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A Non-Commutative Symmetric Algebra via S-Proximit Structure

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ABSTRACT: These axioms easily characterize the Gelfand theory of a commutative symmetric algebra. Numerous characteristics of commutative Gelfand theory are also presented in Gelfand theories of arbitrary symmetric algebras. We proved that symmetric algebras that are homogeneous and unital always have a Gelfand theory. We demonstrated that the identity is the unique Gelfand theory for liminal symmetric algebras with discrete spectrum (subject to a suitable concept of equivalence).

Key Words: Symmetric S-Proximit symmetric algebra, non-commutative Gelfand theory, homogeneous symmetric algebra.

Contents

1. Introduction

The theory of symmetric algebras and related structures has been extensively developed and generalized through various approaches in functional analysis. Several works have contributed to this field, starting with new structures in random approach normed spaces via spaces [19], new kinds of vector spaces via s-proximity structures [4], and results on Q-bounded functionals in symmetric Δ-symmetric algebras [11,13,9]. Other studies have addressed normed approach spaces and β-approach structures [27], as well as new kinds of topological vector spaces based on proximity structures [16,15].

Classical foundations of symmetric algebra theory are provided in well-known works such as those by Bosall and Duncan [6], Jameson [14], and Kaplansky [18], along with more recent contributions to random approach vector spaces [10] and fuzzy soft ordered symmetric algebras [8]. The theory of multipliers [28], Hermitian operators [7], and completion of normed approach spaces [1,26] has also been studied in depth.

Significant developments have been made in δ-character in symmetric t^{ω}-Δ-symmetric algebras [25], topological approach vector spaces [3], and completion and generalization of normed approach spaces [2]. Several works by Lowen and collaborators have explored Lindelöf and separability in approach spaces [5], the fundamental role of approach spaces in topology [21], completion of quasi-metric spaces [21], and local compactness in approach spaces [22,23], as well as approach vector spaces [24].

More recent advancements include the study of sober metric approach spaces [20] and fundamental results in operator algebra theory, in addition to works addressing equivalent locally martingale measures in ordered symmetric algebras [12] and the broad structural aspects of algebras [17].

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2. Preliminary

Definition 2.1 *[18] Let* M *be symmetric algebra. If* $\mathtt{K} \subseteq \mathsf{M}$ *is a linear subspace of* M

i. $\mathtt{K}$ *is called a right ideal of* M *if* $vw \in \mathtt{K}$ *for all* $v \in \mathsf{M}$ *and* $w \in \mathtt{K}$.

ii. $\mathtt{K}$ *is called left ideal of* M *if* $wv \in \mathtt{K}$ *for all* $v \in \mathsf{M}$ *and* $w \in \mathtt{K}$.

iii. $\mathtt{K}$ *is called two side ideal of* M *if it is both a left and right ideal of* M.

Definition 2.2 *[13] Let* M *be an* $\mathtt{S}$*-Proximit symmetric algebra , a maximal ideal in* M *is ideal such that there is no ideal* $\mathfrak{T}$ *in* M *satisfy* $\mathtt{K} \subseteq \mathfrak{T} \subseteq \mathsf{M}$.

Proposition 2.3 *[19] Let* M *be* $\mathtt{S}$*-Proximit symmetric symmetric algebra, then* M *corresponding to the* $\mathtt{C}_\circ(\mathcal{S}ch(\mathsf{M}))$ *where:* $(\mathcal{S}ch(\mathsf{M})) = \{k | k : \mathcal{S}ch(\mathsf{M}) \to \mathbb{C}$, k *is continuous* δ*-character*$\}$ *and* $\mathcal{S}ch(\mathsf{M}) = \{f_a | f_a : \mathsf{M} \to \mathbb{C}\}$.

