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Kind of bipartite graph associated on triple elements of subgroup of group

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ABSTRACT: Assume $\mathbb{G}$ is a non commutative group, $\mathbb{H}$ is a subgroup of $\mathbb{G}$, in this paper we define a kind of bipartite graph which denoted by $\Delta_{\mathbb{H},\ \mathbb{G}}$ and define as a vertex set $V = MUN$ were $M = \{(r,r,r) \in \mathbb{H}\times\mathbb{H}\times\mathbb{H}\}\ /\{(r,r,r) \in \mathbb{H}^3 : [t,3r] = 1,\ \forall t \in \mathbb{G}\}$ & $N =\ \mathbb{G} - \{t \in \mathbb{G} : [t,3r]\ = 1,\ \forall r \in \mathbb{H}\}$. Two vertices t and (r,r,r) are adjacent iff $[t,r,r,r] \neq 1$ this graph has no isolated vertex ,also we introduce the relative 3-Engle degree of $\mathbb{H}$ in ζ and discuss the relation between it and the graph $\Delta_{\mathbb{H},\ \mathbb{G}}$.

Key Words: Bipartite graph, relative 3-Engle degree, diameter, girth, Hamiltonian graph, infinite group.

Contents

1. Introduction

Algebraic graph theory is one of important top in mathematics which is interested for both specialist in the field of algebra and graph theory in the recent years. In algebraic graph theory, every graph associated to a group , ring, module or any other algebraic structure. we many refer to some papers on Engle graph[1], non- cyclic graph[3], Sub ring of graph[10], The AnnihilatingIdeal graph[14],zero-divisor graph [12], intersection graph [16], a graph associated to proper non-small sub-semi-modules f a semi-module [5] to explore more, the following research works may be reviewed [11,13,4,6,9,2,15,7,8]. The derived subgroup $[\mathbb{G},\mathbb{G}]$ of a group $\mathbb{G}$ is a subgroup of $\mathbb{G}$ generated by its commutators, which is to say $[\mathbb{G},\mathbb{G}] = \{[w,z] : w,z \in \mathbb{G}\}$, let $\mathbb{G}$ be a group and $r_1, r_2, \ldots, r_n \in \mathbb{G}$. Then

$[r_1, r_2, \ldots, r_n] = [r_1, r_2, \ldots, r_{n-1}]^{-1} {r_n}^{-1} [r_1, r_2, \ldots, r_{n-1}]\, r_n,\ \textit{for all}\ n > 1$. If $r_2 = \cdots = r_n$,

then we refer $[r_1, r_2, \ldots, r_n]$ by $[r_1,(n-1)r_2]$. The group $\mathbb{G}$ is an Engel group if for each pair $(a,b) \in \Gamma$, there exists an integer number $n = n(a,b)$ such that $[b,na] = 1$. In this article we introduce a new kind of bipartite graph which define as follows, let $\mathbb{G}$ be a group and $\mathbb{H}$ be a subgroup of $\mathbb{G}$ we denoted $(\Delta_{\mathbb{H},\mathbb{G}})$ as a bipartite graph whose vertex set MUN were $M = \{(r,r,r) \in \mathbb{H}\times\mathbb{H}\times\mathbb{H}\}\ /\{(r,r,r) \in \mathbb{H}^3 : [t,3r] = 1,\ \forall t \in \mathbb{G}\}$ & $N =\ \mathbb{G} - \{t \in \mathbb{G} : [t,3r]\ = 1,\ \forall r \in \mathbb{H}\}$. Two vertices t and (r,r,r) are adjacent iff $[t,r,r,r] \neq 1$

We studied and proved some important properties of this graph, such as that it does not have isolated vertices, as well as calculating the diameter and girth of the graph, and we explained when there are not Hamiltonian. Also we define relative 3-Engle degree of $\mathbb{H}$ in $\mathbb{G}$ and discuss the relation between it and the graph $\Delta_{\mathbb{H},\mathbb{G}}$. Also, use this probability to fined lower bounded on the number of edges in this graph.

