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Chromatic Pendant Domination in Graphs

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ABSTRACT: Let $G = (V, E)$ be a undirected, simple, finite graph. A subset S of V is said to be dominating set if for every v in $V - D$ there exist u in D such that u and v are adjacent. A dominating set S in G is said to be a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. We introduce the concept of chromatic pendant dominating set. A subset S of V is said to be chromatic pendant dominating set if S is a pendant dominating set and $\chi(< S >) = \chi(G)$, where $\chi(G)$ is a chromatic number of G. The minimum cardinality of the chromatic pendant dominating set in G is called the chromatic pendant domination number of G, denoted by $\gamma_{cpe}(G)$. We find the chromatic pendant domination number of some standard graphs and characterize the graph for $\gamma_{cpe}(G) = 2$.

Key Words: Dominating set, pendant dominating set, chromatic domination, chromatic pendant domination, chromatic pendant domination number.

Contents

1. Introduction

Let $G = (V, E)$ be any graph with $|V(G)| = n$ and $|E(G)| = m$ edges. Then n, m are respectively called the order and the size of the graph G. For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v and the closed neighborhood of v is the set $N[v]$ containing v and all the vertices u adjacent to v. Let S be any subset of V, then the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \bigcup S$. The minimum and maximum of the degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph G is said to be regular if $\delta(G) = \Delta(G)$. A vertex v of a graph G is called a *cut vertex* if its removal increases the number of components. A *bridge* or *cut edge* of a graph is an edge whose removal increases the number of components. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. The graph containing no cycle is called a tree.

A colouring of G is assignment of colours to the vertices of G such that no two adjacent vertices have the same colour. A chromatic number of G is the minimum number of colours needed for colouring of G and is denote by $\chi(G)$.

The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex. The closed helm CH_n is the graph obtained from helm H_n by joining each pendant vertex to form a cycle. The web graph $W(t, n)$ is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. $W(t, n)$ is the generalized web with t cycles each of order n. A firecracker graph $F_{n,k}$ is formed by concatenation of n $k-$ stars by linking one pendant vertex of a star to pendant vertex of next star. $F_{n,k}$ is isomorphic to centepede graph when $k = 2$. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The minimum cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. For more details about pendant domination refer [5], [6].

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2. Chromatic Pendant Domination In Graphs

Definition 2.1 *Let $G = (V, E)$ be a graph. A subset S of V is said to be chromatic pendant dominating set (or CPD-set) if S is a pendant dominating set and $\chi(< S >) = \chi(G)$, where $\chi(G)$ is a chromatic number of G. The minimum cardinality of the chromatic pendant dominating set in G is called the chromatic pendant domination number of G, denoted by $\gamma_{cpe}(G)$.*

Observation 2.1 *For the non - trivial connected graph G*
(i) *For any graph $\gamma_{cpe}(G) \leq V(G)$*
(ii) *We observe that $2 \leq \gamma_{cpe}(G) \leq n$*
(iii) *Every chromatic pendant dominating set is a pendant dominating set*

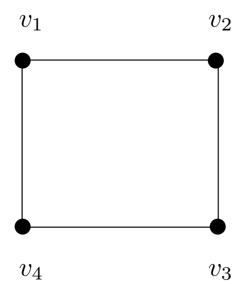

Figure 1. A Cycle Graph with 4 Vertices

Example 2.1 *The chromatic pendant dominating set of the following graph G is $S = \{v_1, v_2\}$ therefore $\gamma_{cpe}(G) =| S |= 2$.*

Theorem 2.1 *Let $G = P_n$ be a path with n vertices. Then*

$$\gamma_{cpe}(P_n) = \begin{cases} \frac{n}{3} + 1, & \text{if } n \equiv 0 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof: Let G be a path with n vertices and $V(G) = \{v_1, v_2, ..., v_n\}$. Clearly $\chi(P_n) = 2$. Let S be a minimal pendant dominating set of G. Then $\chi(< S >) = 2$. Therefore S is a minimal chromatic pendant dominating set of G. Therefore $\gamma_{cpe}(G) =| D |= \gamma_{pe}(G)$.