Definition 2.4 *[18] Let* W *be a subspace of a vector space* M*. the coset of an element* $n \in \mathsf{M}$ *with respect to* W *is denoted by* $n + \mathsf{W}$ *and is defined to be the set:*

$$n + \mathsf{W} = \mathsf{M} : \mathsf{M} = n + m, m \in \mathsf{W}.$$

Define of algebraic operations by:

$$(w + \mathsf{W}) + (n + \mathsf{W}) = (w + n) + \mathsf{W}, \alpha\,(n + \mathsf{W}) = \alpha n + \mathsf{W}.$$

This space is called the quotient space (or sometimes factor space) of M *by* W *and is denoted by* M/W.

Definition 2.5 *[12] Let* M *be an algebra, let* $\mathtt{E}, F$ *are vector spaces, and let* μ_1, μ_2 *be representations of* M *on* $\mathtt{E}$ *and* F*, respectively. We say that* μ_1 *and* μ_2 *are equivalent if there exists an isomorphism* $\vartheta : \mathtt{E} \to F$ *such that* $\vartheta^{-1} \circ \mu_1(a) \circ \vartheta = \mu_1(a)$*, for all* $a \in \mathsf{M}$.

Definition 2.6 *[12] Let* M *be an algebra, let* E *be a linear space, and let* φ *be a representation of* M *on* E*. A vector* $x \in E$ *is called cyclic if* $\varphi(\mathsf{M})x = E$*. A representation* φ *of an algebra* M *on a linear space* E *is cyclic if there exists a cyclic vector for* φ *on* E.

3. S-Proximit Quotient Space

Proposition 3.1 *If* $(\mathsf{M}, \mathtt{S}_\mathsf{M}, \rho_{\epsilon\mathsf{M}})$ *is* $\mathtt{S}$*-Proximit space,* W *is closed* $\mathtt{S}$*-proximit subspace of* M*. Then* $(\mathsf{M}/\mathsf{W}, \mathtt{S}_{\mathsf{M}/\mathsf{W}}, \rho_{\epsilon\mathsf{M}/\mathsf{W}})$ *is symmetric* $\mathtt{S}$*-Proximit symmetric algebra.*

Proof: Define $\mathtt{S}$-Proximit distance $\mathtt{S}_{\mathsf{M}/\mathsf{W}} : 2^{\mathsf{M}/\mathsf{W}} \times 2^{\mathsf{M}/\mathsf{W}} \to [0, \infty]$. By $\mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{z} + \mathsf{W}, \mathcal{D} + \mathsf{W}) = \mathtt{S}(\mathfrak{z} + \mathcal{D})$
First: we have to prove $((\mathsf{M}/\mathsf{W}, \mathtt{S}_{\mathsf{M}/\mathsf{W}}, \rho_{\epsilon\mathsf{M}/\mathsf{W}})$ is $\mathtt{S}$-Proximit space. Let $\mathfrak{A}$, $\mathcal{C} \subset \mathsf{M}$. Then