2. Basic results

This section introduces a new class of bipartite graphs based on properties of set theory, providing illustrative examples to help highlight the features of this type of graph. We then review a number of key properties associated with the proposed graph. $\Delta_{\mathbb{H},\mathbb{G}}$.

Definition [1.1] suppose that $\mathbb{G}$ is a finite non commuting group and $\mathbb{H}$ be a subgroup of $\mathbb{G}$, then we can define a bipartite graph associated on triple elements of a subgroup $\mathbb{H}$ of $\mathbb{G}$ which is denoted by $\Delta_{\mathbb{H},\mathbb{G}}$ as follows the vertices set is $V(\ \Delta_{\mathbb{H},\mathbb{G}}\ ) = MUN\ were\$\ M = \{(r,r,r) \in \mathbb{H}\times\mathbb{H}\times\mathbb{H}\}\ /\{(r,r,r) \in \mathbb{H}^3 :$

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$[t, 3r] = 1,\ \forall t \in \mathbb{G}\}$ & $N = \ \mathbb{G} - \{t \in \mathbb{G} : [t, 3r]\ = 1,\ \forall\ r \in \mathbb{H}\}$. Two vertices t and (r, r, r) are adjacent iff $[t, r, r, r] \neq 1$

Example [1.2] Assume $D_{10} = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ and $\mathbb{H}$=$\{$e,$a^3, b, a^3b\}$ then $\Delta_{\mathbb{H},D_{10}}$ as follows (see figure 1)

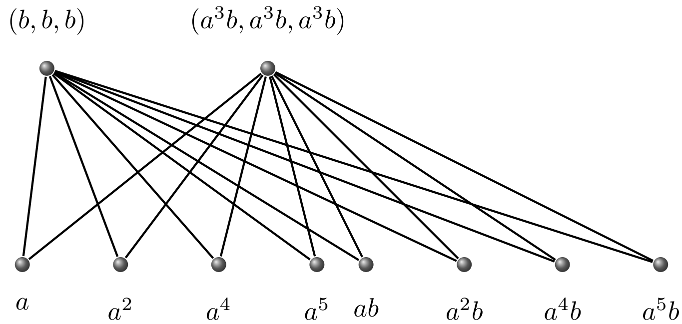

Figure 1: $\Delta_{\mathbb{H},D_{10}}$

Lemma [1.3] $\Delta_{\mathbb{H},\mathbb{G}}$ has no isolated vertices.

Proof: let x be arbitrary vertices in $\Delta_{\mathbb{H},\mathbb{G}}$ then there are two cases $x \in N\ or\ x\ \in M,\ if\ x \in N$ then from definition of $\Delta_{\mathbb{H},\mathbb{G}}$there exist $r \in \mathbb{H}$ such that $[x, 3r] \neq 1$, thus x is adjacent to (r, r, r) and this gives that x is not isolated vertices if x$\in M$ by similar method then x is not isolated vertices and the proof is complete.

Theorem [1.4] let $\mathbb{H}$ abelian subgroup of $\mathbb{G}$ and $[\mathbb{G} :\ \mathbb{H}]\ = 2$ Then daim ($\Delta_{\mathbb{H},\mathbb{G}}) = 2$ if and only if $|M| \geq 1$.

Proof

since $\mathbb{H}$ is an abelian subgroup of $\mathbb{G}$ and $[\mathbb{G} : \mathbb{H}] = 2$, then $\mathbb{G} = \mathbb{H} \cup X_1\mathbb{H}$, were $X_1 \in \mathbb{G}$ and $|N| \leq |X_1\ \mathbb{H}|$, also $|M| \geq 1$, this implies that there exist $b \in N$ such that b is adjacent to an element of M, $a \in M$, $a = (r, r, r),\quad b = (X_1t)$ and this gives that $[X_1t, r, r, r] \neq e$, so $[X_1, r, r, r] \neq e$, for all $(r, r, r) \in A$, this deduce that element of N is adjacent of all elements of M because $[X_1t, r, r, r]$= $[[[X_1t, r], r], r] = [[[x_1, r]^t, r], r]$=$[X_1, r, r, r] \neq e$, Thus the graph $\Delta_{\mathbb{H},\mathbb{G}}$ is a complete bipartite graph and the result holds. The converse follows from daim($\Delta_{\mathbb{H},\mathbb{G}}$)=2.

