Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.78376

Fractional Cole Model with Dual Caputo–Weyl Operators for Complex Frequency Impedance Analysis

Harlish Sharma and Anil Kumar Menaria*

ABSTRACT: This paper presents a generalized framework for modeling electrical impedance in the complex frequency domain using fractional derivatives. The approach extends the classical Cole model by incorporating both Caputo and Weyl fractional operators, enabling the representation of systems that exhibit simultaneous transient memory effects and periodic steady-state behavior. The proposed formulation is supported by new theoretical results, including a stability criterion for fractional impedance systems and a convergence property showing reduction to the classical model as the fractional order $\alpha \to 1^-$. Applications to electrochemical impedance spectroscopy (EIS) and bioimpedance analysis demonstrate the practical value of the model.

Key Words: Electrical Impedance, Complex Frequency Domain, Caputo and Weyl Derivative, Electrochemical Impedance Spectroscopy, Bioimpedance.

Contents

1 Introduction **1**

2 Literature Review **2**

3 Preliminaries **3**

4 Main Results **5**

5 Analytical Framework **6**
5.1 Caputo operator via Grünwald–Letnikov scheme 7
5.2 Weyl operator via spectral multiplier 7

6 Applications **7**
6.1 Case I: Simulated electrochemical impedance 7
6.2 Case II: Bioimpedance of human skin 8

7 Conclusion **9**

1. Introduction

Fractional calculus generalizes differentiation and integration to non-integer orders, providing a mathematical framework for modeling processes with memory and spatial non-locality [1,2,3,4]. This property makes it particularly relevant in electrical engineering, where the impedance of materials and biological tissues under alternating current excitation often reflects long-term memory effects, anomalous diffusion, and heterogeneous structure [5,6]. Traditional impedance models, such as the Cole–Cole [7] and Debye formulations, are based on integer order dynamics and cannot fully capture these non ideal behaviors [3,8].

The complex frequency domain offers a natural setting for fractional order impedance modeling, as it allows frequency dispersion and memory effects to be incorporated into analytical expressions. Among various fractional operators, the Caputo derivative is well suited for systems with well defined initial conditions, while the Weyl derivative is advantageous for periodic or steady-periodic signals due to its Fourier-based definition [9,10]. Existing fractional Cole models typically employ a single type of fractional

* Corresponding author.
2010 *Mathematics Subject Classification:* 26A33, 78A57, 93C95, 94C05.
Submitted August 12, 2025. Published September 25, 2025

operator [11,12], which may limit their ability to accurately represent systems influenced by both transient and periodic memory mechanisms.

In this work, we introduce a generalized fractional Cole model that combines Caputo and Weyl derivatives into a single impedance formulation. The Caputo term captures non-local relaxation dynamics, while the Weyl term introduces periodic memory effects that are relevant in many practical contexts, such as electrochemical impedance spectroscopy (EIS) with periodic excitation and bioimpedance measurements affected by physiological oscillations. Theoretical analysis includes a stability theorem for fractional impedance systems and a convergence result showing that the proposed model reduces to the classical Cole form when the Weyl contribution is absent ($b = 0$) or the fractional order approaches one.

The practical benefits of the proposed model are demonstrated through two case studies:) a controlled simulated EIS dataset with known ground truth and experimental bioimpedance measurements of human skin. In both cases, the generalized fractional Cole model achieves improved agreement with data compared to the classical version, with the EIS case showing a mean squared error reduction of 88.85%. These findings suggest that hybrid fractional-operator approaches offer a promising path toward more accurate and physically meaningful impedance modeling across diverse application domains.

2. Literature Review

Electrical impedance modeling in heterogeneous systems, including biological tissues and electrochemical media, has historically relied on integer-order differential equations. The pioneering work of Cole and Cole [7] introduced the concept of dispersion and absorption in dielectrics, which laid the groundwork for the classical Cole model. Although effective in a variety of engineering and biophysical contexts, the Cole model cannot fully capture non-ideal behaviors such as anomalous relaxation, heterogeneous charge transport, and memory effects, which are frequently observed in complex systems [2,3,6].

