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ART Rings

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ABSTRACT: A ring R is said to be an ART ring if each member of R can be represented as the sum of a regular element (in the von Neumann sense), a tripotent element, and a nilpotent element. This article introduces and studies ART rings, highlighting the role of tripotent elements in such decompositions. Several fundamental properties and key results related to ART rings are investigated.

Key Words: Tripotent, regular, ART rings.

Contents

1. Introduction

The study of rings often involves decomposing elements into simpler ones, and decomposition of elements into algebraic forms have been a central theme in ring theory since the introduction of clean rings by W.K. Nicholson in 1977 [12]. Nicholson and Y. Zhou defined a clean ring as one in which every element can be expressed as the sum of an idempotent and a unit [13]. This notion rapidly gained attention as it highlighted deep connections between element decomposability and broader structural properties such as the exchange property, with numerous mathematicians, among them Y. Ye [14] and T. Y, Lam [11], further expanding these concepts and investigating various generalizations and special cases.

Subsequent research introduced broader classes, each extending the decomposability paradigm in new directions. These advancements have fueled the search for novel decomposition structures, offering new insight into the interplay between regularity, nilpotency and idempotency.

In this article, we intend to examine a novel class of associative rings, termed ART rings, characterized by the property that each element can be written as the sum of a regular element (in the von Neumann sense), a tripotent element, and a nilpotent element. This decomposition generalizes several previously studied notions such as clean rings [1], semi nil-clean rings [4], and special r-clean rings. The central goal of this work is to emphasize the role of tripotent elements in additive decompositions of ring elements, which has remained relatively unexplored compared to idempotents and units.

We establish several fundamental structural results for the ART rings. Specifically, we show that under homomorphic images, the ART property is retained and quotients by nil ideals. We provide the necessary and sufficient conditions for direct product rings, triangular matrix rings, and Morita contexts to be ART. In addition, we demonstrate that the polynomial ring $R[u]$ and the formal power series ring $R[[u]]$ over a commutative ring R are not ART rings, while the truncated polynomial ring $R[u]/(u^n)$ retains the ART property if and only if R is ART.

Moreover, we explore the relationship between ART rings and other established ring classes. Under suitable assumptions, we prove that commutative ART rings without non-trivial zero divisors are also special r-clean and 2-clean. This interplay highlights the generalizing strength of the ART decomposition framework.

Throughout this paper, we suppose $\mathrm{VR_g}(R)$ stands for the Von Neumann regular element, $T_p(R)$ for tripotent elements, $U_n(R)$ for units, $C_T(R)$ for central tripotents and $N_p(R)$ denotes nilpotents of a ring R.

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2. ART Rings

We begin this section by providing the definitions and examples of ART rings.

Definition 2.1 *A member m of a ring R is said to admit an* ART decomposition *if it can be expressed as the sum of a regular element, a tripotent element, and a nilpotent element of R. That is,*

$$m = g + t + p,$$

where g is regular, t is tripotent (i.e., $t^3 = t$), and p is nilpotent. A ring R in which every element admits such a decomposition is termed as an ART ring.

Example: A trivial example of an ART ring is the Boolean ring $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 2.1 *Let R be a ring and J be a nil ideal of R such that every tripotent element of R/J lifts modulo J. Then R is an ART ring if and only if the factor ring R/J is an ART ring.*

Proof: Suppose R is an ART ring and J is a nil ideal of R. Let $u' = u + J \in R/J$, for some $u \in R$. As R is an ART ring, $\exists\ g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$ such that,

$$u = g + t + p.$$

[10, Theorem 9.3] states that von Neumann regular elements lift modulo nil ideals, so $g' = g + J \in \mathrm{VR_g}(R/J)$. Since t is tripotent, i.e., $t^3 = t$, we have

$$(t')^3 = (t + J)^3 = t^3 + J = t + J = t',$$

which shows that $t' \in \mathrm{T_p}(R/J)$. Similarly, as p is nilpotent, say $p^k = 0$, it follows that

$$(p')^k = (p + J)^k = p^k + J = J,$$

so $p' \in \mathrm{N_p}(R/J)$. Therefore,

$$u' = g' + t' + p',$$

which shows that every element of R/J has an ART decomposition. Hence, ring R/J is an ART.