Theorem [1.5] Assume x is a vertex of $\Delta_{\mathbb{H},\mathbb{G}}$ and $|x| \geq 2$,then girth ($\Delta_{\mathbb{H},\mathbb{G}}) = 4$

Proof if x $\in N$, since $|x| \geq 2$ So there exist at least two vertices of M ,x_1 , $x_2 \in M$,x_1=(r_1, r_1, r_1), $x_2 = (r_2, r_2, r_2)$. such that x adjacent with x_1 and x_2 ,so $[x, r_1, r_1, r_1] \neq e$ and $[x, r_2, r_2, r_2] \neq e$ and this given that $[x_z$,$r_1, r_1, r_1]$= $[[[x_z, r_1], r_1], r_1] = [[[[x, r_1]^z[z, r_1], r_1], r_1]$= $[x, r_1, r_1, r_1] \neq e$,also $[x_z, r_2, r_2, r_2] = [x, r_2, r_2, r_2]\ \neq e$ .there for we have

$$x_1 \backsim x \backsim x_2 \backsim x_z \backsim x_1$$

So the result holds.

If $x \in M,\ x = (r, r, r)$ then there exist $x_1, x_2 \in N$,such that x adjacent with $x_1 and x_2$ and this give that $[x_1, r, r, r] \neq e$ and $[x_2, r, r, r] \neq e$ we have that $x_1 and x_2$ adjacent with $x_3 = [zr, zr, zr]$ because $[x_1, zr, zr, zr]$= $[[x_1, zr], zr], zr]$= $[x_1, r, r, r] \neq e$,so for x_2 ,thus we have

$$x_1 \backsim x_3 \backsim x_2 \backsim x \backsim x_1$$

Lemma [1.6] Assume $\mathbb{H}$ is a commutative subgroup of a group $\mathbb{G}$, then graph $\Delta_{\mathbb{H}\ ,\mathbb{G}}$ is star graph, if $|M| = 1$ or $|N| = 1$,

Proof: since $\mathbb{H}$ is a commutative sub group of $\mathbb{G}$, so $[\mathbb{G}, \mathbb{H}] = 2$ and this given that $|N| = |x\mathbb{H}| = |\mathbb{H}|$, for some $x \notin \mathbb{H}$, therefor if $|N| = 1$, this contradiction, so $|M| = 1$. If A singleton set since this graph with out isolated vertex, so the proof is complete.

Theorem[1.7] Assume $\mathbb{H}$ is a commutative subgroup of a group $\mathbb{G}$, then $\Delta_{\mathbb{H}\ ,\mathbb{G}}$ is not Hamilton if $|M| < |N|$

Proof: *let* $M = \{m_1, m_2, \ldots, m_i\}$, $N = \{n_1, n_2, \ldots, n_{i-1}, n_i, n_{i+1}, \ldots, n_j\}$, let $\Delta_{\mathbb{H}\ ,\mathbb{G}}$ has Hamilton cycle C, we can see that after 2i edges, were C start at $m_1 = (r_1, r_1, r_1)$

$$m_1 \backsim n_i \backsim m_i \backsim n_{i-1} \backsim m_{i-1} \backsim \cdots \backsim n_2 \backsim m_2 \backsim n_1 \backsim m_1$$

Hence j-i vertices in N out of the cycle C, so C is not Hamilton cycle

Definition [1.8] suppose that $\mathbb{H}$ is a subgroup of $\mathbb{G}$, then the relative 3-Engle group of $\mathbb{H}$ in $\mathbb{G}$ is denoted by $d_3(\mathbb{H}, \mathbb{G})$ and define as follows

$d_3(\mathbb{H}, \mathbb{G}) = \frac{1}{|\mathbb{H}||\mathbb{G}|}\ |\{(r,t) \in \mathbb{H} * \mathbb{G} : [t, r, r, r] = 1\}|$, it's clear that $\mathbb{G}$ is 3-Engle group were $d_3(\mathbb{H}, \mathbb{G}) = 1$ and $\mathbb{H} = \mathbb{G}$

