: The authors declare no conflict of interest.

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