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Viscosity Variation With Velocity Slip Effect Using Couple Stress Fluid Between Rough Conical Bearings

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ABSTRACT: Viscosity variation and velocity slip between porous rough conical bearings with couple stress fluid is investigated. Darcy's law is employed to describe fluid flow through a porous medium while the Stokes couple stress fluid theory is utilized to account for couple stress effects in the lubricant. This work examines the influence of viscosity variation along with the impact of velocity slip at the boundary. The analytical derivation of the modified Reynolds equation that account for roughness is derived using a stochastic random variable with non-zero mean, variance and skewness. This study numerically examined the impact of couple stress fluid, viscosity variation with slip velocity for pressure, load carrying capacity and time leading to significant improvements in bearings characteristics when roughness effects and slip velocity are present. It was concluded that fluids with micro-structure additives are significantly apparent and the inclusion of couple stress fluid effects enhance the lubrication performance.

Key Words: Viscosity variation, couple stress fluid, velocity slip, roughness, porous.

Contents

1. Introduction

In hydrodynamic lubrication theory, the assumptions of constant viscosity and no-slip boundary conditions often lead to limitations when dealing with practical applications involving micro structured fluids or complex surfaces. To capture this behaviour viscosity variation models such as the Barus law [1] are frequently introduced reflecting the exponential dependence of viscosity on operating conditions. When viscosity variation and slip velocity are integrated into the Stokes couple stress framework [2] the lubrication mechanism undergoes significant modifications resulting in altered pressure fields, load support and stability characteristics. Such considerations are vital in the analysis of advanced hydrodynamic bearings, squeeze-film systems and porous media flows where higher performance and reliability are required. In recent years, the velocity slip with viscosity variation phenomenon has many industrial applications. Numerous theoretical analyses have been undertaken to investigate the impact of velocity slip in relation to viscosity variation using couple stress lubrication [3,4].Vinutha et.al [5] revealed that as slip velocity parameter rise, there is a corresponding increase in workload that enhances fluid flow. This enhancement effectively counteracts the negative impacts of surface roughness by facilitating smoother lubrication between surfaces thus improving the load-bearing capacity of the bearing system. Salma et al. [6] focused on curved circular and rough porous flat plates considering viscosity variation in the context of lubrication with a conducting couple stress fluid and magnetohydrodynamics (MHD). The roughness characterized by surface asperities allows for the support of substantial loads and improves the duration of squeezing. Notably, the azimuthal roughness pattern significantly influences both load-carrying capability and squeezing time more than the radial roughness pattern. Tyrone Dass et al. [7] investigated the characteristic of journal bearing's load-carrying capacity, attitude angle, and friction parameter were all significantly impacted by the piezo-viscosity parameter. Reaching maximum pressure, reducing friction, and increasing the journal bearings load-carrying capability all depended on the slip condition. Umadevi