We define the sets of all elements of $\mathbb{H}$ and $\mathbb{G}$, where $[t, r, r, r]$ =1 as follows $C_{\mathbb{H}}(t)$= $\{r \in \mathbb{H} :\ [t, r, r, r] = 1\}$ and $C_{\mathbb{G}}(r)$=$\{t \in \mathbb{G} :\ [t, r, r, r] = 1\}$.So $d_3\ (\mathbb{H}, \mathbb{G}) = \frac{1}{|\mathbb{H}||\mathbb{G}|} \sum_{r\in\mathbb{H}} |C_{\mathbb{G}}(r)|$

=$d_3\ (\mathbb{H}, \mathbb{G}) = \frac{1}{|\mathbb{H}||\mathbb{G}|} \sum_{t\in\mathbb{G}} |C_{\mathbb{H}}(t)|$ .we can see that $C_{\mathbb{H}}(t)$ and $C_{\mathbb{G}}(r)$ are not necessary subgroups of $\mathbb{H}$ and $\mathbb{G}$ resp. Put $C_{\mathbb{G}}\ (\mathbb{H}) = \bigcap_{r\in\mathbb{H}} C_{\mathbb{G}}(r)\ and\ C_{\mathbb{H}}\ (\mathbb{G}) = \bigcap_{t\in\mathbb{G}} C_{\mathbb{H}}(t)$

Let κ_1 denoted the set of all centralizers $C_{\mathbb{H}}(t)\ in$ $\mathbb{H}$ and $C_{\mathbb{G}}\ (r)\ in$ $\mathbb{G}$ where both $C_{\mathbb{H}}(t)$ and $C_{\mathbb{G}}\ (r)$ are subgroups of their respective groups."

Example[1.9]: Let $\mathbb{G} = D_6$;$\mathbb{H} = \{e, a, a^2\}$ then $d_3(\mathbb{H}, \mathbb{G}) = \frac{1}{|\mathbb{H}||\mathbb{G}|}\ |\{(r,t) \in \mathbb{H} * \mathbb{G}\ : [t, r, r, r] = 1\}|$ So $d_3(\mathbb{H}, \mathbb{G}) = 1$, D_6 is a relative 3-Engle group for subgroup $\mathbb{H}$

Theorem[1.10]: Suppose that $\Delta_{\mathbb{H},\mathbb{G}}$ is a graph, then if $C_{\mathbb{G}}(t)\ \cap C_{\mathbb{G}}(\mathbb{H}) \neq \emptyset$ where $t \in N$, girth ($\Delta_{\mathbb{H},\mathbb{G}}$) = 4

Proof: since $t \in N$ there exist $x_1 \in M$, $x_1 = (r, r, r)$ such that t adjacent to x_1 and $C_{\mathbb{G}}(t)\ \cap C_{\mathbb{G}}(\mathbb{H})$ $\neq \emptyset$,$\exists\ h \in C_{\mathbb{G}}(t) \cap C_{\mathbb{G}}(\mathbb{H})$ and we have $t_h \backsim x_1$ because $[t_h, r, r, r]$= $[[[t_h, r], r], r] = [[[t_h, r]^h [h, r], r], r]$ =$[[[t_h, r], r], r] \neq e$

With a similar method, we can see that t and t_h are adjacent to (r_c, r_c, r_c) and the proof is complete.

Theorem [1.11]: Assume $\mathbb{H}$ is a subgroup of finite group $\mathbb{G}$, then $|E\ (\ \Delta_{\mathbb{H},\mathbb{G}})| = |\mathbb{H}|\ |\mathbb{G}|\ (1 - d_3\ (\mathbb{H}, \mathbb{G}))$.