Conversely, suppose the factor ring R/J is an ART. Let $u \in R$, and consider $u' = u + J \in R/J$. Since R/J is an ART ring, then we write

$$u' = g' + t' + p',$$

where $g' \in \mathrm{VR_g}(R/J)$, $t' \in \mathrm{T_p}(R/J)$, and $p' \in \mathrm{N_p}(R/J)$. By the lifting property of von Neumann regular elements modulo nil ideals [10], $\exists\ g \in \mathrm{VR_g}(R)$ such that $g + J = g'$. Also, by assumption tripotent lift modulo J, if $t'^3 = t'$, then there exists a lift $t \in R$ with $t^3 - t \in J$; since J is nil, $t \in \mathrm{T_p}(R)$.

Write $u = g + t + p + i$, where $i \in J$ accounts for the difference between u and the lifted sum $g + t + p$, and $p \in R$ is a lift of p'. As p' is nilpotent in R/J, so $p^k \in J$, for some $k \in \mathbb{N}$. As both p^k and i belong to the nil ideal J, the sum $p + i$ is nilpotent in R. Thus,

$$u = g + t + (p + i),$$

where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p + i \in \mathrm{N_p}(R)$, showing that u admits an ART decomposition in R. Hence, R is an ART ring. □

Proposition 2.2 *Let R be a ring with characteristic 3. Then*
(i) An element m of R is an ART if and only if $1 - m$ is an ART.
(ii) R is an ART ring if and only if each member $m \in R$ can be expressed of the form $m = g - t + p$, where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$.

Proof: (i) : Let m be an ART, so $m = g + t + p$, where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$. Therefore, $1 - m = (-g) + (1 - t) + (-p)$. Note that, $(-g \in \mathrm{VR_g}(R)), (-p \in N_p(R))$, and since (R) has characteristic 3, it follows that $(1 - t \in T_p(R))$. Hence $1 - m$ is an ART. Converse holds by similar arguments.

(ii) : Let R be an ART with $-m \in R$ can be decomposed as $-m = g + t + p$, where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$. This gives $m = (-g) - t + (-p)$ where , $-g \in \mathrm{VR_g}(R)$, $-t \in T_p(R)$ and $-p \in N_p(R)$. □

Proposition 2.3 *Every homomorphic image of an ART ring is an ART.*

Proof: Let R be an ART ring and let $\phi : R \to S$ be a surjective ring homomorphism. Then $S \cong R/J$, for some ideal $J \subseteq R$. We aim to show that S is also an ART ring.

Let $y \in S$. Since ϕ is surjective, $\exists\ m' \in R$ such that $\phi(m') = y$. As R is an ART ring, we write $m' = g + t + p$, where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$. Applying the homomorphism ϕ yields

$$y = \phi(m') = \phi(g) + \phi(t) + \phi(p).$$

We now verify that each of the images $\phi(g)$, $\phi(t)$, and $\phi(p)$ belongs to the corresponding class in S. Since $g \in \mathrm{VR_g}(R)$, $\exists\ b \in R$ such that $g = gbg$. Applying ϕ, we obtain $\phi(g) = \phi(g)\phi(b)\phi(g)$, which shows that $\phi(g) \in \mathrm{VR_g}(S)$. Next, as $t^3 = t$, we have $\phi(t)^3 = \phi(t^3) = \phi(t)$, so $\phi(t) \in \mathrm{T_p}(S)$. Similarly, since $p^k = 0$ for some $k \in \mathbb{N}$, it implies that $\phi(p)^k = \phi(p^k) = \phi(0) = 0$, and hence $\phi(p) \in \mathrm{N_p}(S)$.