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Devani et al. [8] conducted a study on a curved circular plate with slip velocity utilizing the Simpson's 1/3 rule for numerical analysis. Their research reveals that an increase in permeability corresponds to a decrease in pressure distribution. The results show that couple stresses and magnetic effects play a crucial role in the performance of bearings. The findings highlight the significant role of couple stresses and magnetic effects in influencing bearing performance. The conclusion drawn is that fluids with couple stresses are more effective. The research emphasizes the substantial impact of couple stresses and magnetic effects on bearing performance while higher permeability adversely affects performance. This finding pertains to a wide rectangular plate of viscosity variation, employing couple stress and ferrofluid lubricant as reported by Dass et al. [9]. It was observed that, with a couple stress parameter the parietal shear stresses escalated as the fraction of particles in the base lubricant enhanced. This effect was especially notable in the porous disc with a sealed boundary. In contrast, the side leakage flow for the unsealed condition diminished over the complete oscillation cycle as reported by Bilal Boussaha et al. [10]. Kousar et al. [11] analyzed the squeeze film between a cylinder and plane plate using a couple-stress lubricant, and reported that slip velocity along with pressure-dependent viscosity enhanced load capacity and squeeze-film time. Extending this approach, Salma and Hanumagowda [12] examined a curved circular plate against a flat porous plate under magnetohydrodynamic (MHD) conditions, and found that variable viscosity combined with slip effects improved film stability and resistance. In a subsequent study, Kousar et al. [13] incorporated surface roughness into a cylinder–plate system and showed that slip velocity and viscosity variation helped counteract roughness-induced performance losses. Similarly, Asha et al. [14] developed a stochastic Reynolds equation for micropolar MHD lubricants between rough parallel plates, where viscosity variation and slip velocity played a central role in maintaining load support under uncertainty. Byeon et al. [15] also emphasized the importance of slip and viscosity effects in rough curved–flat plate systems, demonstrating their positive impact on squeeze-film pressure distribution. Suresha et al. [16] reported that the combined presence of slip velocity, viscosity variation, and MHD effects enhanced film strength while reducing friction losses in cylinder–rough plate geometries. Kempepatil et al. [17] investigated annular circular plates and observed that slip and viscosity variation significantly influenced squeeze-film response, particularly under couple-stress conditions. Vinutha et al. [18] confirmed similar improvements in lubrication performance for long cylinder–rough plate systems, noting higher pressure peaks and improved stability when slip velocity and viscosity variation were accounted for. Dass et al. [19] further broadened the scope by integrating slip velocity, variable viscosity, magnetic fluid, and surface texturing into journal bearing analysis, concluding that these parameters cooperatively enhanced load capacity and friction reduction. Finally, Patil et al. [20] studied curved circular and flat plates with non-Newtonian lubricants, and established that slip boundary conditions coupled with viscosity variation significantly modified pressure profiles and improved overall squeeze-film efficiency. Ramesh Kempepatil et al. [21] explored the characteristics of conical bearings. Their numerical analysis indicates that the presence of MHD couple stress improves pressure distribution, load carrying capacity and the duration of the squeeze film. Additionally, a decrease in these attributes was noted with higher slip velocity and the cone angle. The research conducted by Vadher et al. [22] the findings suggest that the efficiency of the bearing system improves when utilizing a conventional lubricant. This article demonstrates that the adverse impact caused by porosity and standard deviation can be entirely mitigated by the beneficial influence of the magnetization parameter and conductivities, provided that an appropriate aspect ratio and semi-vertical angle are selected in scenarios involving negatively skewed roughness in truncated conical plates. Thus, the maximum pressure is considerably increased as a result of the influence of slip-velocity and the magnetic parameter. However, the percentage increase is slightly less pronounced when both effects are considered together than when only slip-velocity is present. The study conducted by Sharma et al. [22] investigated the performance of a double-layer conical porous hybrid journal bearing system using a non-Newtonian lubricant governed by a cubic law model. The analysis emphasizes how parameters such as semi-cone angle, permeability, and lubricant nonlinearity affect the overall bearing behaviour. Results indicate that the presence of a double porous layer improves stability margins, enhances load carrying capacity, and reduces frictional losses when compared with conventional single-layer configurations. The use of a non-Newtonian lubricant further contributes to performance improvement by offering higher film pressures and better damping characteristics. The findings highlight the importance of porous structures and non-Newtonian effects in the design of advanced conical bearings

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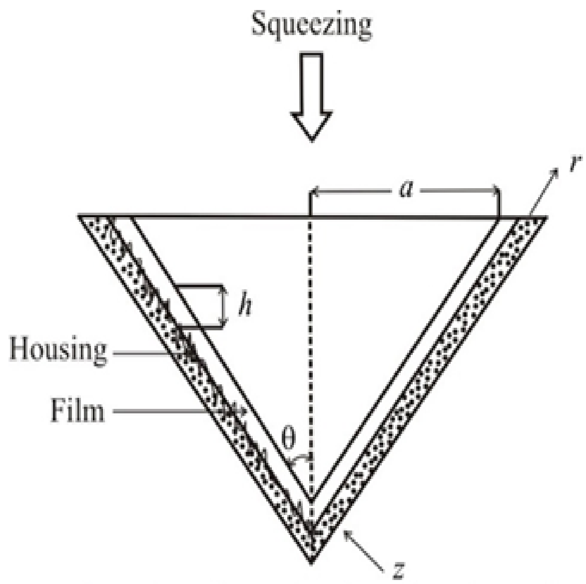

Figure 1: Geometrical representation of the conical bearing

for achieving superior operating efficiency and reliability. The literature review indicates that no studies have been conducted using conical bearings a rough porous surface lubricated by couple stress lubricants and viscosity variation in the presence of velocity slip. Therefore, this article analyses the combined effects of all these factors.