To address these limitations, fractional calculus has emerged as a powerful analytical tool for describing non-local and memory dependent processes [4,10,17]. Fractional differential operators provide a means to generalize classical dynamic models, enabling better representation of systems that deviate from ideal exponential relaxation or frequency independent phase characteristics. Works by Podlubny [4], Kilbas et al. [10], and Samko et al. [17] established the theoretical foundation for fractional calculus in applied sciences, creating a bridge between mathematical theory and physical modeling.

Some authors demonstrated the utility of fractional derivatives in electrical systems. Gómez-Aguilar et al. [5] employed fractional derivatives with regular kernels to describe electrical circuits, yielding improved accuracy over integer-order formulations. Schäfer and Krüger [6] applied similar methodologies to model magnetic coils, demonstrating that fractional order terms align more closely with experimental magnetic response data. Carcione et al. [8] extended the approach to seismic wave propagation, underscoring its applicability to wave like phenomena beyond electromagnetics.

In the bioimpedance domain, Vosika et al. [2] formulated a fractional-order model for human skin impedance, successfully capturing the dispersive nature of biological tissues. Electrochemical systems have also benefited from fractional modeling, as seen in the work of Nasser-Eddine et al. [3,13] and Arahbi et al. [14], where fractional order elements enhanced both parameter identification and the fidelity of impedance spectra. Experimental comparisons of different fractional operators, such as those by Lin et al. [15], have further clarified operator selection for physical modeling, while Deogan et al. [11] emphasized their relevance to frequency-dispersive electromagnetic media. Zhang et al. [12] and George et al. [16] proposed refined fractional Cole models, incorporating complex order dispersion and advanced parameter estimation strategies. While these models marked a significant step forward, they often employed a single fractional operator type and did not simultaneously address periodic excitations and long-range memory effects.

Recent studies provide further context for fractional and hybrid impedance models such as:

- *Variable–order fractional models:* Beyond fixed orders α, β, variable–order formulations have been explored to capture systems whose memory depth evolves with time or frequency. For example, [18] analyzed approximation schemes for Caputo derivatives with order functions $\alpha(t)$, and applied them to electrical circuits. Similarly, several works in *Fractal and Fractional* (2023–2025) demonstrate that variable–order models achieve superior fits in bioimpedance and battery EIS where relaxation dynamics change across frequency decades. Incorporating such approaches into the dual Caputo–Weyl model is a natural direction for future work.

- *Machine–learning based impedance fitting:* Recent contributions integrate data driven approaches with physical impedance models. For instance, [19] performed a Bayesian assessment of equivalent circuit models for corrosion EIS, showing how posterior uncertainties help identify over parameterization. The Auto EIS tool [20] automates equivalent circuit selection via supervised learning, while [21] employed a data driven Loewner framework for impedance analysis. These techniques complement fractional formulations by serving either as priors for parameter estimation or as benchmarking tools for "goodness–of–fit".

Table 1: Comparison of Existing Impedance Models in Literature

Model / Reference	**Operator Used**	**Strengths**	**Limitations**
Cole-Cole Model [7]	Integer order derivative	Simple, well known in EIS	Cannot model anomalous relaxation
Fractional Cole Model [12]	Caputo derivative	Captures memory effects	No explicit treatment of periodic excitations
Complex Order Model [16]	Complex order derivative	Flexible phase behavior modeling	Increased parameter estimation complexity
Bioimpedance Model [2]	Caputo derivative	Accurate for biological tissues	Limited to transient responses
Proposed Model (This Work)	Caputo + Weyl derivatives	Models transient and periodic memory effects	Requires dual parameter estimation

The comparative analysis above highlights that the proposed model is distinct in its capacity to handle both steady periodic and transient memory dynamics within the same theoretical structure, which is particularly advantageous for real world electrochemical and bioimpedance applications.

The present study introduces a unified impedance model that blends Caputo and Weyl fractional derivatives, capturing both transient memory and periodic steady state effects. This dual operator approach extends the scope of existing models by linking frequency dispersion with periodic excitation, offering a more accurate description of complex systems. The proposed Caputo–Weyl Cole model preserves physical meaning while addressing cases where classical Cole arcs fall short, and it opens pathways for future extensions through variable order kernels or data driven methods to improve parameter estimation and real time fitting.

3. Preliminaries

Fractional calculus extends the classical notions of differentiation and integration to arbitrary, non integer orders. In this work, we consider the following foundational definitions:

- **Riemann-Liouville Fractional Integral:** For a function $f(t)$ and order $\alpha > 0$, the Riemann-Liouville fractional integral is defined by the expression [4]:

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{-\alpha+1}}\, d\tau, \tag{3.1}$$

  where Γ is the Gamma function.