1. If $\mathtt{S}_\mathsf{M}(\mathfrak{A}, \mathcal{D}) = 0$, so that $\cap\mathcal{D} \neq \emptyset$.
2. If $\mathfrak{A} = \emptyset$ or $\mathcal{D} = \emptyset$, then $\mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A} + \mathsf{W}, \mathcal{D} + \mathsf{W}) = \infty$.
3. $\mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A} + \mathsf{W}, (\mathcal{D} + \mathsf{W}) \cap (\mathcal{C} + \mathsf{W})) = \max\ \mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A} + \mathsf{W}, \mathcal{D} + \mathsf{W}), \mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A} + \mathsf{W}, \mathcal{C} + \mathsf{W})$.
4. $\mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A} + \mathsf{W}, \mathcal{D} + \mathsf{W}) \leq \mathtt{S}_{\mathsf{M}/\mathsf{W}}(\mathfrak{A}^\epsilon + \mathsf{W}, \mathcal{D}^\omega + \mathsf{W}) + \epsilon + \omega$.Then $(\mathsf{M}/\mathsf{W}, \mathtt{S}_{\mathsf{M}/\mathsf{W}}, \rho_{\epsilon X/\mathsf{W}})$ is $\mathtt{S}$-Proximit space.
5. $(\mathsf{M}/\mathsf{W}, +)$ is group.
6. Assume that $n \in (\rho_\epsilon(\mathcal{A}) + \mathsf{W}, \rho_\epsilon(\mathcal{D}) + \mathsf{W})$ then $\rho_{\epsilon\mathsf{M}/\mathsf{W}}(\{\mathfrak{z}\} + \mathsf{W}, \mathfrak{A} + \mathsf{W}) = \rho_{\epsilon\mathsf{M}}(\{\mathfrak{z}\}, \mathfrak{A}) \leq \epsilon$ and $\rho_{\epsilon\mathsf{M}/\mathsf{W}}(\{t\} + \mathsf{W}, \mathcal{D} + \mathsf{W}) = \rho_{\epsilon\mathsf{M}}(\{t\}, \mathcal{D}) \leq \epsilon$. There for $n \in \rho_\epsilon(\mathtt{F}(\mathfrak{A} + \mathcal{D}))$.
7. Suppose that $n \in (\rho_\epsilon(\mathfrak{A}) + \mathsf{W})$. Assume that $n = -\mathfrak{z} + \mathsf{W}$ such that $\mathfrak{z} + \mathsf{W} \in (\rho_\epsilon(\mathfrak{A}) + \mathsf{W})$. Then $\mathsf{M}/\mathsf{W}, \mathtt{S}_{\mathsf{M}/\mathsf{W}}, +)$ is $\mathtt{S}$-Proximit group. There for $(\mathsf{M}/\mathsf{W}, \mathtt{S}_{\mathsf{M}/\mathsf{W}}, +, .)$ is $\mathtt{S}$-Proximit vector space.

Second: We have to prove $\|\cdot\|$ is norm on $^{\mathsf{M}}/_{\mathsf{W}}$ define by $\|n+\mathsf{W}\| = \inf\|x+y\| : y \in \mathsf{W}$.

1. For $n+\mathsf{W} \in {}^{\mathsf{M}}/_{\mathsf{W}}$ such that $\mathsf{W} \neq 0$. So we get $\|n+\mathsf{W}\| > 0$.
2. $\|\partial(n+\mathsf{W})\| = \inf\|\partial x + y\| = |\partial| \inf\|x+y\| = |\partial|\,\|n+\mathsf{W}\|$.
3. $\lim_{\beta\to 0}\|n+\mathsf{W}\| = \lim_{\beta\to 0}\inf\|\beta x + y\| = 0$.
4. Let $n+\mathsf{W}, m+\mathsf{W} \in {}^{\mathsf{M}}/_{\mathsf{W}}$ and $|c| \geq 1\|n+\mathsf{W}+m+\mathsf{W}\| = \inf\{\|x+y+\ z+y\| : y \in \mathsf{W}\ \} \leq \inf c\,[\|x+z+y\|]$
5. If $n+\mathsf{W} = m+\mathsf{W}$, then $\mathtt{S}(\mathfrak{A},\mathsf{B}) = 0$. If $n+\mathsf{W} \neq m+\mathsf{W}$, then $\mathtt{S}(\mathfrak{A},\mathsf{B}) = \sup|(n+\mathsf{W})-(m+\mathsf{W})|$.
6. $\|(n+\mathsf{W})(m+\mathsf{W})\| \leq \|n+\mathsf{W}\|\|m+\mathsf{W}\|$.
7. Suppose that $^{\mathsf{M}}/_{\mathsf{W}}$ with identity, then $\|(n+\mathsf{W})(e)\| = \inf\|x+y)(e)\| = \|e\| = 1$.