2. Mathematical Formulation

Figure 1 depicts the integration of a non-Newtonian couple stress fluid with a velocity slip field present in the conical bearing. Thickness of the fluid film situated between the two conical bearings is represented by the variable h. The uniform squeezing velocity is expressed as $w = \sin\theta \left(\frac{\partial h}{\partial t}\right)$.

$$\mu \frac{\partial^2 u}{\partial z^2} - \eta \frac{\partial^4 u}{\partial z^4} = \frac{\partial p}{\partial r} \tag{2.1}$$

$$\frac{\partial p}{\partial z} = 0 \tag{2.2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \tag{2.3}$$

As stated in Naduvinamani [24] the flow of a couple stress fluid in a porous medium is regulated by Darcy's law, which takes polar effects into consideration.

$$u^* = \frac{-k}{\mu(1-\beta)} \frac{\partial p^*}{\partial r^*} \tag{2.4}$$

$$w^* = \frac{-k}{\mu(1-\beta)} \frac{\partial p^*}{\partial z^*} \tag{2.5}$$

$$\frac{\partial u^*}{\partial r^*} + \frac{\partial w^*}{\partial z^*} = 0 \tag{2.6}$$

The boundary conditions are at $z=0$

$$\frac{\alpha}{\sqrt{k}}\left(u-u^*\right)=\frac{\partial u}{\partial z}, w=w^*, \frac{\partial^2 u}{\partial z^2}=0 \tag{2.7}$$

The boundary conditions are at $z=h\sin\theta$

$$u=0, \frac{\partial^2 u}{\partial z^2}=0, w=\sin\theta\frac{\partial h}{\partial t} \tag{2.8}$$

$$u=-\frac{h_o^2}{\mu_0 e^{\beta p}}\frac{\partial p}{\partial r}\left\{\begin{array}{l}(z-h\sin\theta)\left[z+\frac{1}{3}h\sin\theta\xi_1-2\xi_2 l\tanh\left(\frac{h\sin\theta}{2l}\right)\right]+\\ 2l^2\left[1-\frac{\cosh\frac{(2z-h\sin\theta)}{2l}}{\cosh\left(\frac{h\sin\theta}{2l}\right)}\right]\end{array}\right\} \tag{2.9}$$

where

$$\chi_2=\frac{3}{(1-\beta)}\left[\frac{2s^2\alpha^2}{\left(h^2\sin^2\theta+sh\sin\theta\right)}+\frac{(1-\beta)\,s}{(h\sin\theta+s)}\right], \chi_1=\frac{s}{h\sin\theta+s}$$

To obtain the equation for pressure, we first substitute equation 2.9 into equation 2.3 and then carry out the integration. The inclusion of conditions 2.7 and 2.8 allows us to finalize the equation.

$$\frac{\partial}{\partial r}\left\{\left[k\left(h,\chi_1,\chi_2,l\right)e^{-\beta p}+\frac{12k\delta}{(1-\beta)}\right]r\frac{\partial p}{\partial r}\right\}=12r\frac{\partial h}{\partial t}\sin\theta \tag{2.10}$$

where

$$k\left(h,\chi_1,\chi_2,l\right)=\left(h\sin\theta\right)^3\left(1+\chi_2\right)-6(h\sin\theta)^2 l\chi_1\tanh\left(\frac{h\sin\theta}{2l}\right)-12l^2\left(h\sin\theta-2tanh\left(\frac{h\sin\theta}{2l}\right)\right) \tag{2.11}$$

The representation of film thickness in relation to longitudinal one-dimensional roughness follows the approach outlined by Naduvinamani [24]

$$H=h\left(t\right)+h_s\left(x,\xi\right) \tag{2.12}$$

$$k(h_s)=\begin{cases}\frac{35}{32b^7}\left(b^2-h_s^2\right)^3, & -b\le h_s\le b,\\ 0, & \text{elsewhere.}\end{cases} \tag{2.13}$$

$$\alpha^*=E\left(h_s\right) \tag{2.14}$$

$$\sigma^*=E\left[\left(h_s-\alpha^*\right)^2\right] \tag{2.15}$$

$$\varepsilon^*=E\left[\left(h_s-\alpha^*\right)^3\right] \tag{2.16}$$

$$E\left(.\right)=\int_{-\infty}^{\infty}(.)k\left(h_s\right)dh_s \tag{2.17}$$

In equation (10) and (12) the dimensionless form is obtained as

$$\frac{\partial}{\partial r^*}\left\{e^{-Gp^*}k\left(H^*,\chi_1,\chi_2,l^*,\alpha^*,\varepsilon^*,\sigma^*,\psi,\theta\right)r^*\frac{\partial p^*}{\partial r^*}\right\}=-12r^* \tag{2.18}$$