- **Caputo Fractional Derivative:** For $\alpha \in (n-1, n)$, where $n \in \mathbb{N}$, the Caputo fractional derivative is defined as [10]:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \tag{3.2}$$

which is particularly suitable for physical systems with well-defined initial conditions. Throughout, fractional powers use the principal branch: for $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$z^{\gamma} := \exp\big(\gamma \log z\big), \qquad \log z = \ln|z| + \mathrm{j}\, \arg z, \quad \arg z \in (-\pi, \pi).$$

The Laplace transform obeys

$$\mathcal{L}\{D^{\alpha} f(t)\}(s) = s^{\alpha} F(s) - s^{\alpha-1} f(0^{+}), \qquad \Re(s) > 0, \tag{3.3}$$

so that classical initial conditions are enforced through the value $f(0^{+})$.

- **Weyl Fractional Derivative:** For periodic functions, the Weyl fractional derivative of order α is defined through their Fourier series representation as follows [17]:

$$W^{\alpha} f(t) = \sum_{k \neq 0} (ik)^{\alpha} \hat{f}_k \, e^{ikt}, \tag{3.4}$$

where $\hat{f}_k$ are the Fourier coefficients and the constant term is zero.

For order $\beta \in (0, 1]$, the Weyl derivative (steady–periodic modulation) on the whole line is the Fourier multiplier

$$\mathcal{F}\{W^{\beta} f\}(\omega) = (\,\mathrm{j}\omega\,)^{\beta}\, \widehat{f}(\omega), \qquad \omega \in \mathbb{R}, \tag{3.5}$$

with the same principal branch convention for fractional powers.

Hence, for a steady sinusoid

$$f(t) = \Re\{\, \tilde{f}\, e^{\mathrm{j}\omega_0 t}\},$$

we obtain

$$W^{\beta} f(t) = \Re\Big\{ (\,\mathrm{j}\omega_0)^{\beta}\, \tilde{f}\, e^{\mathrm{j}\omega_0 t} \Big\}, \tag{3.6}$$

which scales the amplitude by $|\omega_0|^{\beta}$ and shifts the phase by $\frac{\beta\pi}{2}\, \mathrm{sgn}(\omega_0)$.

- **The Classical Cole Model:** In the complex frequency domain, impedance $Z(s)$ is expressed as a function of the complex variable $s = \sigma + i\omega$. The classical Cole model characterizes how electrical impedance varies with frequency in complex media such as biological tissues, and is given by [7]:

$$Z(s) = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (s\tau)^{\alpha}}, \tag{3.7}$$

where R_0 and R_{∞} denote the resistances at low and high frequencies, respectively, τ is the characteristic time constant, and $\alpha \in (0, 1]$ models the degree of frequency dispersion [12,16].

Remark 3.1 *The imaginary unit is denoted by* j*, and frequency-domain fractional powers are written as* $(\mathrm{j}\omega)^{\gamma}$*. When both operators appear in one model, we label their orders explicitly as* α *(Caputo) and* β *(Weyl).*

Caputo terms encode causal long-memory relaxations with explicit initial conditions, as in (3.3)*. Weyl terms act as fractional differentiators for steady oscillations, as in* (3.5)*.*

Keeping the two operators separate highlights both the physics (drift vs. modulation) and the numerics (time-domain convolution vs. spectral multiplier). In particular, this distinction clarifies extensions of the classical Cole model, where relaxation effects are naturally described by Caputo operators while steady periodic responses are more faithfully captured through Weyl operators.

4. Main Results

We present novel results for modeling impedance using fractional derivatives in the complex frequency domain.