Proof : From definition of $d_3\ (\mathcal{H}, \mathfrak{G})$ we have the following :

$d_3\ (\mathbb{H}, \mathbb{G})\ |\mathbb{H}|\ |\mathbb{G}| = |\{(r,t) \in \mathbb{H} * \mathbb{G} : [t, r, r, r] = 1\}|$, also we have

$$|\mathbb{H}|\ |\mathbb{G}| = |\{(r,t) \in \mathbb{H} * \mathbb{G} : [t, r, r, r] = 1\}| + |\{(r,t) \in \mathbb{H} * \mathbb{G} : [t, r, r, r] \neq 1\}|$$

Thus $|\mathbb{H}|\ |\mathbb{G}| = d_3\ (\mathbb{H}, \mathbb{G})\ |\mathbb{H}|\ |\mathbb{G}| + |E\ (\ \Delta_{\mathbb{H},\mathbb{G}})|$. There for $|E\ (\Delta_{\mathbb{H},\mathbb{G}})| = |\mathbb{H}|\ |\mathbb{G}|\ (1 - d_3\ (\mathbb{H}, \mathbb{G}))$.

Example [1.12]: Let $\mathbb{G} = S_3$ and $\mathbb{H}$ be a subgroup of $\mathbb{G}$ defined as $\mathbb{H} = \{e, (123)\ , (132)\}$. Then $|E\ (\Delta_{\mathbb{H},S_3})| = |\mathbb{H}|\ |S_3|\ (1 - d_3\ (\mathbb{H}, S_3)) = (3)\ (6)\ \left(1 - \frac{3}{2}\right) = 6.$

Theorem [1.13]: Let $\mathbb{H}$ be a commutative subgroup of a non-commutative group $\mathbb{G}$. Then

$$\frac{|\mathbb{H}|}{2}\ (|\mathbb{G}| - |C_{\mathbb{G}}\ (\mathbb{H})|) \leq |E\ (\Delta_{\mathbb{H},\mathbb{G}})|\ .$$

Proof: Let $|E\ (\Delta_{\mathbb{H},\mathbb{G}})| = |\mathbb{H}|\ |\mathbb{G}|\ (1 - d_3\ (\mathbb{H}, \mathbb{G}))$.......$_{(**)}$

Since $d_3\ (\mathbb{H}, \mathbb{G}) = \frac{1}{|\mathbb{H}||\mathbb{G}|}\ \sum_{t\in\mathbb{G}} |C_{\mathbb{H}}(t)|$

$$= \frac{1}{|\mathbb{H}|\ |\mathbb{G}|} \sum_{t\in C_{\mathbb{G}}(\mathbb{H})} |C_{\mathbb{H}}(t)| + \sum_{t\notin C_{\mathbb{G}}(\mathbb{H})} |C_{\mathbb{H}}(t)|$$

$$\leq \frac{1}{|\mathbb{H}|\ |\mathbb{G}|}\ \left[|\mathbb{H}|\ |C_{\mathbb{G}}\ (\mathbb{H})| + (|\mathbb{G}| - |C_{\mathbb{G}}\ (\mathbb{H})|)\ \frac{|\mathbb{H}|}{2}\right]$$

$$= \frac{1}{|\mathbb{H}|\ |\mathbb{G}|}\ \left[\frac{1}{2}\ |\mathbb{H}|\ |C_{\mathbb{G}}\ (\mathbb{H})| + \left(\frac{|\mathbb{G}|\ |\mathbb{H}|}{2}\right)\right]$$

$= \frac{1}{2}\left(1 + \frac{|C_{\mathbb{G}}(\mathbb{H})|}{|\mathbb{G}|}\right)$ ....$_{(***)}$

By creating a $_{(**)}$ in $_{(***)}$ equation we get :

$$|E\ (\Delta_{\mathbb{H},\mathbb{G}})| \geq |\mathbb{H}|\ |\mathbb{G}|\ \left(1 - \frac{1}{2}\left(1 + \frac{|C_{\mathbb{G}}\ (\mathbb{H})|}{|\mathbb{G}|}\right)\ \right)$$

$$= \frac{|\mathbb{H}|}{2}\ (|\mathbb{G}| - |C_{\mathbb{G}}\ (\mathbb{H})|)\ \ .$$

3. Conclusions

A novel family of bipartite graphs linked to subgroups of groups with three components each was presented and thoroughly investigated in this work. The algebraic and structural features of the graph $\Delta_{\mathbb{H},\mathbb{G}}$ were investigated once it was characterized in terms of subgroup-group relations and commutator conditions. A number of significant outcomes were confirmed: 1. It was established that there were no isolated vertices, demonstrating $\Delta_{\mathbb{H},\mathbb{G}}$ intrinsic connectivity. 2. Under various group-theoretic assumptions, the graph's diameter and girth were determined, paying particular attention to situations in which the subgroup H is abelian.