The pressure boundary conditions are:

$$\frac{dp}{dr} = 0 \quad \text{at} \quad r = 0 \tag{2.19}$$

$$p = 0 \quad \text{at} \quad r = \cos\theta \tag{2.20}$$

using the non- dimensional quantities

$$l^* = \frac{l}{h_o}, r^* = \frac{r}{h_o}, G = \frac{\beta\mu_o a^2\left(-\frac{\partial h}{\partial t}\right)}{h_0^3}, p^* = \frac{ph_o^3}{R^2\mu_0\left(-\partial h/\partial t\right)}, \psi = \frac{k\delta}{h_0^3}, s = \frac{\sigma^*}{h_o}, \sigma^* = \frac{\sqrt{k}}{\alpha}, H^* = \frac{h}{h_o}$$

$$\chi_2 = \frac{3}{(1-\beta)}\left[\frac{2s^2\alpha^{*2}}{\left((H^*\sin\theta)^2 + sH^*\sin\theta\right)} + \frac{(1-\beta)\,s}{(h\sin\theta + s)}\right], \chi_1 = \frac{s}{H^*\sin\theta + s}$$

$$k\left(H^*, \chi_1, \chi_2, l^*, \alpha^*, \varepsilon^*, \sigma^*, \psi, \theta\right) = (1+\chi_2)\,n_1 - 6l^*(H^*\sin\theta)^2\chi_1 n_2 n_3 - 12^* l^2\left(H^*\sin\theta + \alpha^*\right) + n_4 + \frac{12\psi}{1-\beta}$$

$$n_1 = (H^*\sin\theta)^3 + 3(H^*\sin\theta)^2\alpha^* + 3H^*\sin\theta\left(\sigma^{*2} + \alpha^{*2}\right) + \varepsilon^* + 3\alpha^*\sigma^{*2} + \alpha^{*3}$$

$$n_2 = (H^*\sin\theta)^2 + \sigma^{*2} + \alpha^{*2} + 2H^*\sin\theta\alpha^*$$

$$n_3 = \tanh\left(\frac{H^*\sin\theta}{l^*}\right) + \left(1 - \tanh^2\left(\frac{H^*\sin\theta}{2l}\right)\right)\frac{1}{24l^3}\left(12l^2\alpha^* - \varepsilon^* - 3\alpha^*\sigma^{*2} - \alpha^{*3}\right)$$

$$n_4 = 24l^3\tanh\left(\frac{H^*\sin\theta}{l^*}\right) + \left(1 - \tanh^2\left(\frac{H^*\sin\theta}{2l}\right)\right)\left(12l^2\alpha^* - \varepsilon^* - 3\alpha^*\sigma^{*2} - \alpha^{*3}\right)$$

The pressure boundary conditions are:

$$\frac{dp^*}{dr^*} = 0 \quad \text{at} \quad r^* = 0 \tag{2.21}$$

$$p^* = 0 \quad \text{at} \quad r^* = 1 \tag{2.22}$$

By performing an integration of the nondimensional modified lubrication equation in relation to r^* and incorporating the pressure boundary conditions

$$p^* = -\frac{1}{G}\ln\left\{\frac{3G(r^{*2}-1)}{k\left(H^*, \chi_1, \chi_2, l^*, \alpha^*, \varepsilon^*, \sigma^*, \psi, \theta\right)} + 1\right\} \tag{2.23}$$

The load carrying capacity equation in dimensionless form is acquired as

$$w^* = \left\{wh_0^3/\mu_o a^4\left(-dh/dt\right)\csc^2\theta\right\} = \frac{2\pi}{G}\int_0^1 \ln\left\{\frac{3G(r^{*2}-1)}{k\left(H^*, \chi_1, \chi_2, l^*, \alpha^*, \varepsilon^*, \sigma^*, \psi, \theta\right)} + 1\right\} r^* dr^* \tag{2.24}$$