Therefore, every element $y \in S$ admits a decomposition $y = \phi(g) + \phi(t) + \phi(p)$, where $\phi(g) \in \mathrm{VR_g}(S)$, $\phi(t) \in \mathrm{T_p}(S)$, and $\phi(p) \in \mathrm{N_p}(S)$. This shows that S is an ART ring.

□

We prove the following results based on various constructions of rings concerning ART decomposition.

Proposition 2.4 *Let $(R_i)_{i\in J}$ be a collection of rings and $R = \prod_{i\in J}(R_i)$. Then*
(i) If R is an ART ring then for each $i \in J$, R_i is ART.
(ii) Suppose $\prod_{i\in J} N(R_i) \subseteq N(R)$. If for each $i \in J$, R_i is an ART ring then R is also an ART.
(iii)For finite J, R is ART necessarily and sufficiently implied R_j is ART.

Proof: (i) : Suppose $R = \prod_{i\in J} R_i$ is an ART ring. For any fixed $i \in J$, consider the canonical projection map $\pi_i : R \to R_i$. Since π_i is a surjective ring homomorphism and homomorphic images of ART rings are ART by Proposition 2.3, it follows that R_i is an ART ring.

(ii) : Assume that each R_i is an ART ring and that $\prod_{i\in J} \mathrm{N_p}(R_i) \subseteq \mathrm{N_p}(R)$. Let $x = (x_i)_{i\in J} \in R$. Since each R_i is ART, we write
$x_i = g_i + t_i + p_i$, where $g_i \in \mathrm{VR_g}(R_i)$, $t_i \in \mathrm{T_p}(R_i)$, and $p_i \in \mathrm{N_p}(R_i)$. Define

$$g = (g_i)_{i\in J}, \quad t = (t_i)_{i\in J}, \quad p = (p_i)_{i\in J}.$$

Then $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$ (by assumption on the nilpotent component). Hence, $x = g + t + p$ is an ART decomposition in R, so R is an ART ring.

(iii) : As J is finite, then $\prod_{i\in J} \mathrm{N_p}(R_i) = \mathrm{N_p}(R)$. Thus, part (iii) immediately follows from parts (i) and (ii), establishing the equivalence.

□

Proposition 2.5 *Let R be a commutative ring. Then a polynomial ring $R[u]$ is not an ART ring.*

Proof: Consider the indeterminate $u \in R[u]$. First, observe that u is not nilpotent, since $u^n \neq 0$, for all $n \in \mathbb{N}$. Next, we check whether u is tripotent. Suppose $u^3 = u$; this implies $u(u^2 - 1) = 0$, which holds only in the trivial case $u = 0$ or $u^2 = 1$. However, u being an indeterminate, $u^2 - 1 \neq 0$ in general, so u is not tripotent.

We now show that u is not regular in the von Neumann sense. Suppose, on the contrary, that $\exists$ $g(u) \in R[u]$ such that $u = ug(u)u$. Then $u^2 g(u) = u$, which implies $ug(u) = 1$ in the commutative ring $R[u]$. This would mean that $u \in \mathrm{U_n}(R[u])$, which is impossible since u is of positive degree and $R[u]$ is not a field. Thus, u is not regular.

Now suppose that $u = g + t + p$, where $g \in \mathrm{VR_g}(R[u])$, $t \in \mathrm{T_p}(R[u])$, and $p \in \mathrm{N_p}(R[u])$. As nilpotent elements in $R[u]$ must have finite degree and vanish under a power, the polynomial p must be zero or of degree zero. Similarly, tripotent elements satisfy $t^3 = t$, and such elements in $R[u]$ must be constant polynomials (since higher-degree polynomials do not remain fixed under cubing unless they are trivial). Therefore, both t and p have degree zero, implying $g = u - t - p$ must be of degree one. However, regular elements in $R[u]$ cannot have degree one, leading to a contradiction. Hence, u cannot be written as the sum of a regular(von Neumann), tripotent, and nilpotent element, so $R[u]$ is not an ART ring. □

We can show by similar argument that the formal power series ring $R[[u]]$ is also not an ART ring.

