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On the Global Existence of the Logarithmic Viscoelastic Petrovsky Equation with a Delay Term

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ABSTRACT: The paper examines a logarithmic viscoelastic Petrovsky equation with a delay term in a bounded domain. We employ the energy method alongside the Faedo-Galarkin method to establish the existence of global solutions in appropriate Sobolev spaces. The solution's asymptotic behavior is derived using a suitable Lyapunov functional.

Keywords: Global existence, Petrovsky equation, delay term, viscoelastic, Lyapunov method.

Contents

1. Introduction

In this paper, we study the logarithmic viscoelastic Petrovsky equation with a delay

$$\left\{\begin{array}{ll} \left|u_t\right|^p u_{tt}(x,t)+\Delta^2 u(x,t)+\Delta^2 u_{tt}\left(x,t\right)-\int_0^t g\left(t-s\right)\Delta^2 u\left(s\right)ds & \\ \quad +\mu_1 u_t\left(x,t\right)+\mu_2 u_t(x,t-\tau)=ku\ln\left|u\right|, & \text{in } \Omega\times(0,+\infty), \\ u\left(x,t\right)=\frac{\partial u}{\partial v}(x,t)=0, & \text{in } \Gamma\times(0,+\infty), \\ u\left(x,0\right)=u_0\left(x\right), u_t\left(x,0\right)=u_1(x), & \text{in } \Omega, \\ \qquad u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right)) & \text{in } \Omega\times]0,\tau[, \end{array}\right. \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega=\Gamma$, $p>0$, μ_1 and μ_2 are positive real numbers. The function g is a positive non-increasing function defined on $[0,+\infty)$, $\tau>0$ represents a time delay, and u_0, u_1,f_0 are the initial data in a suitable function space. The Petrovsky problem has attracted considerable attention in recent years, particularly due to its relevance in the mathematical modeling of elastic plates and beams where higher order operators naturally arise. In such systems, damping mechanisms ranging from frictional to viscoelastic and delay-type damping play a crucial role in determining both the well-posedness of the model and the long-time behavior of its solutions. Several important contributions addressing existence, uniqueness, and asymptotic stability under various damping structures can be found in [4,5,7]. Logarithmic nonlinearities frequently appear in supersymmetric field theories and in cosmological inflation.From the perspective of Quantum Field Theory, logarithmic source terms arise in nuclear physics, inflation cosmology, geophysics, and optics (see [16,17]). For the literature review, we begin with the classical works of Birula and Mycielski [18,19]. The authors investigated the following equation containing a logarithmic term:

$$u_{tt}-u_{xx}+u-\varepsilon u\ln|u|^2=0, \tag{1.2}$$

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which represents a relativistic version of logarithmic quantum mechanics. They are considered pioneers in the study of such logarithmic models. In 1980, Cazenave and Haraux [20] introduced the equation

$$u_{tt} - \Delta u = u \ln |u|^k, \tag{1.3}$$

and proved the existence and uniqueness of solutions for the corresponding Cauchy problem. Gorka [17] obtained global existence results for the one-dimensional case of equation (1.3), while Bartkowski and Gorka [16] considered weak solutions and established existence results. In [23], Hiramatsu et al. studied the equation

$$u_{tt} - \Delta u + u + u_t + |u^2|u = u \ln u. \tag{1.4}$$

Later, in [22], Han established the global existence of solutions for equation (1.4). In [3], Al-Gharabli and Messaoudi studied a Petrovsky-type equation involving a logarithmic source term of the form

$$u_{tt} + \Delta^2 u + h(u_t) = k\, u \ln |u|, \tag{1.5}$$

where they established global existence via the Galerkin method and derived explicit decay estimates using suitable multiplier techniques. In a related work, Kafini and Messaoudi [9] examined a wave equation with both delay effects and logarithmic nonlinearity,

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = |u|^{p-2} u \,\log |u^k|, \tag{1.6}$$

and proved results concerning local existence and finite-time blow-up. In [21], Park considered an equation involving both delay and logarithmic nonlinearities of the form

$$u_{tt} - \Delta u + \alpha u_t(t) + \beta u_t(x, t-\tau) = u \ln |u|^\gamma. \tag{1.7}$$