Hence $^{\mathsf{M}}/_{\mathsf{W}}$ is symmetric S-Proximit symmetric algebra. □

Definition 3.2 *For a given* S*-Proximit symmetric algebra* M*, let* σ_{M} *denote the set of all maximal modular left ideal of* M*. A Gelfand theory for* M *is any pair* $(\Psi,\mathfrak{A})$ *that satisfies the following conditions:*
$(\mathfrak{S}_1)$*:* $\mathfrak{A}$ *is* S*-Proximit symmetric algebra and* $\Psi : \mathsf{M} \longrightarrow \mathfrak{A}$ *is algebra homeomorphism.*
$(\mathfrak{S}_2)$ *: the assignment* $\sigma_{\mathsf{M}} \ni \mathsf{W} \to \Psi^{-1}(\mathsf{W})$ *is a bijection between* $\sigma_{\mathfrak{A}}$ *and* σ_{M}*.*
$(\mathfrak{S}_3)$ *: For each* $\in \sigma_{\mathfrak{A}}$*, the linear map* $\Psi_{\mathsf{W}} : {}^{\mathsf{M}}/_{\Psi^{-1}(\mathsf{W})} \to {}^{\mathfrak{A}}/_{\mathsf{W}}$ *induced by* Ψ *has dense range.*

Theorem 3.3 *Let* M *be a* S*-Proximit symmetric algebra and* $(\Psi,\mathfrak{A})$ *be a Gelfand theory for* M *then* Ψ *is* S*-contraction .*

Proof: Suppose that $x \in \Psi(\rho_\epsilon(\mathscr{A}))$ then $x \in \Psi(\rho_\epsilon(\mathtt{S}(\mathscr{A})))$ such that $\rho_\epsilon(x,\mathscr{A}) \leq \epsilon$ implies $\rho_\epsilon(x,\Psi(\mathscr{A})) \leq \epsilon$ so that $x \in \rho_\epsilon(\Psi(\mathscr{A}))$ hence Ψ is S-contraction. □

Proposition 3.4 *Let* M *be a* S*-Proximit symmetric algebra and* $(\Psi,\mathfrak{A})$ *be a Gelfand theory for* M*, let* $\mathsf{W} \in \sigma_{\mathfrak{A}}$ *and let* $\mathtt{f}_{\mathsf{W}} : \mathfrak{A} \to \mathscr{L}({}^{\mathfrak{A}}/_{\mathsf{W}})$ *the corresponding irreducible representation of* $\mathfrak{A}$ *then* $(\mathtt{f}_{\mathsf{W}} \circ \Psi)$ *is* S*-contraction.*

Proof: $\mathtt{Q}_{\mathsf{W}}$ be the image of Ψ_{W} in $^{\mathfrak{A}}/_{\mathsf{W}}$, we have Thus $^{\mathfrak{A}}/_{\mathsf{W}}$ is pre-Hilbert space since $\mathtt{Q}_{\mathsf{W}} \subset {}^{\mathfrak{A}}/_{\mathsf{W}}$ then $\mathtt{Q}_{\mathsf{W}}$ is pre-Hilbert space ,by $(\mathfrak{S}_2)$ we have $\Psi^{-1}(\mathsf{W})$ is maximal modular ideal of S so we get $\mathsf{M} \to \mathscr{L}({}^{\mathfrak{A}}/_{\mathsf{W}}), a \to (\mathtt{f}_{\mathsf{W}} \circ\ \Psi)(a)\mid_{\mathtt{Q}_{\mathsf{W}}}$ is an irreducible representation of M. □

Proposition 3.5 *Let* M *be* S*-Proximit symmetric algebra, and let* $(\Psi,\mathfrak{A})$ *be a Gelfand theory for* M*. Then for* $\mathfrak{a} \in \mathsf{M}$ *the element* $\Psi\mathfrak{a}$ *is quasi-invertible in* $\Psi\mathsf{M}$ *if and only if it is quasi-invertible in* $\mathfrak{A}$*.*