Theorem 2.2 *If $K_{m,n}$ be a complete bipartite graph with $m + n$ vertices. Then $\gamma_{cpe}(K_{m,n}) = 2$ for all $m, n \geq 2$.*

Proof: Let $K_{m,n}$ be a complete graph with $m + n$ vertices and $V(K_{m,n}) = \{v_1, v_2, ..., v_m, u_1, u_2, ..., u_n\}$. Clearly $\chi(K_{m,n}) = 2$.
Let $S = \{\{v_i, v_j\}/1 \leq i \leq m, 1 \leq j \leq n\}$. The $< S >= P_2$. Clearly S is a pendant dominating set and $\chi(< S >) = 2 = \chi(K_{m,n})$. Hence S is a chromatic pendant dominating set. Therefore $\chi(K_{m,n}) \leq 2$.
Suppose that if $\gamma_{cpe}(K_{m,n}) < 2$, then the set S is not a pendant dominating, which is contradiction. Therefore $\gamma_{cpe}(K_{m,n}) = 2$.

Theorem 2.3 *Let $K_{1,n}$ be a star graph with $n + 1$ vertices. Then $\gamma_{cpe}(K_{1,n}) = 2$.*

Proof: Let G be a star graph with $n + 1$ vertices. $\gamma_{pe}(K_{1,n}) = 2$ and $\chi(K_{1,n}) = 2$. Let v be a vertex of $K_{1,n}$ with $deg(v) = n - 1$. Then, the vertex v dominates all the other vertices of G. Let $x \in V(K_{1,n}) - \{u\}$ such that $ux \in E(K_{1,n})$. Let the set $D = \{v, x\}$ will be the pendant dominating set of G. Then $\chi(< D >) = 2 = \chi(K_{1,n})$. D is a chromatic pendant dominating set. Therefore $\gamma_{cpe}(K_{1,n}) =| D |= 2$.

Theorem 2.4 *Let $D_{r,s}$ be a double star graph with $r+s$ vertices. Then $\gamma_{cpe}(D_{r,s}) = 2$, for all $r, s \geq 2$.*

Proof: Let $D_{r,s}$ be a double star graph with $r+s$ vertices and $V(D_{r.s}) = \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_s\}$. Clearly $\chi(D_{r,s}) = 2$. Let $S = \{v_1, u_1\}$, since $deg(v_1) = r, deg(u_1) = s$. Then the induced subgraph of S is P_2. Clearly S is a minimum pendant dominating set and $\chi(< S >) = 2 = \chi(D_{r,s})$. Hence $\gamma_{cpe}(D_{r,s}) = 2$.

Theorem 2.5 *Let G be a Generalised butterfly graph, then $\gamma_{cpe}(G) = 4$.*

Proof: Let G be a Generalised butterfly graph with vertex set $V(G) = \{v, v_1, v_2, ..., v_n, u_1, u_2, ..., u_m\}$. Let v be the central vertex with $deg(v) = 2mn$, $\gamma_{pe}(G) = 2$ and $\chi(G) = 3$. Let $v_1, v_2 \in V(G)$ such that $v_1v_2 \in E(G)$. Let $S = \{v, v_1, v_2\}$. The induced subgraph of S forms a triangle. So for pendant vertex in the induced subgraph of S we add any one vertex form $u_i, 1 \leq i \leq m$. Then $S = \{v, v_1, v_2\} \cup \{u_1\}$ and $\chi(< S >) = \chi(G)$. Therefore S is a chromatic pendant dominating set of G. Therefore $\gamma_{cpe}(G) \leq\mid S \mid= 4$.

Theorem 2.6 *Any superset of a chromatic pendant dominating set is a chromatic pendant dominating set of a graph G.*

Proof: Let S be a chromatic pendant dominating set of the connected graph G. Then by definition of a dominating set any vertex in $V - S$ is adjacent to at least one vertex in S. If any of vertex in $V - S$ is included in S then let S_1 be the new vertex set obtained by $S \cup \{v\} = S_1$ where $v \in V - S$. Since S_1 is a chromatic pendant dominating set and any vertex in $V - S$ is adjacent to a vertex in S_1, it follows that S_1 is also chromatic pendant dominating set.