3. It was determined under what circumstances the graph turns into a star graph or a complete bipartite graph. 4. It was shown that $\Delta_{\mathbb{H},\mathbb{G}}$ is non-Hamiltonicity under specific cardinality requirements of the vertex sets. 5. The notion of a subgroup's relative 3-Engel degree within a group was presented, along with its relationship to the graph's structure. A lower constraint on the graph's edge count was also obtained using this relation. 6. The basic results were confirmed by providing precise computations of $\Delta_{\mathbb{H},\mathbb{G}}$ the girth, and the Engel degree using a number of instructive instances, such as dihedral and symmetric groups. A graph structure that exhibits higher commutator interactions connects group theory with graph theory, expanding algebraic graph theory to Engel-related subgroup characteristics.

References

1. J.A.Abdollahi, Engel graph associated with a group, J. Algebra 318(2007), 680-691.
2. A.Abdollahi, S. Akbari, H.R. Maimani, Non-commutative graph of a group, J. Algebra 298(2006), 468-492.
3. A.Abdollahi, A. Mohammadi Hassanabadi, Non-cyclic graph of a group, Comm. Algebra 35(2007), 2057-2081..
4. A. Erfanian, R. Rezaei and P. Lescot, On the relative commutativity degree of a subgroup of a finite group, Comm. Algebra 35 (2007), no. 12, 41834197.
5. A. H. Alwan, A graph associated to proper non-small sub-semi-modules f a semi-module, International Journal f Nonlinear Analysis and Applications, vol. 12, no. 2 (2021), 499-509.
6. A.H. Alwan, Maximal ideal graph of commutative semirings, International Journal of Nonlinear Analysis and Applications, 12(1) (2021) 913-926.
7. A. H. Alwan, g-small intersection graph of a module, Baghdad Sci. J., 21(8), 2024, 2671-2680.
8. A. H. Alwan N. K. Tuama, , Nilpotent graph of a semiring, Asia Pacific Journal of Mathematics, 11:77, 2024.
9. A. Iranmanesh, and Jafarzadeh, A., On commuting graph associated with the symmetric and alternating groups, J. Alg. Appl. 7(1) (2008) 129–146.
10. A. Nada Laabi, Ahmed. Hameed. Kamil, Yaqoob. A. Farawi, Subring in Graph Theory, Advances in Nonlinear Variational Inequalities. J., 27(2024), 284-287.
11. B. Tolue, Erfanian, A. and Jafarzadeh, A., A kind of non-commuting graph of finite groups, { J. Sci. Islam. Repub. Iran} 25(4) (2014) 379–384
12. D, F., Anderson and Philip S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra, 217(1999),434-447.
13. D. Malik, S. John, M. and Mordeson,. Sen, M. K., {Fundamentals Of Abstract ALgebra. IV.} Series QA162. M346 1997.
14. F. Aliniaeifard, M. Behboodi, E. Mehdi-Nezhad, A. M. Rahimi: The Annihilating-Ideal graph of a commutative ring with respect to an ideal, Comm. In Algebra 42:5,(2014),2269-2284.. Numer. Simul., vol. 90, 2020, doi: 10.1016/j.cnsns.2020.105361.
15. M. Nasiri, Ahmad. Erfanian, Yaqoob. A. Farawi, I. Muchtadi-Alamsyah: A Kind of Graph Associated to a Fixed Element and a Subgroup of Group, Southeast Asian Bulletin of Mathematics. J, 44(2020), 813-818.
16. R. Shen: Intersection graph of subgroup of finite groups, Czechoslovak Math. J.,60(2010), 945-950.

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