Theorem 3.6 *Let* M *be* S*-Proximit symmetric algebra ,and let* $(\Psi,\mathfrak{A})$ *be a Gelfand theory for* M*.Then if* M *is unital :* $\varphi_{\mathsf{M}}(\mathfrak{a}) =\ \varphi_{\mathfrak{A}}(\Psi\mathfrak{a})$*,* $\mathfrak{a} \in \mathsf{M}$*. And if* M *is non-unit and* $\varphi_{\mathsf{M}}(\mathfrak{a}) =\ \varphi_{\mathfrak{A}}(\Psi\mathfrak{a}) \cup 0, \mathfrak{a} \in \mathsf{M}$*.*

Proof: from the Previous lemma ,we have that $\varphi_{\mathsf{M}}(\mathfrak{a}) \cup 0 =\ \varphi_{\mathfrak{A}}(\Psi\mathfrak{a}) \cup 0, \mathfrak{a} \in \mathsf{M}$. If M is non-unital S-Proximit ,then $\varphi_{\mathsf{M}}(\mathfrak{a}) =\ \varphi_{\mathfrak{A}}(\Psi\mathfrak{a}) \cup 0, \mathfrak{a} \in \mathsf{M}$. Now assume that $0 \notin \varphi_{\mathfrak{A}}(\Psi\mathfrak{a})$ but $0 \in \varphi_{\mathsf{M}}(\mathfrak{a})$ this mean $\mathfrak{a}$ is not invertible in M. Suppose that $\mathfrak{a}$ doesn't have a left invers ,assume that $\mathfrak{a}$ has a left invers $\mathfrak{b} \in \mathsf{M}$ then $(\Psi\mathfrak{a})^{-1} = \Psi\mathfrak{b}$, so that $0 \notin \varphi_{\mathfrak{A}}(\Psi\mathfrak{b})$ since $\mathfrak{a}$ is right invers of $\mathfrak{b}$ then $\mathfrak{b}$ cannot be left invertible in M ,in other wise $\mathfrak{b}$ would be invertible with invers $\mathfrak{a}$ thus $0 \notin \varphi_{\mathsf{M}}(\mathfrak{a})$ and this is contraction with our assumption. □

4. Existence of Gelfand Theories

Proposition 4.1 *Let* M *be an* $\mathtt{S}$*-proximit symmetric algebra such that* $\mathtt{K}$ *be an ideal of* M *and let* φ *be an irreducible representation of* $\mathtt{K}$ *on a linear space* E*,then* φ *extends to a unique irreducible representation of* M *on* E*.Conversely if* φ *is an irreducible representation of* M *on a linear space* E *such that* $\varphi|_{\mathtt{K}} \neq 0$*, then* $\varphi|_{\mathtt{K}}$ *is an irreducible representation of* $\mathtt{K}$ *on* E*.*

Proof: Let $a \in \mathsf{M}$ and $x \in E$. Since φ is irreducible, there is $y \in \mathtt{K}$ and $\mathfrak{b} \in E$ such that $\varphi(y)\,\mathfrak{b} = x$, define $\varphi(a)\,x = \varphi(ay)\,\mathfrak{b}$. Note that this does not depend on the selection of y and $\mathfrak{b}$, because, if $\varphi(y)\,\mathfrak{b} = \varphi(\mathfrak{z})\,c$ with $\mathfrak{z} \in \mathtt{K}$, $c \in E$, then $\varphi(y)\,\mathfrak{b} - \varphi(\mathfrak{z})\,c = 0$. hence $\varphi(y)\,\mathfrak{b} - \varphi(\mathfrak{z})\,c = 0$ and so $\varphi(y)\,\mathfrak{b} - \varphi(\mathfrak{z})\ \varphi(k)\,\mathfrak{b} = 0$. Thus $\varphi(ay - a\mathfrak{z}k)\mathfrak{b} = 0$ for every $a \in \mathsf{M}$. Moreover, this extension is unique because if we assume that there are two such extensions, say φ_1, φ_2, then $\varphi_1(a)\,x = \varphi(a)\,\varphi(\mathfrak{b})\,y = \varphi(a\mathfrak{b})\,y = \varphi_2(a\mathfrak{b})\,y = \varphi_2(a)\,\varphi(\mathfrak{b})\,y = \varphi_2(a)\,x$, hence $\varphi_1 = \varphi_2$. □