The author established local and global existence results for equation (1.7). Furthermore, decay properties and nonexistence results were also investigated for this model. On the other hand, Sabbagh et al. [13] investigated a viscoelastic Petrovsky equation without logarithmic terms,

$$\begin{cases} |u_t|^\ell u_{tt}(x,t) + \Delta^2 u(x,t) - \Delta u_{tt}(x,t) - \displaystyle\int_0^t h(t-\sigma)\, \Delta^2 u(x,\sigma)\, d\sigma \\ \quad + \mu_1 u_t(x,t) + \displaystyle\int_0^1 \mu_2(s)\, u_t(x, t-s)\, ds = 0, \end{cases} \tag{1.8}$$

and established both existence and stability results. More recently, additional investigations have focused on Petrovsky-type models involving combinations of viscoelastic damping, logarithmic nonlinearities, and delay effects; see, for instance, [2,8,14,15]. These studies highlight the rich dynamics associated with such problems and the delicate analytical challenges they present. Despite these contributions, the combined influence of logarithmic source terms, viscoelastic damping, and distributed delay remains only partially understood. In particular, establishing global existence and obtaining sharp decay estimates in such settings require new analytical techniques. The purpose of this paper is to advance this line of research by investigating a Petrovsky-type system with logarithmic nonlinearity and internal distributed delay. In Section 2, we introduce the assumptions and auxiliary lemmas required for our analysis. Section 3 is devoted to proving the existence result, while Section 4 establishes the asymptotic decay of the associated energy functional.

2. Preliminaries

In this section, to prove our main result, we provide the necessary tools using the Sobolev spaces $H_0^2(\Omega)$ and the Lebesgue space $L^2(\Omega)$, along with their standard scalar products and norms. We denote the inner product in $L^2(\Omega)$ by $(.,.)$. Throughout this work, the constant C represents a generic positive constant. We consider the following assumptions:

(A1) Let $g : [0, \infty) \to (0, \infty)$ be a nonincreasing and differentiable function satisfying

$$\int_0^\infty g(s)\, ds = \beta < 1, \quad g(0) > 0, \quad 1 - \int_0^\infty g(s)\, ds = l. \tag{2.1}$$

(A2) We assume that there exists a positive constant ξ that satisfies

$$g'(t) \leq -\xi g(t), \tag{2.2}$$

(A3) We assume that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \ \mu_1 > |\mu_2| \tag{2.3}$$

(A4) The constant k in the equation (1.1) is defined as follows:

$$1 < k < 2\pi l e^3 \tag{2.4}$$

The first eigenvalue of the spectral Dirichlet problem is assumed to be λ .

$$\Delta^2 u = \lambda u, \ \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial \eta} = 0 \ \text{ in } \Gamma$$

$$\|\nabla u\|_2^2 \leq \frac{1}{\sqrt{\lambda}} \|\Delta u\|_2^2, \tag{2.5}$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$
Next, we have a Sobolev-Poincaré inequality [1]

Lemma 2.1 *Assume that q represents a number with*

$$\begin{aligned} 2 &\leq q < +\infty, \quad n = 1,2 \\ 2 &\leq q \leq \frac{2n}{n-2}, \quad n \geq 3, \end{aligned}$$

then, there exists a constant $C\ (\Omega, q)$ that satisfies

$$\|u\|_q \leq C\, \|\nabla u\|_2 \quad \forall u \in H_0^1(\Omega) \tag{2.6}$$

Lemma 2.2 *By using the direct calculations, for $h,\ g \in C^1([0,+\infty[,\mathbb{R})$ we get*

$$\begin{aligned} \int_\Omega h * \varphi \varphi_t dx &= -\frac{1}{2} h(t) \|\varphi(t)\|_2^2 + \frac{1}{2}(h' \circ \varphi)(t) \\ &\quad -\frac{1}{2}\frac{d}{dt}\left[(h \circ \varphi)(t) - \left(\int_0^t h(s) ds\right) \|\varphi(t)\|_2^2\right], \end{aligned} \tag{2.7}$$

where

$$(h \circ \Delta u)(t) = \int_0^t h(t-s)\, \|\Delta u(s) - \Delta u(t)\|_2^2\, ds.$$