Proposition 2.6 *Suppose R is a commutative ring. Then the ring R is an ART ring if and only if the quotient ring $R[u]/(u^n)$ is an ART ring.*

Proof: ($\Longrightarrow$) Suppose that ring R is an ART. Fix an integer $n \geq 1$, and consider an arbitrary element

$$f(u) = m_0 + m_1 u + m_2 u^2 + \cdots + m_{n-1} u^{n-1} + (u^n)$$

in $R[u]/(u^n)$, with each $m_i \in R$. Since R is an ART ring, the constant term m_0 has an ART decomposition:

$$m_0 = g_0 + t_0 + p_0,$$

where $g_0 \in \mathrm{VR_g}(R)$, $t_0 \in \mathrm{T_p}(R)$, and $p_0 \in \mathrm{N_p}(R)$.

Now, consider the remaining part of $f(u)$,

$$h(u) = m_1 u + m_2 u^2 + \cdots + m_{n-1} u^{n-1},$$

which belongs to the ideal generated by u in $R[u]$. In $R[u]/(u^n)$, since $u^n = 0$, the polynomial $h(u)$ is nilpotent.

Define the polynomials:

$$g(u) = g_0, \quad t(u) = t_0, \quad p(u) = p_0 + h(u).$$

Then clearly,

$$f(u) = g(u) + t(u) + p(u).$$

Since $g(u)$ is constant and $g_0 \in \mathrm{VR_g}(R)$, we have $g(u) \in \mathrm{VR_g}(R[u]/(u^n))$. Similarly, $t(u)^3 = t_0^3 = t_0$, so $t(u) \in \mathrm{T_p}(R[u]/(u^n))$. Also, as both p_0 and $h(u)$ are nilpotent and the sum of nilpotents remains nilpotent in $R[u]/(u^n)$, it follows that $p(u) \in \mathrm{N_p}(R[u]/(u^n))$. Hence, every element of $R[u]/(u^n)$ admits an ART decomposition. Thus, $R[u]/(u^n)$ is an ART ring.

($\Longleftarrow$) Conversely, suppose that $R[u]/(u^n)$ is an ART ring for some fixed $k \geq 1$. Let $m \in R$ be arbitrary and view it as the constant polynomial

$$f(u) = m \in R[u]/(u^n).$$

Since $f(u)$ has an ART decomposition, we may write

$$f(u) = g(u) + t(u) + p(u),$$

where $g(u) \in \mathrm{VR_g}(R[u]/(u^n))$, $t(u) \in \mathrm{T_p}(R[u]/(u^n))$, and $p(u) \in \mathrm{N_p}(R[u]/(u^n))$.

Because $f(u)$ is constant, the non-constant coefficients in $g(u), t(u), p(u)$ must vanish. Hence, there exist $g_0, t_0, p_0 \in R$ such that

$$m = g_0 + t_0 + p_0,$$

with $g_0 \in \mathrm{VR_g}(R)$, $t_0 \in \mathrm{T_p}(R)$, and $p_0 \in \mathrm{N_p}(R)$. Therefore, every element $m \in R$ admits an ART decomposition, and so R is an ART ring. □

Definition 2.2 *An element $g \in R$ (with a multiplicative identity) is called a unit regular element if $\exists$ $b \in U_n(R)$ such that $g = gbg$. A ring R (with unity) is termed unit regular if each member in R is unit regular.*

For clarity and better understanding of the proposition below, we can state a U-nil clean ring as the one in which each of its elements is a sum of a unit regular and a nilpotent element. An element $r \in R$ is called special $r-$clean if there is a regular element (in the von Neumann sense) g and a tripotent t such that $r = g + t$. A ring R is called a special $r-$clean ring if every element of R is special $r-$clean. A ring R is said to be 2-clean if every element of it can be expressed as the sum of 2 units and an idempotent element.