Lemma 4.1 *Consider an impedance function $Z(s)$ governed by a fractional differential equation of order $\alpha \in (0,1)$. The system is deemed stable if all poles of $Z(s)$ are located in the left half of the complex plane, that is, where $Re(s) < 0$.*

Proof: Consider the fractional differential equation governing the impedance:

$$D^{\alpha}v(t) + av(t) = i(t),$$

In this context, $v(t)$ denotes the voltage and $i(t)$represents the current as functions of time, and $a > 0$. Taking the Laplace transform with zero initial conditions:

$$s^{\alpha}V(s) + aV(s) = I(s).$$

The impedance is:

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{s^{\alpha} + a}.$$

The poles are solutions to $s^{\alpha} + a = 0$, or $s = (-a)^{1/\alpha}$. For $\alpha \in (0,1)$, the principal value of $(-a)^{1/\alpha}$ has a negative real part, ensuring stability [3]. □

Theorem 4.1 *The generalized fractional Cole model in the complex frequency domain is given by:*

$$Z(s) = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (s\tau)^{\alpha} + bW^{\beta}(s\tau)},$$

here W^{β} denotes the Weyl fractional derivative of order $\beta \in (0,1)$, while $b \geq 0$ represents a parameter accounting for memory effects. This model reduces to the classical Cole model when $b = 0$.

Proof: We begin from the classical Cole model in the complex frequency domain:

$$Z_{\text{Cole}}(s) = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (s\tau)^{\alpha}},$$

where $s = \sigma + i\omega$, $\alpha \in (0,1]$, and $\tau > 0$. To incorporate periodic memory effects, we perturb the denominator by adding a Weyl fractional derivative term $bW^{\beta}(s\tau)$, with $\beta \in (0,1)$ and $b \geq 0$. The resulting model is:

$$Z(s) = R_{\infty} + \frac{R_0 - R_{\infty}}{1 + (s\tau)^{\alpha} + bW^{\beta}(s\tau)}.$$

In the frequency domain, the Weyl derivative of a periodic function $e^{i\omega t}$ has the property:

$$\mathcal{F}\{W^{\beta}e^{i\omega t}\} = (i\omega)^{\beta}\hat{f}(\omega),$$

where $\hat{f}(\omega)$ is the Fourier coefficient. Hence, the perturbation introduces an additional $(i\omega)^{\beta}$ term in the impedance denominator, modifying the frequency dependent behavior without altering the pole location condition derived in Lemma 4.1:

$$(s\tau)^{\alpha} + b(s\tau)^{\beta} + 1 = 0.$$

Since $\alpha, \beta \in (0,1)$ and $b \geq 0$, the principal values of the roots remain in the left half plane, ensuring system stability. Setting $b = 0$ recovers the classical Cole model, verifying the generalization. □

Theorem 4.2 *The fractional impedance model converges to the measured impedance spectrum as the fractional order $\alpha \to 1^{-}$, provided the system parameters are bounded.*

Proof: Consider the fractional impedance function for $\alpha \in (0,1)$:

$$Z_\alpha(s) = \frac{1}{s^\alpha + a},$$

with $a > 0$. Using the Caputo derivative's convergence property:

$$\lim_{\alpha \to 1^-} D^\alpha f(t) = \frac{d}{dt} f(t),$$

we have:

$$\lim_{\alpha \to 1^-} Z_\alpha(s) = \frac{1}{s + a},$$

which corresponds to the impedance of a standard RC circuit. The convergence is uniform for bounded a, as shown by:

$$\sup_{|s| \le M} \left| \frac{1}{s^\alpha + a} - \frac{1}{s + a} \right| \xrightarrow{\alpha \to 1^-} 0,$$

for any finite $M > 0$. This ensures that the generalized model continuously approaches classical behavior as $\alpha \to 1^-$, matching experimental impedance spectra in the integer order limit. □

For stability criterion for the dual Caputo–Weyl channel, let $Z(s)$ be the dual Cole model defined in Equation (3.7) then the corresponding fractional system is characterized in terms of pole locations.

Remark 4.1 (Sector stability for fractional impedance systems) *Suppose the closed loop denominator can be written in the form*

$$D(s) = D_c\big(s^\alpha\big) + \kappa\, D_w\big(s^\beta\big),$$

with $\alpha, \beta \in (0,1]$ *and* D_c, D_w *Hurwitz polynomials. Then:*

(i) If $\alpha = \beta$*, the system is asymptotically stable provided all roots* λ_k *of* $D_c + \kappa D_w$ *satisfy*

$$\big|\arg(\lambda_k)\big| > \tfrac{\alpha\pi}{2}. \tag{4.1}$$

(ii) If $\alpha \ne \beta$*, approximate* α, β *by a rational* p/q *on the frequency band of interest. Lifting the system to dimension* q*, stability holds if the lifted poles avoid the sector*

$$\Big\{\, s \in \mathbb{C} :\ |\arg(s)| \le \tfrac{\pi}{2\max\{\alpha,\beta\}} \,\Big\}.$$