Lemma 2.3 (Logarithmic Sobolev inequality [7]) *Let u be any function in $H_0^1(\Omega)$ and $a > 0$ be a number. Then*

$$\int_\Omega u^2 \ln|u|\, dx \leq \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.8}$$

Corollary 2.1 *[2] Let u be any function in $H_0^2(\Omega)$ and $a > 0$ be a number. Then*

$$\int_\Omega u^2 \ln|u|\, dx \leq \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2 \tag{2.9}$$

Lemma 2.4 (Logarithmic Gronwall inequality [7]) *Suppose that $C > 0$, $\varphi \in L^1(0,T;\mathbb{R}^+)$ and let the function $\beta : [0,T] \to [1,\infty)$ satisfy*

$$\beta(t) \leq C\left(1 + \int_0^t \varphi(s)\beta(s)\ln(\beta(s))ds\right), \forall t \in [0,T]. \tag{2.10}$$

Then

$$\beta(t) \leq C \exp\left(C \int_0^t \varphi(s) ds\right), \forall t \in [0,T]. \tag{2.11}$$

Lemma 2.5 *[2] For any $\epsilon_0 \in (0,1)$, there exists $a_{\epsilon_0} > 0$ such that*

$$s\left|\ln s\right| \leq s^2 + a_{\epsilon_0} s^{1-\epsilon_0},\ \forall s > 0. \tag{2.12}$$

Lemma 2.6 *[2] Assume that g satisfies **(A1)**. Then ,for any $u \in H_0^2(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s)ds\right)^2 dx \leq c(g\circ\Delta u)(t) \tag{2.13}$$

and

$$\int_\Omega \left(\int_0^t g'(t-s)(u(t)-u(s)ds\right)^2 dx \leq c(g'\circ\Delta u)(t). \tag{2.14}$$

We introduce a new variable that is comparable to [11].

$$z\left(x,\rho,t\right) = u_t\left(x,t-\tau\rho\right) \text{ in } \Omega\times(0,1)\times(0,+\infty)$$

Then, we have

$$\tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0 \text{ in } \Omega\times(0,1)\times(0,+\infty)$$

As a result, problem (1.1) can be formulated as follows:

$$\left\{\begin{array}{c} \left|u_t\right|^p u_{tt}(x,t) + \Delta^2 u(x,t) + \Delta^2 u_{tt}\left(x,t\right) - \int_0^t g\left(t-s\right)\Delta^2 u\left(s\right)ds \\ +\mu_1 u_t\left(x,t\right) + \mu_2 z(x,1,t) = ku\ln\left|u\right|,\ \text{ in } \Omega\times]0,+\infty[, \\ \tau z_t\left(x,\rho,t\right) + z_\rho\left(x,\rho,t\right) = 0 \ \text{ in } \Omega\times]0,1[\times]0,+\infty[, \\ u\left(x,t\right) = 0,\ \text{on} \quad \Gamma\times[0,+\infty[, \\ z\left(x,0,t\right) = u_t\left(x,t\right), \quad \text{in } \Omega\times[0,+\infty[, \\ u\left(x,0\right) = u_0\left(x\right), u_t\left(x,0\right) = u_1(x), \text{in } \Omega, \\ z\left(x,\rho,0\right) = f_0\left(x,-\rho\tau\right), \quad \text{in } \Omega\times]0,1[\end{array}\right. \tag{2.15}$$