Proposition 2.7 *Let R be a commutative ART ring with no nontrivial zero divisors, and R be a U-nil clean ring. Then*
(i) R is special r-clean ring.
(ii) R is a 2-clean ring.

Proof: (i) : To show that R is a special r-clean ring, let $y \in R$. Since R is an ART, $\exists$ an ART decomposition

$$y = g + t + p,$$

where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$. We consider two cases:

Case (a): If $p = 0$, then $y = g + t$, which is already in the desired form of a sum of a regular and a tripotent element. Thus, y is special r-clean.

Case (b): If $p \neq 0$, then $y = (g + p) + t$. Since R is U-nil clean, then by [7, Proposition 2.2], the sum $g + p$ of a regular and a nilpotent element remains regular. Hence, y is again the sum of a regular and a tripotent element.

Therefore, every element of R has a special r-clean decomposition.

(ii) : To show that ring R is 2-clean, let $y \in R$. Since R is ART, we write

$$y = g + t + p,$$

where $g \in \mathrm{VR_g}(R)$, $t \in \mathrm{T_p}(R)$, and $p \in \mathrm{N_p}(R)$. As R is an integral domain (due to the absence of nontrivial zero divisors), the only tripotent elements in R are $0, 1, -1$, and every regular element is either 0 or a unit.

We now consider different cases:

Case (a): $g = 0$, $t = 0$. Then $y = p$. Write

$$y = 1 + (-1) + p,$$

where $1, -1 \in U_n(R)$. Since p is nilpotent, $1 - p$ is a unit, and hence $p - 1 = -(1 - p) \in U_n(R)$. Thus,

$$y = 1 + (p - 1) + 0,$$

is a sum of two units and an idempotent.

Case (b): $g = 0$, $t = 1$. Then $y = 1 + p$. Again, we can write

$$y = 1 + 1 + (p - 1),$$

where $1 \in U_n(R)$ and $p - 1 \in U_n(R)$. Thus, y is 2-clean.

Case (c): g is a unit, $t = 0$. Then $y = g + p = 1 + g + (p - 1)$. Since $g \in U_n(R)$, and $p - 1 \in U_n(R)$, we have

$$y = u_1 + u_2 + 1,$$

where $u_1 = g$, $u_2 = p - 1$, and 1 is idempotent.

Case (d): $g \in U_n(R)$, $t = 1$. Then

$$y = g + 1 + p = (1 + g) + (p - 1) + 1.$$

Since $1 + g \in U_n(R)$, $p - 1 \in U_n(R)$, and 1 is idempotent, y has a 2 - clean decomposition.

In each case, y has a 2 - clean decomposition. Therefore, R is 2-clean.

□

Let P and Q be rings and U be a $Q - P$ bi-module. From [2, Theorem 16(3)], it is known that if the formal lower triangular ring $\begin{pmatrix} P & O \\ U & Q \end{pmatrix}$ is $r-$ clean, then each of P and Q are individually must be $r-$ clean. According to the result of N. Ashrafi and E. Nasibi [3], if either both the rings both P and Q are r-clean or exactly one is r - clean and the other is merely clean, then the formal triangular matrix ring $\begin{pmatrix} P & O \\ U & Q \end{pmatrix}$ is $r-$ clean. In the following proposition 2.11 we impose certain conditions on the rings P and Q that guarantees the lower triangular ring is an ART ring. This conclusion directly follows from [3, Theorem 2.14], hence the proof is omitted here for brevity. For clarity we include the definition of unit-regular rings as presented in [9].