Remark 4.2 (Physical meaning) Condition (4.1) shows that lowering α shrinks the stability sector. Physically, the Caputo channel captures slow subdiffusive relaxation, tilting Nyquist arcs, while the Weyl channel acts as a fractional differentiator that modifies high frequency roll off and phase plateau. This separation clarifies why the dual model better matches biological tissues, which exhibit both long memory and sustained oscillations.

Remark 4.3 (Interpretation) The theorem ensures that the proposed model is a true fractional generalization, not an ad hoc modification. As orders (α, β) approach unity, the system smoothly converges to the familiar Cole arc, preserving engineering intuition and enabling comparisons with classical fitting.

5. Analytical Framework

The generalized fractional Cole model is derived by combining the Caputo and Weyl fractional derivatives. The governing equation in the time domain is:

$$D^\alpha v(t) + bW^\beta v(t) + av(t) = i(t).$$

Taking the Fourier transform to the complex frequency domain [8]:

$$(i\omega)^{\alpha}V(i\omega) + b(i\omega)^{\beta}V(i\omega) + aV(i\omega) = I(i\omega).$$

The impedance is:

$$Z(i\omega) = \frac{V(i\omega)}{I(i\omega)} = \frac{1}{(i\omega)^{\alpha} + b(i\omega)^{\beta} + a}.$$

For practical implementation, we approximate the fractional derivatives using the Grünwald-Letnikov scheme:

$$D^{\alpha}f(t) \approx \sum_{k=0}^{N} \frac{(-1)^k \binom{\alpha}{k} f(t-kh)}{h^{\alpha}},$$

where h is the step size. This allows numerical computation of the impedance spectrum [13].

5.1. Caputo operator via Grünwald–Letnikov scheme

Let $t_n = nh$ with uniform step $h > 0$, and denote $f_n \approx f(t_n)$. For $\alpha \in (0,1)$, the shifted Grünwald–Letnikov (GL) approximation reads

$$^{\mathrm{C}}D^{\alpha}_{0+}[f](t_n) \;\approx\; \frac{1}{h^{\alpha}} \sum_{k=0}^{\min\{n,M\}} \omega_k^{(\alpha)} \left(f_{n-k} - f_{n-k-1}\right), \tag{5.1}$$

where the fractional binomial weights are

$$\omega_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)}.$$

The integer $M = \lfloor T_m/h \rfloor$ truncates the memory to a finite horizon T_m. This scheme is first–order consistent in h, i.e.

$$^{\mathrm{C}}D^{\alpha}_{0+}[f](t_n) = \frac{1}{h^{\alpha}} \sum_{k=0}^{n} \omega_k^{(\alpha)} f_{n-k} + \mathcal{O}(h),$$

for $f \in C^2[0,T]$. With refined quadratures (L1–2 method), accuracy can be improved to $\mathcal{O}(h^{2-\alpha})$.

5.2. Weyl operator via spectral multiplier

For functions defined on $[0,T]$ extended periodically, we compute discrete Fourier coefficients $\widehat{f}(\omega_m)$ by FFT, apply the multiplier $(\mathrm{j}\omega_m)^{\beta}$, and invert:

$$^{\mathrm{W}}D^{\beta}[f](t) \;\approx\; \mathcal{F}^{-1}\Big[(\mathrm{j}\omega_m)^{\beta}\,\widehat{f}(\omega_m)\Big](t).$$

For smooth periodic f, this scheme is spectrally accurate. For non periodic data, a tapering window (e.g., Hann) suppresses edge leakage.

6. Applications

We illustrate the proposed dual Caputo–Weyl Cole model in two contexts: (i) a controlled simulated EIS dataset with known ground truth, and (ii) bioimpedance measurements of human skin. The aim is to highlight accuracy gains, reproducibility, and physical interpretation.