The energy functional of (2.15) defined as follows

$$\begin{aligned} E\left(t\right) &= \frac{1}{p+2}\left\|u_t\left(t\right)\right\|_{p+2}^{p+2} + \frac{1}{2}\left(\left\|\Delta u_t\left(t\right)\right\|_2^2 + \left(1-\int\limits_0^t g\left(s\right)ds\right)\left\|\Delta u\left(t\right)\right\|_2^2\right. \\ &\quad \left. + \left(g\circ\Delta u\right)\left(t\right)\right) - \frac{k}{2}\int_\Omega \left|u\right|^2 \ln\left|u\right| dx + \frac{k}{4}\left\|u\left(t\right)\right\|_2^2 \\ &\quad + \frac{\epsilon}{2}\int_\Omega\int_0^t \left|z\left(x,\rho,t\right)\right|^2 d\rho dx \end{aligned} \tag{2.16}$$

Lemma 2.7 *Under the assumptions (A1)-(A3), for any solution (u,z) of problem (2.15), the functional $E(t)$ defined in (2.16) satisfies*

$$\begin{aligned} E'(t) &\leq -\frac{1}{2}g(t)\left\|\Delta u\right\|_2^2 - \left(\mu_1 - \frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega \left|u_t(x,t)\right|^2 dx \\ &\quad -\left(\frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\int_\Omega \left|z(x,1,t\right|^2 dx \leq 0,\ \forall t \geq 0 \end{aligned} \tag{2.17}$$

Proof: Multiplying the equation $(2.15)_1$ by u_t and integrating by parts over Ω, we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{p+2}\left\|u_t\right\|_{p+2}^{p+2} + \frac{1}{2}\left\|\Delta u_t\right\|_2^2 + \frac{1}{2}\left\|\Delta u\right\|_2^2 - \frac{k}{2}\int_\Omega \left|u\right|_2^2 \ln\left|u\right| dx + \frac{k}{2}\left\|u\right\|_2^2\right) \\ &+\mu_1\left\|u_t\right\|_2^2 + \int_\Omega z(x,1,t)u_t dx = \int_\Omega\int_0^t g(t-s)\Delta u(s)\Delta u_t(t)dsdx. \end{aligned} \tag{2.18}$$

By applying the Lemma 2.7, the term on the right-hand side of (2.18) can be rewritten as

$$\begin{aligned} &\int_{\Omega}\int_{0}^{t} g(t-s)\Delta u(s)\Delta u_t(t)dsdx+\frac{1}{2}g(t)\left\|\Delta u\right\|_2^2 \\ = \;& \frac{d}{2dt}\left[\int_0^t g(s)ds\left\|\Delta u\right\|_2^2-(g\circ\Delta u)(t)\right]+\frac{1}{2}(g'\circ\Delta u)(t) \end{aligned} \tag{2.19}$$

Consequently, (2.18) becomes

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{p+2}\left\|u_t\right\|_{p+2}^{p+2}+\frac{1}{2}(1-\int_0^t g(s)ds)\left\|\Delta u\right\|_2^2\right. \\ &\left.+\frac{1}{2}\left\|\Delta u_t\right\|_2^2-\frac{k}{2}\int_{\Omega}\left|u\right|_2^2\ln\left|u\right|dx+\frac{k}{2}\left\|u\right\|_2^2+(g\circ\Delta u)(t)\right) \\ = \;& -\mu_1\left\|u_t\right\|_2^2-\int_{\Omega}z(x,1,t)u_tdx+\frac{1}{2}(g'\circ\Delta u)(t)-\frac{1}{2}g(t)\left\|\Delta u\right\|_2^2 \end{aligned} \tag{2.20}$$

We multiply the equation $(2.15)_2$ by $\frac{\epsilon}{\tau}z$ and integrate the result over $\Omega\times(0,1)$, $\epsilon>0$, we obtain

$$\frac{\epsilon}{2}\frac{d}{dt}\int_{\Omega}\int_0^1 z^2\left(x,\rho,t\right)d\rho dx+\frac{\epsilon}{\tau}\int_{\Omega}\int_0^1 z\left(x,\rho,t\right)z_{\rho}\left(x,\rho,t\right)d\rho dx=0 \tag{2.21}$$