Corollary 4.2 *Let* M *be* $\mathtt{S}$*-proximit symmetric algebra and let* $\mathtt{K}$ *be a closed ideal of* M*.Then* $\{\mathsf{W} \in \sigma_{\mathsf{M}} : \mathtt{K} \not\subset \mathsf{W}\} \to \sigma_{\mathtt{K}}, \mathsf{W} \to \mathsf{W} \cap \mathtt{K}$ *is a bijective.*

Theorem 4.3 *Let* M *be a non-unital* C^**-algebra and let* $\mathsf{M}^\sharp$ *be the unitization of* M*. Endow* $\mathsf{M}^\sharp$ *with the following norm:* $\|a \ \oplus \lambda\| = \sup\{\|ab + \lambda b\| : \|b\| \leq 1, b \in \mathsf{M}\}$*.Then* $\mathsf{M}^\sharp$ *with this norm is a* C^**-algebra.*

Theorem 4.4 *Let* M *be* $\mathtt{S}$*-Proximit a symmetric algebra without unit. Define* $\mathsf{M}^\sharp = \mathsf{M} \oplus \mathbb{C}$*, with addition given by* $a_1 \oplus \lambda_1 + a_2 \oplus \lambda_2 = (a_1 + a_2) \oplus (\lambda_1 + \lambda_2)$ *and multiplication defined by* $(a_1 \oplus \lambda_1)(a_2 \oplus \lambda_2) = (a_1a_2 + \lambda_1 a_2 + \lambda_2 a_1) \oplus \lambda_1 + \lambda_2$*, for all* $a_1 a_2 \in \mathsf{M}, \lambda_1, \lambda_2 \in \mathbb{C}$*. Then* $\mathsf{M}^\sharp$ *with norm* $\|a \oplus \ \lambda\| = \|a\| + |\ \lambda|$ *is a symmetric* $\mathtt{S}$*-Proximit a symmetric algebra with unit* $0 \oplus 1$*.*

Proof: Define $\mathtt{S}$-Proximit distance $\mathtt{S}_{\mathsf{M}^\sharp} : 2^{\mathsf{M}^\sharp} \times 2^{\mathsf{M}^\sharp} \to [0, \infty]$ as follows: for all $\mathfrak{A}, \mathscr{D} \in 2^{\mathsf{M}}$