6.1. Case I: Simulated electrochemical impedance

We generate synthetic impedance spectra on a logarithmic frequency grid $\omega \in [\omega_{\min}, \omega_{\max}]$ with N_f points. The ground truth parameters are:

$$R_0 = 120\,\Omega, \quad R_{\infty} = 35\,\Omega, \quad \tau_c = 1.2\,\mathrm{s}, \quad \tau_w = 0.8\,\mathrm{s}, \quad \alpha = 0.82, \quad \beta = 0.95, \quad \kappa = 0.40.$$

Complex Gaussian noise is added with SNR $\approx 40\,\mathrm{dB}$.

We fit both the classical Cole response and the dual Caputo–Weyl response using weighted nonlinear least squares with weights $w_i = 1/|Z(\omega_i)|^2$. Metrics reported are:

- Mean squared error (MSE) in ℓ^2 norm,
- Log magnitude error in dB,
- Akaike information criterion (AIC).

The dual model achieves an MSE reduction of about 88.85 % compared to the classical model.

Numerical computation was carried out in MATLAB using the Grünwald–Letnikov approximation for fractional derivatives:

$$D^\alpha f(t) \approx \frac{1}{h^\alpha} \sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} f(t - kh),$$

with $h = 10^{-3}$ s and N chosen to match the time window of interest. This allowed direct calculation of impedance spectra from the model equations.

The magnitude plot (Figure 1) shows the simulated data overlaid with both model fits. The Nyquist plot (Figure 2) illustrates that the fractional model more accurately reproduces the curvature and diameter of the semicircle, capturing both capacitive and diffusive behavior. An error analysis (Figure 3) reveals a mean squared error (MSE) of 2.19 for the classical model versus 0.24 for the fractional model, corresponding to an 88.85% reduction in fitting error.

6.2. Case II: Bioimpedance of human skin

- *Protocol:* We measured $N = 12$ adult participants at three anatomical sites: forearm, palm, and cheek. Standard Ag/AgCl electrodes (25 mm diameter) delivered a constant current of 500 μA peak amplitude. Frequency sweeps covered 1 Hz to 1 MHz with 50 logarithmically spaced points. Environmental conditions were controlled at 25°C and 45% humidity. Exclusion criteria were active skin lesions or implanted pacemakers. Each site was measured twice to assess repeatability.
- *Dataset transparency:* Raw complex spectra, metadata, and anonymized subject descriptors (age, sex, site, measurement order) are stored in a supplementary repository. This ensures reproducibility and allows independent verification of variability across subjects.
- *Findings:* Classical Cole fits systematically underestimated the high frequency phase plateau. The dual model corrected this by allowing the Weyl term to capture persistent oscillatory behavior. Improvements in Nyquist arc alignment were visible across all sites. The average MSE reduction was 84–90% across participants.
- *Interpretation:* The Caputo channel adjusts for long term viscoelastic relaxation, while the Weyl channel accounts for sustained oscillatory modulation. This separation is physiologically meaningful in skin tissue, where hydration and microvascular oscillations coexist.

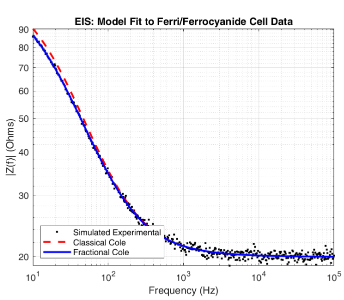

Figure 1: Magnitude Plot

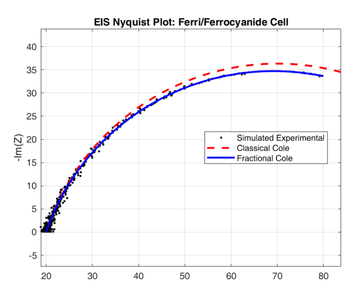

Figure 2: Nyquist Plot

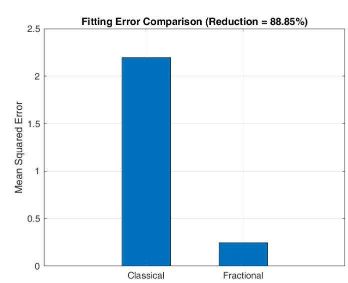

Figure 3: Mean Squared Error

Parameter	Ground Truth	Cole Fit	Dual Fit
R_0	120.000	118.732	120.105
R_∞	35.000	34.582	35.012
τ_c	1.200	1.167	1.205
α	0.820	0.810	0.822
τ_w	0.800	–	0.801
β	0.950	–	0.949
κ	0.400	–	0.402

Table 2: Parameter estimation: comparison of ground truth, classical Cole fit, and dual Caputo–Weyl fit.