Thesqueeze film time equation in dimensionless form is acquired as

$$T^* = \left\{wh_0^2 dt/\mu_o a^4\csc^2\theta\right\} = \frac{2\pi}{G}\int_{h_f}^1\int_0^1 \ln\left\{\frac{3G(r^{*2}-1)}{k\left(H^*, \chi_1, \chi_2, l^*, \alpha^*, \varepsilon^*, \sigma^*, \psi, \theta\right)} + 1\right\} r^* dr^* dH^* \tag{2.25}$$

3. Results and Discussion

In this work, we investigate the effects of roughness and slip velocity on the couple-stress fluid with viscosity variation between conical bearings. We investigate the squeezing film between the conical bearings using the principle of Stokes couple stress fluid theory. The existence of slip velocity is taken into consideration in the theory. Here, we present the results of non-dimensional parameters such as slip velocity, roughness, couple stress, and viscosity variation.

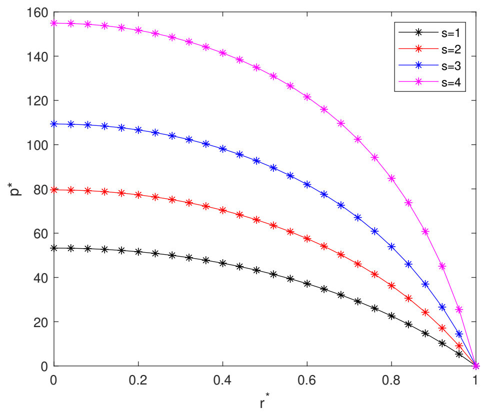

Figure 2: Combined influence of slip parameter s vs r^* with $\alpha^* = 0.05$, $\sigma^* = 0.1$, $\psi = 0.01$, $H^* = 0.6$, $\varepsilon^* = 0.01$, $\beta = 0.3$, $G = 0.01$, $\theta = \frac{\pi}{6}$, $l^* = 0.1$

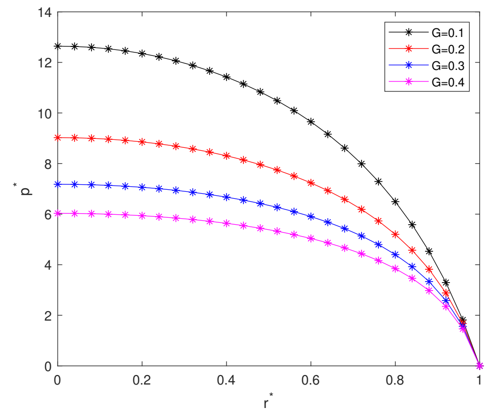

Figure 3: combined influence of viscosity variation G vs r^* with $\alpha^* = 0.05$,$\sigma^* = 0.1$,$\psi = 0.01$,$H^* = 0.6$,$\varepsilon^* = 0.01$,$\beta = 0.3$,$s = 0.1$,$\theta = \frac{\pi}{6}$,$l^* = 0.1$

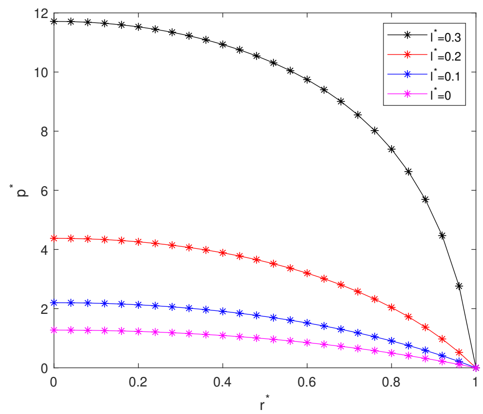

Figure 4: Combined influence of slip parameter s vs r^* with $\alpha^* = 0.05$, $\sigma^* = 0.1$, $\psi = 0.01$, $H^* = 0.6$, $\varepsilon^* = 0.01$, $\beta = 0.3$, $G = 0.01$, $\theta = \frac{\pi}{6}$, $l^* = 0.1$