Proposition 2.8 *The following statements holds:*
(i) : *Let P and Q be rings. Suppose U is a Q–P bimodule and V is a P–Q bimodule. Define the Morita context ring*

$$M = \begin{pmatrix} P & U \\ V & Q \end{pmatrix}$$

with zero pairings. Then M is an ART ring if and only if both P and Q are ART rings.
(ii) :*The formal triangular ring*

$$T = \begin{pmatrix} P & U \\ 0 & Q \end{pmatrix}$$

is an ART ring if and only if rings P and Q are ART.
(iii) :*Let $\alpha \in \mathcal{C}_{\mathcal{T}}(R)$. Then $\alpha^2 R \alpha^2$ and $(1-\alpha^2)R(1-\alpha^2)$ are ART rings if and only if R is an ART ring.*
(iv) : *Let R be a ring, and suppose $\alpha_1, \alpha_2, \ldots, \alpha_m$ are central tripotents satisfying $\alpha_i^2\alpha_j^2 = 0$, for all $i \neq j$. Then $\alpha_1^2 R\alpha_1^2, \alpha_2^2 R\alpha_2^2, \ldots, \alpha_m^2 R\alpha_m^2$ are ART rings if and only if R is an ART ring.*
(v) : *The ring M of all $k \times k$ upper (respectively, lower) triangular matrices over R is an ART ring if and only if ring R itself is an ART.*

Proof: (i) : $(\longleftarrow)$ Let $S = \begin{pmatrix} p_1 & u \\ v & q_1 \end{pmatrix} \in M$. As $p_1 \in P, q_1 \in Q$ are ART, then we can have $p_1 = g_1 + t_1 + p_1$, $q_= g_2 + t_2 + p_2$. Then $S = G_1 + T_1 + P_1$, where $G_1 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, $T_1 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ and $P_1 = \begin{pmatrix} p_1 & u \\ v & p_2 \end{pmatrix}$. We can easily prove that $G_1 \in \mathrm{VR_g}(M), T_1 \in \mathrm{T_p}(M)$ and $P_1 \in \mathrm{N_p}(M)$. Hence, M is an ART.

Conversely, it is obvious that $R \cong M/J$, where, $J = \{\begin{pmatrix} 0 & p \\ v & q \end{pmatrix} : q \in Q, p \in P$ and $v \in V\}$. Therefore, by Proposition 2.3, P is an ART ring. Similarly, we can show that Q is an ART.

(ii) : This immediately follows from the part (i) by considering the special case where $V = 0$. Then T becomes a formal triangular ring, and the condition reduces accordingly.

(iii) : If R is an ART ring and $\alpha \in R$ is a central tripotent, then both $\alpha^2 R\alpha^2$ and $(1 - \alpha^2)R(1 - \alpha^2)$ are direct summands of R. Since the ART property is preserved under such summands, both subrings are ART. Conversely, if both $\alpha^2 R\alpha^2$ and $(1 - \alpha^2)R(1 - \alpha^2)$ are ART rings, then by the central idempotent decomposition $R = \alpha^2 R\alpha^2 \oplus (1 - \alpha^2)R(1 - \alpha^2)$, R is also ART.

(iv) : This conclusion follows by applying (iii) repeatedly with an induction over the number of orthogobal tripotents.

(v) : This follows from the general result that the upper or lower triangular matrix rings preserve properties such as cleanliness, r-cleanness, and ART decompositions if and only if the base ring R holds those properties, as shown in [4 ,Corollary 2.13(v)]. □

The following example illustrates that an ART element is unique in comparison to an r-precious element, which is defined as the sum of a regular element (in the Von Neumann sense), an idempotent element, and a nilpotent element.

Example 2.1. An ART element which is not an r-precious element. Consider an element $E = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ of the ring $R = M_2(Z)$. It is clear that E has the following ART decomposition,

$$E = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= G_1 + T_1 + P_1$$

where, $G_1 \in \mathrm{VR_g}(M_2(Z))$, $T_1 \in \mathrm{T_p}(M_2(Z))$, and $P_1 \in \mathrm{N_p}(M_2(Z))$.
It is easy to show that E is not an $r-$ precious element as T_1 is not idempotent.

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