Theorem 2.7 *For a complete bipartite graph, $\gamma_{cpe}(G) = diam(G)$.*

Proof: Let G be a complete bipartite graph with $m + n$ vertices. Let S be the chromatic pendant dominating set of G with $\mid S \mid= 2$ and the diameter of complete bipartite graph is two. Therefore $\gamma_{cpe}(G) = diam(G)$.

Theorem 2.8 *For a complete bipartite graph, $\gamma_{cpe}(G) < \delta(G) + 1$.*

Proof: Let G be a complete bipartite graph with $m + n$ vertices. Let $v_1 \in V(G)$ and $deg(v_1) = \delta(G)$. Since $diam(G) = 2$, then the set $S = \{v_1, u_1\}$ is a pendant dominating set for G and $\mid S \mid= \delta(G)$ with $\chi(G) = \chi(< S >) = 2$. Hence $\gamma_{cpe}(G) < \delta(G) + 1$.

Theorem 2.9 *If G is a graph of order n with no isolated vertex then $\gamma_{cpe}(G) \geq \frac{n}{\Delta(G)}$.*

Proof: Let S be a $\gamma_{cpe}(G)$ set. Each vertex is chromatic pendant dominated by the set S, every vertex belongs to the open neighbourhood of minimum one vertex in S. Hence,

$$\begin{aligned}
V(G) &= \cup_{v \in S} N_G(v) \\
\mid V(G) \mid &=\mid \cup_{v \in S} N_G(v) \mid \\
n &=\mid \cup_{v \in S} N_G(v) \mid \\
&\leq \sum_{v \in S} \mid N_G(v) \mid \\
n &=\mid S \mid .\Delta(G) \\
\frac{n}{\Delta(G)} &\leq\mid S \mid \\
\gamma_{cpe}(G) &\geq \frac{n}{\Delta(G)}
\end{aligned}$$

Theorem 2.10 *For any graph G, $\gamma_{cpe}(G) = 2$ if and only if G belongs to the family A.*

Proof: Let G belongs to the family A. Let $S = \{u, v\}$. Since $< S >= K_2$, S is a pendant dominating set and $\chi < S >= 2 = \chi(G)$. Then S is a chromatic pendant dominating set of G. Hence $\gamma_{cpe}(G) = 2$. Conversely. Let G be a chromatic pendant dominating graph with $\gamma_{cpe}(G) = 2$. Let $S = \{u, v\}$ be a

$\gamma_{cpe}-$set of G. Let $N(u) = \{v, u_1, u_2, ..., u_n\}$ and $N(v) = \{u, v_1, v_2, ..., v_m\}$ then $V(G) = N(u) \cup N(v)$. Otherwise if there exist a vertex z in $V(G) - \big(N(u) - N(v)\big)$ then the vertex z can be dominated by no vertex in S. Also no two vertices v, z in $V(u)$ are adjacent to each other. Otherwise $V(< u, v, z >)$ forms a triangle. This form a contradiction that $\gamma_{cpe}(G) \geq 3$. Similarly, no two vertices in $N(v)$ are adjacent to each other. But the vertices in $N(u) - \{v\}$ may be adjacent to the vertex in $N(v)$. The above discussion shows that G belongs to the family A