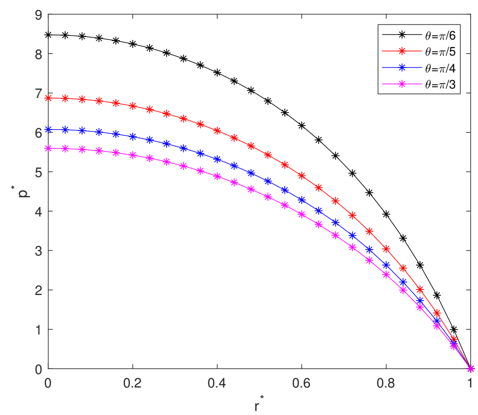

Figure 5: Combined influence of θ vs r^* with $\alpha^* = 0.05$,$\sigma^* = 0.1$,$\psi = 0.01$,$H^* = 0.6$,$\varepsilon^* = 0.01$,$\beta = 0.3$,$s = 0.1$,$l^* = 0.1$, $G = 0.1$

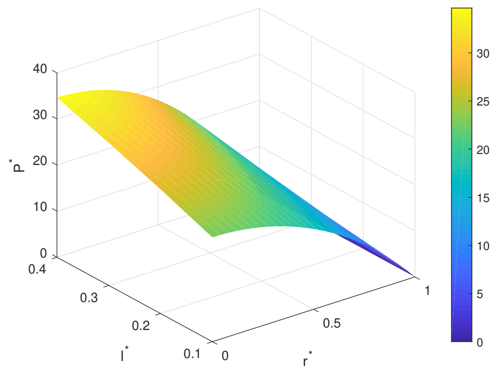

Figure 6: 3D graph for Combined impact of couple stress parameter and permeability parameter on pressure

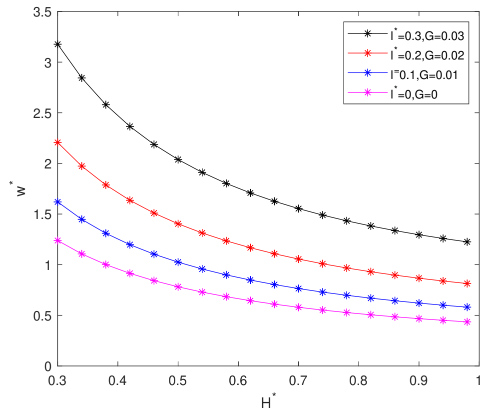

Figure 7: Combined influence of couple stess l^* vs H^* with $\alpha^* = 0.05$,$\sigma^* = 0.1$,$\psi = 0.01$,$\varepsilon^* = 0.01$,$\beta = 0.3$,$s = 0.1$,$l^* = 0.1$

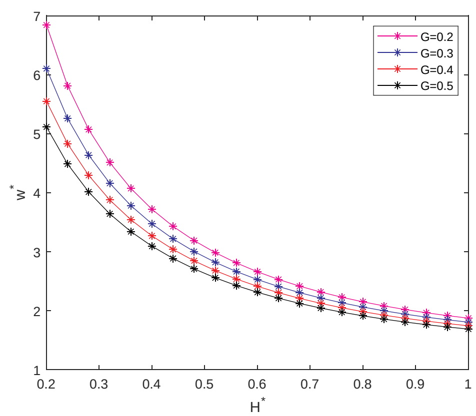

Figure 8: Combined influence of viscosity variation G vs H^* with $\alpha^* = 0.05$,$\sigma^* = 0.1$,$\psi = 0.01$,$\varepsilon^* = 0.01$,$\beta = 0.3$,$s = 0.1$,$l^* = 0.1$

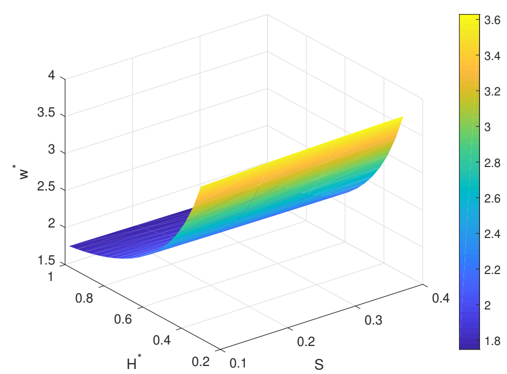

Figure 9: 3D graph for Combined impact of slip parameter and permeability parameter on work load