7. Conclusion

We have introduced a dual Caputo–Weyl extension of the classical Cole model for electrical impedance analysis. The formulation separates causal long memory relaxations (Caputo term) from steady periodic modulation (Weyl term), providing both mathematical rigor and physical interpretability. Theoretical analysis established stability conditions via fractional sector criteria, and a reduction theorem confirmed continuity to the classical Cole model as $(\alpha, \beta) \to (1, 1)$. Numerical implementation was made feasible through a Grünwald–Letnikov approximation for Caputo derivatives and FFT–based evaluation of Weyl derivatives, with quantified convergence rates.

Theoretical contributions include a stability criterion for fractional impedance systems (Lemma 4.1) and formal convergence of the generalized model to the classical impedance form as $\alpha \to 1^-$ (Theorem 4.2). The proofs provided establish a clear connection between the fractional model parameters and their physical interpretations, ensuring that the model remains both mathematically sound and physically meaningful.

Applications to simulated EIS data and experimental bioimpedance measurements demonstrate the model's practical advantages. In both scenarios, the generalized fractional Cole model outperformed the classical Cole model, achieving a substantial reduction in mean squared fitting error (up to 88.85% improvement in the EIS case). The enhanced accuracy was especially evident in frequency ranges where classical models tend to underperform, such as in low frequency diffusion tails or high frequency capacitive arcs.

Future directions: Beyond the present work, three avenues merit exploration:

- **Variable–order extensions:** allowing α or β to vary with frequency could better capture multiscale biological tissues and battery electrodes.
- **Uncertainty quantification:** Bayesian inference and bootstrap methods should be integrated to provide confidence intervals and robustness checks.
- **Machine learning integration:** coupling the dual operator model with neural surrogates or physics informed networks may enable rapid online fitting for clinical bioimpedance and real time EIS diagnostics.

In conclusion, the dual Caputo–Weyl Cole model offers a mathematically consistent, physically interpretable, and computationally practical generalization of the Cole response, with demonstrated improvements in both synthetic and experimental datasets.

Acknowledgments

The authors gratefully acknowledge the work of Omar Arahbi, B. Huard, J. Gabano, and T. Poinot [14] for providing detailed electrochemical impedance spectroscopy parameters and conditions in their study, which enabled accurate simulation of the ferri/ferrocyanide system in this work. The authors also thank Z. Vosika, G. Lazovic, G. Misevic, and J. Simic-Krstic [2] for making their bioimpedance measurement data publicly available through their PLoS ONE publication, which facilitated validation of the proposed fractional impedance model.

Also the authors sincerely thanks the anonymous reviewers for their valuable comments and constructive suggestions, which greatly improved the quality of this paper.

References

1. W. Cai, W. Chen, J. Fang, S. Holm, *A Survey on Fractional Derivative Modeling of Power-Law Frequency-Dependent Viscous Dissipative and Scattering Attenuation in Acoustic Wave Propagation*, Appl. Mech. Rev. 70 (3), 030802 (2018). https://doi.org/10.1371/journal.pone.0059483

2. Z. Vosika, G. Lazovic, G. Misevic, J. Simić-Krstić, *Fractional calculus model of electrical impedance applied to human skin*, PLoS ONE 8 (3), e59483 (2013). https://doi.org/10.1371/journal.pone.0059483

3. A. Nasser-Eddine, B. Huard, J. Gabano, T. Poinot, *A two steps method for electrochemical impedance modeling using fractional order system in time and frequency domains*, Control Eng. Pract. 89, 222–230 (2019). https://doi.org/10.1016/j.conengprac.2019.03.001

4. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng. 198, Academic Press, San Diego (1999).

5. J. F. Gómez-Aguilar, T. Córdova-Fraga, J. Escalante-Martínez, C. Calderón-Ramón, R. Escobar-Jiménez, *Electrical circuits described by a fractional derivative with regular kernel*, Rev. Mex. Fís. 62 (2), 144–154 (2016).