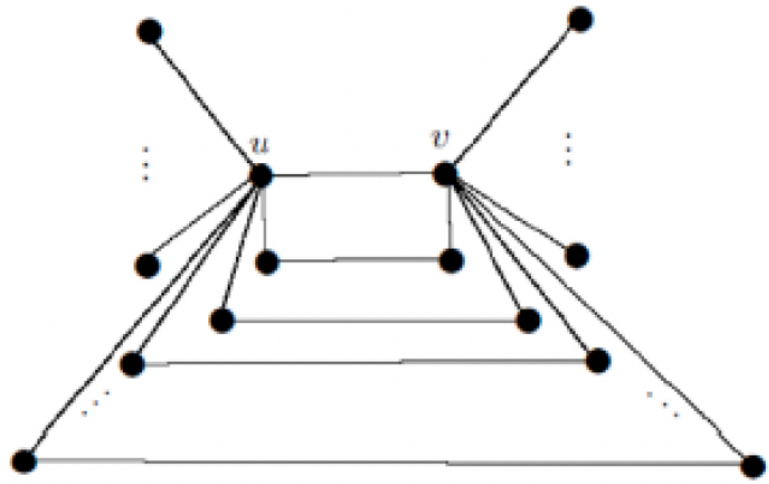

Figure 1: Graph in Family A

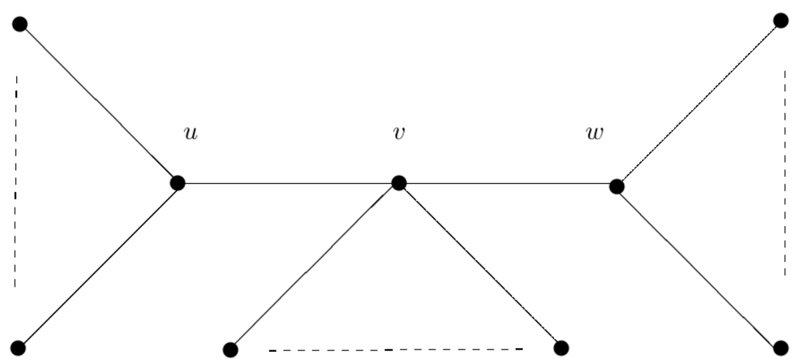

Figure 2: A Graph of Family τ

Theorem 2.11 *For any tree T, $\gamma_{cpe}(T) = 3$ if and only if T belongs to the family τ.*

Proof: Let T be a tree belongs to τ. Let $S = \{u, v, w\}$. Since $< S >= P_3$ and dominates all the vertices of T, the set S is a pendant dominating set of T and $\chi(< S >) = 2 = \chi(T)$. Then S is a chromatic pendant dominating set of T. Hence, $\gamma_{cpe}(T) = 3$.
Conversely, let T be a tree such that $\gamma_{cpe}(G) = 3$. Let $S = \{u, v, w\}$ be a $\gamma_{cpe}-$ set of T. Then by definition the induced subgraph of S contains a pendant vertex and $\chi(< S >) = \chi(T) = 2$. Hence there exists exactly 2 edges in $< S >$. Without loss of generality, let us assume that $uv \in E(G)$; $vu \in E(T)$ and $uw \notin E(T)$. Hence $< S >= P_3$. Let $N(u) = \{v, v_1, v_2, ..., v_r\}$; $N(v) = \{u, w, v_1, v_2, ..., v_s\}$ and $N(w) = \{v, v_1, v_2, ..., v_t\}$ where r, s and t are the non negative integers. Then its clear that no two vertices in $N(u)$ are adjacent to each other. otherwise $G(< u, v, w >)$ form a triangle. This implies the contradiction of T is a tree. Similarly, no two vertices in $N(v)$ and $N(u)$ are adjacent to each other. Its

clear that $V(T) = N(u) \cup N(v) \cup N(w)$. Otherwise if there exists a vertex x in $V(T) - (N(u) \cup N(v) \cup N(w))$ then x can be dominated no one vertex of S. Let x, y be two vertices of u, v and w. If $xy \in E(T)$ then T contains the cycle C_4. Which is contradiction to T is a tree. Therefore $xy \notin E(T)$. The above discussion shows that T belongs to the family τ.

3. Conclusion

Nowadays, the study of theory of domination is an interesting area in Graph theory and also remarkable research is going on in this area. In recent years many scholars are working in this area and also they are introducing new domination parameters. In this way, we are now introduced a new domination invariant called chromatic pendant domination. The investigation of this parameter has begun with this article. We have computing the exact values for few standard family graphs and we have determined few bounds for this parameter in terms order, degree etc.

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