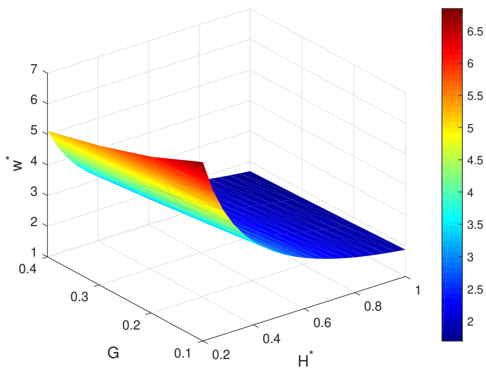

Figure 10: 3D graph for Combined impact of viscosity variation parameter on work load

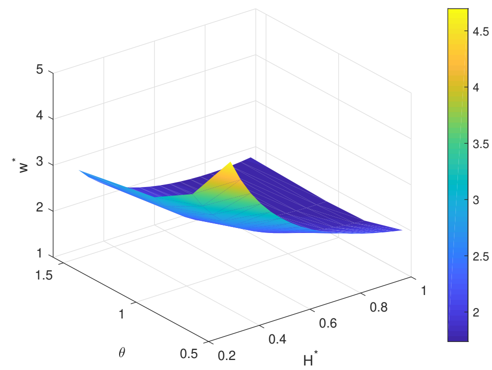

Figure 11: 3D graph for Combined impact of θ on work load

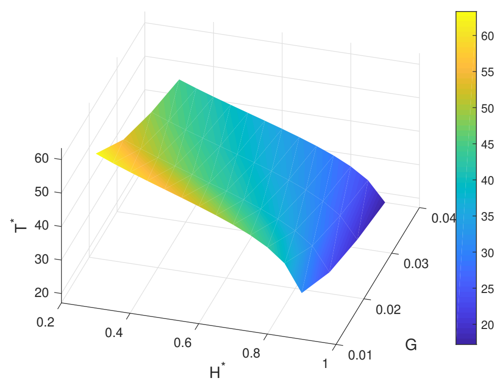

Figure 12: 3D graph for Combined impact of viscosity variation parameter on time

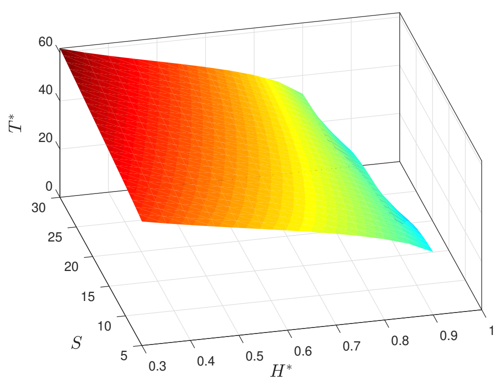

Figure 13: 3D graph for Combined impact of s on time

Figure 2 illustrates the relationship between pressure p^* and radial coordinate r^* for a range of values of the slip parameter while other parameters remain unchanged it is observed that the graph indicates that changes in G and l^* contribute to a rise in the pressure. When the slip parameter rises with couple stress fluid it has a more impact on fluid flow. This often causes a decline in fluid velocity and can change the pressure gradient. Thus, an increase in velocity slip can lead to an elevation in pressure.

Viscosity variation with porosity greatly reduces the bearing systems performance as seen in Figure 3. Furthermore, it is evident that the pressure decreases as the viscosity variation parameter rises. Figure 4 presents the relationship between p^* and r^* for varying values of l^*. The impact of couple stress leads to greater viscosity at elevated shear rates, which may facilitate the maintenance of a thicker lubricating film. The relationship between pressure p^* and radius r^* for various cone angles θ is presented in Figure 5. It is evident that pressure increases with a rise in the cone angle θ. This is due to the fact that a larger cone angle creates more space between the bearings which allows for improved fluid flow better lubrication and reduced wear. The combined influence highlights a competitive behavior while larger values of l^* strengthen the film and elevate pressure, higher permeability counteracts this effect by promoting fluid leakage. These results are consistent with earlier studies, which reported that couple stress fluids improve film thickness and load support, whereas higher permeability tends to reduce the pressure due to leakage losses. Overall, the findings suggest that maintaining a higher couple stress parameter alongside