1. If $\mathtt{S}_{\mathsf{M}^\sharp}(\mathfrak{A} \oplus \mathbb{C}, \mathscr{D} \oplus \mathbb{C}) = 0$, then $(\mathfrak{A} \oplus \mathbb{C}) \cap (\mathscr{D} \oplus \mathbb{C}) \neq \emptyset$.
2. If $\mathfrak{A} \oplus \mathbb{C} = \emptyset$ or $\mathscr{D} \oplus \mathbb{C} = \emptyset$, then $\mathtt{S}_{\mathsf{M}^\sharp}(\mathfrak{A} \oplus \mathbb{C}, \mathscr{D} \oplus \mathbb{C}) = \infty$.
3. $\mathtt{S}_{\mathsf{M}^\sharp}(\mathfrak{A} \oplus \mathbb{C}, (\mathscr{D} \oplus \mathbb{C}) \cap (\mathcal{C} \oplus \mathbb{C})) = \max\ \mathtt{S}_{\mathsf{M}^\sharp}(\mathfrak{A} \oplus \mathbb{C}, \mathscr{D} \oplus \mathbb{C}), \mathtt{S}_{\mathsf{M}^\sharp}(\mathfrak{A} \oplus \mathbb{C}, \mathcal{C} \oplus \mathbb{C})$.
   Then $(\mathsf{M}^\sharp \mathtt{S}_{\mathsf{M}^\sharp}, \rho_{\epsilon\ \mathsf{M}^\sharp})$ is $\mathtt{S}$-Proximit space . It clear that $\mathsf{M}^\sharp$ is an algebra with identity (0,1) and M is sub algebra of $\mathsf{M}^\sharp$. Then $\|\,(a \oplus \alpha)(b \oplus \beta)\,\| = \|\,(ab + \beta a + \alpha b) \oplus \alpha\beta\|$. Then $\mathsf{M}^\sharp$ be a normed algebra.
4. Let $u = (u_1, u_2, \ldots, u_k)\ \ v = (v_1, v_2, \ldots, v_k)$
5. $(\lambda u + \mu v)^* = ((\lambda u_1 + \mu v_1), \ldots, (\lambda u_k + \mu v_k))^*$. Then $(\lambda u + \mu v)^* = \bar{\lambda} u^* + \bar{\mu} v^*$.
6. $u^{**} = (u^*)^* = (u_1{}^*, u_2{}^*, \ldots, u_k{}^*)^* = (u_1{}^{**}, u_2{}^{**}, \ldots, u_k{}^{**})$
7. $(uv)^* = (u_1 v_1, u_2 v_2, \ldots, u_k v_k)^* = ((u_1 v_1)^*, (u_2 v_2)^*, \ldots, (u_k v_k)^*)$. Therefore $(\mathsf{M}, \mathsf{M}^\sharp)$ is a symmetric algebra. Hence $(\mathsf{M}, \mathsf{M}^\sharp)$ is a $\mathtt{S}$-Proximit symmetric algebra.

□

Proposition 4.5 *Let* M *be a non-unital symmetric algebra which has a Gelfand theory, then* $\mathsf{M}^\sharp$ *has also a Gelfand theory.*

Proof: Let $(\Psi, \mathfrak{A})$ be a Gelfand theory of M when $\mathsf{M}^\sharp$ denote the unconditional unitization of $\mathfrak{A}$ this mean if $\mathfrak{A}$ has an identity. Define a homomorphism:

$$\Psi^\sharp : \mathfrak{A}^\sharp \to \mathfrak{A}^\sharp, \mathfrak{a} + \gamma e_{\mathsf{M}^\sharp} \to \Psi\mathfrak{a} + \gamma e_{\mathfrak{A}^\sharp}$$

If $\mathfrak{A}$ is unital, we have a symmetric isomorphism $\mathfrak{A}^\sharp \to \mathfrak{A} \bigoplus \mathbb{C}, \mathfrak{a} + \gamma e_{\mathsf{M}^\sharp} \to \mathfrak{a} + \gamma e_{\mathsf{M}^\sharp} \bigoplus \gamma$. Hence $(\Psi^\sharp, \mathfrak{A}^\sharp)$ is a Gelfand theory for $\mathsf{M}^\sharp$. □

5. Conclusion

This work presents several results, including a Non-Commutative Symmetric symmetric algebra employing proximit structure, a quotient space and $\mathsf{M}^\sharp$ with a suitable norm are C^*-algebra that follows Golfand's theory. If $(\Psi, \mathfrak{A})$ is a Gelfand theory for M, then Ψ is an irreducible representation of $\mathfrak{A}$. Finally, if φ is an irreducible representation of M on a linear space E, then is an irreducible representation of $\mathtt{K}$ on E and extended extends to a unique irreducible representation of M on E.In contrast, if φ is an irreducible representation of M on a linear space E, then it is an irreducible representation of $\mathtt{K}$ on E and may be extended to a unique irreducible representation of M on E.

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