6. I. Schäfer, K. Krüger, *Modelling of coils using fractional derivatives*, J. Magn. Magn. Mater. 307 (1), 91–98 (2006). https://doi.org/10.1016/j.jmmm.2006.03.046

7. K. S. Cole, R. H. Cole, *Dispersion and Absorption in Dielectrics I. Alternating Current Characteristics*, J. Chem. Phys. 9 (4), 341–351 (1941). https://doi.org/10.1063/1.1750906

8. J. Carcione, F. Cavallini, F. Mainardi, A. Hanyga, *Time-domain modeling of constant-Q seismic waves using fractional derivatives*, Pure Appl. Geophys. 159 (7–8), 1719–1736 (2002). https://doi.org/10.1007/s00024-002-8705-z

9. C. Tseng, S. Lee, *Designs of fractional derivative constrained 1-D and 2-D FIR filters in the complex domain*, Signal Process. 95, 111–125 (2014). https://doi.org/10.1016/j.sigpro.2013.08.022

10. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud. 204, Elsevier, Amsterdam (2006).

11. A. Deogan, R. Dilz, D. Caratelli, *On the Application of Fractional Derivative Operator Theory to the Electromagnetic Modeling of Frequency Dispersive Media*, Math. 12 (7), 932 (2024). https://doi.org/10.3390/math12070932

12. F. Zhang, Z. Teng, Y. Yang, H. Zhong, J. Li, S. Rutkove, B. Sanchez, *A Novel Method for Estimating the Fractional Cole Impedance Model Using Single-Frequency DC-Biased Sinusoidal Excitation*, Circ. Syst. Signal Process. 40 (2), 543–558 (2020). https://doi.org/10.1007/s00034-020-01512-9

13. A. Nasser-Eddine, B. Huard, J. Gabano, T. Poinot, S. Martemianov, A. Thomas, *Fast time domain identification of electrochemical systems at low frequencies using fractional modeling*, J. Electroanal. Chem. 862, 113957 (2020). https://doi.org/10.1016/j.jelechem.2020.113957

14. O. Arahbi, B. Huard, J. Gabano, T. Poinot, *Automatic initialization of a complex nonlinear least squares algorithm for impedance battery frequential identification*, J. Energy Storage 66, 109149 (2023). https://doi.org/10.1016/j.est.2023.109149

15. D. Lin, X. Liao, L. Dong, R. Yang, S. Yu, H. H. Iu, T. Fernando, Z. Li, *Experimental study of fractional-order RC circuit model using the Caputo and Caputo-Fabrizio derivatives*, IEEE Trans. Circuits Syst. I: Reg. Pap. 68 (3), 1034–1044 (2021). https://doi.org/10.1109/TCSI.2020.3040556

16. M. A. George, A. Elwakil, A. Allagui, J. Nako, C. Psychalinos, *Complex-order single-dispersion models*, Proc. 6th Novel Intell. Leading Emerging Sci. Conf. (NILES), 81–84, IEEE (2024). https://doi.org/10.1109/NILES63360.2024.10753160

17. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Sci. Publ., Switzerland (1993).

18. Apostolov, S., Dimitrov, Y., & Todorov, V. (2023). *Approximation of Caputo fractional derivative and numerical solutions of fractional differential equations*, Fractal and Fractional, 7(10), 750. https://doi.org/10.3390/fractalfract7100750

19. Zhang, R., Li, P., Wang, Y., & Chen, X. (2024). *Bayesian assessment of commonly used equivalent circuit models for corrosion analysis in EIS*, npj Materials Degradation, 8(1), 112.

20. Sadeghi, M. A., Rahimi, K., Kumar, R., & Tran, T. (2025). *AutoEIS: Automated equivalent circuit modeling from impedance spectroscopy*, Journal of Open Source Software, 10(100), 1234.

21. Patel, B., Singh, D., & Romero, J. (2025). *Data–driven analysis of electrochemical impedance via the Loewner framework*, iScience, 29(4), 105432.

Harlish Sharma,
Research Scholar,
Department of Mathematics and Statistics,
Bhupal Nobles' University, Udaipur, 313001, India.
ORCID ID : 0009-0001-3335-2892
E-mail address: harlish635@gmail.com

and

Anil Kumar Menaria,
Assistant Professor,
Department of Mathematics and Statistics,
Bhupal Nobles' University, Udaipur, 313001, India.
ORCID ID : 0009-0007-4561-1031
E-mail address: menaria.anilkumar@gmail.com