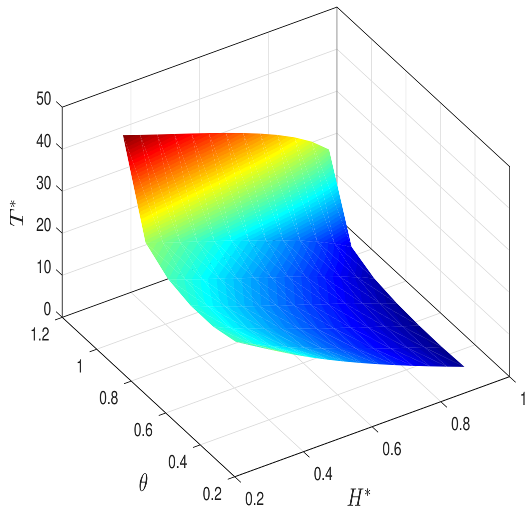

Figure 14: 3D graph for Combined impact of θ on time

lower permeability is crucial for achieving optimum pressure distribution and ensuring stable lubrication performance in conical bearing systems is represented in figure 6.

Figure 7 illustrates that increased values of G and l^*. raises the load carrying capacity within the lubricant film. Although, roughness contributes to improved load support and bearing performance by stabilizing the fluid film between the plates it is important to note that the load carrying capacity of a lubrication system can be greatly improved with a higher viscosity of the lubricant. This enhancement is primarily due to the lubricant's ability to create a thicker and more resilient lubricating film between the moving surfaces. Figure 8 illustrates how the load carrying capacity changes with height for various values of the viscosity variation parameter G. It is noted that the influence of viscosity variation lead to a decrease in load carrying capacity.

Figure 9 illustrates the effect of slip parameter s on load carrying capacity shows a clear increasing trend. At higher slip values, the lubricant adheres more closely to the solid surfaces, which enhances hydrodynamic action and allows the film to sustain greater loads. Figure 10 illustrates the influence of viscosity variation parameter G on the load carrying capacity of a conical bearing lubricated with a couple stress fluid. The results indicate that an increase in G leads to a noticeable reduction in load capacity.Figure 11 presents the influence of cone angle θ on the load carrying capacity in conical bearings operating with a couple stress fluid. The trend indicates that a rise in the cone angle enhances the ability of bearing to sustain load. This improvement can be attributed to the fact that a larger cone angle increases the clearance space within the bearing, which facilitates better lubricant to flow

Figure 12 represents that an increase in slip reduces squeeze film time, as greater slip lowers fluid resistance and accelerates film drainage, whereas smaller slip values sustain the film for a longer duration. Figure 13 indicates thatlLarger cone angles extend squeeze film time by creating more clearance for fluid retention, while smaller angles restrict flow and shorten film duration. Figure 14 illustrate that Higher viscosity variation decreases squeeze film time due to unstable fluid behavior, while lower G values maintain consistent viscosity and prolong the film.

4. Conclusions

The conclusions are drawn from numerical analysis.

- The graphs are used to illustrate how physical parameters influence the velocity slip, viscosity, and roughness characteristics. The results showed that the fluid's velocity is slowed down by viscosity, although the pressure distribution and time profile were improved by the porous parameter.

- The permeability of the porous layer diminishes the efficiency of the squeeze film performance. As the height of the porous layer increases along with variations in viscosity there is a notable decline in the effectiveness of the squeeze film.
- • The utilization couple fluid is to elevate pressure, increase the load-carrying capacity, and extend the squeezing time. This also results in improved performance for conical bearings that operate with the permeability parameter.

Nomenclature

G	Viscosity parameter
δ	Porous layer thickness
E	Expectancy operator
α	Mean of the stochastic film thickness
β	Coefficient of pressure-dependent viscosity
σ	Standard deviation of the film thickness
σ^2	Variance
h	Film thickness $= (h + h_e)$
H^*	Non-dimensional film thickness $= \dfrac{h}{h_0}$
β	Percolation parameter $= \dfrac{\eta}{\mu k}$
l	Couple stress parameter $= \left(\dfrac{\eta}{\mu}\right)^{1/2}$
l^*	Non-dimensional couple stress parameter
η	Material constant characterizing couple stress
ψ	Permeability parameter $= \dfrac{k\delta}{h_0^3}$
μ	Viscosity coefficient
w^*	Non-dimensional load carrying capacity
T^*	Non-dimensional time

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