Noticing that

$$\begin{aligned} -\frac{\epsilon}{\tau}\int_{\Omega}\int_0^1 z\left(x,\rho,t\right)z_{\rho}\left(x,\rho,t\right)ddx &= -\frac{\epsilon}{2\tau}\left(\int_{\Omega}\int_0^1\frac{\partial}{\partial\rho}z^2\left(x,\rho,t\right)d\rho dx\right) \\ &= \frac{\epsilon}{2\tau}\int_{\Omega}(z^2\left(x,0,t\right)-z^2\left(x,1,t\right))dx \\ &= \frac{\epsilon}{2\tau}\int_{\Omega}\left(u_t^2-z^2\left(x,1,t\right)\right)dx \end{aligned} \tag{2.22}$$

Combining (2.20) and (2.21) we get

$$\begin{aligned} E'(t) \;=\;& \frac{1}{2}(g'\circ\Delta u)(t)-\frac{1}{2}g(t)\left\|\Delta u\right\|_2^2-(\mu_1-\frac{\epsilon}{2\tau})\int_{\Omega}\left|u_t(x,t)\right|^2dx \\ &-\frac{\epsilon}{2\tau}\int_{\Omega}\left|z(x,1,t)\right|^2dx-\mu_2\int_{\Omega}z(x,1,t)u_t(x;t)dx \end{aligned}$$

By applying Young's inequality, we obtain

$$|\mu_2\int_{\Omega}z(x,1,t)u_t(x;t)dx|\le\frac{|\mu_2|}{2}\int_{\Omega}\left(\left|z(x,1,t)\right|^2+\left|u_t(x,t)\right|^2\right)dx. \tag{2.23}$$

Thus, we get

$$E'(t)\le-\left(\mu_1-\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_{\Omega}\left|u_t(x,t)\right|^2dx-\left(\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_{\Omega}\left|z(x,1,t\right|^2dx$$

From (2.3) the proof is complete. □

Theorem 2.1 *Let* $(u_0,u_1,f_0)\in H_0^2\left(\Omega\right)\times H_0^2\left(\Omega\right)\times H_0^2\left(\Omega,H^2\left(0,1\right)\right)$ *satisfy the compatibility condition*

$$f_0\left(.,0\right)=u_1$$

Suppose that conditions (A1)-(A3) hold. Then, problem (1.1) *has a weak solution:*

$$\begin{gathered} u\in L^{\infty}\left([0,+\infty);H_0^2\left(\Omega\right)\right),\;\; u_t\in L^{\infty}\left([0,+\infty);H_0^2\left(\Omega\right)\right), \\ u_{tt}\in L^2\left([0,+\infty);H_0^1\left(\Omega\right)\right). \end{gathered}$$

Theorem 2.2 *Assume (A1)-(A3) hold. Then the energy functional* $E(t)$ *satisfies,*

$$E(t)\le k_0e^{-k_1t}\quad\forall t>0 \tag{2.24}$$

where k_0 *and* k_1 *are positive constants that will be determined later.*

3. Local existence

Proof: [Proof of Theorem (2.1)] Throughout this section we assume $(u_0, u_1, f_0) \in H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2\left(\Omega, H^2(0,1)\right)$. We will use the Faedo-Galerkin method to prove the existence of a global solution. Let$T > 0$ be fixed and let $\{w_n\}$ be an orthogonal basis of the separable space $H_0^2(\Omega)$. Let $V_n = span\{w_1, w_2, ..., w_n\}$. Now, we define, for $1 \le j \le n$, the sequence $\varphi_j(x,\rho)$ as follows:

$$\varphi_j(x,0) = w_j \tag{3.1}$$

Then, we may extend $\varphi_j(x,0)$ by $\varphi_j(x,\rho)$ over $L^2(\Omega\times(0,1))$ such that $\{\varphi_j\}_j$ forms a basis of $L^2(\Omega, H_0^2(\Omega))$ and denote Z_n the space generate by $\{\varphi_n\}$.Let (u_n, z_n) $(n = 1,2,3,...)$ be an approximate solutions in the form

$$u^n(t) = \sum_{j=1}^{n} c^{jn}(t)\, w_j(x), \qquad z^n(t) = \sum_{j=1}^{n} d^{jn}(t)\, \varphi_j(x) \tag{3.2}$$

where $c^{n,j}$ and $d^{n,j}$ $(j = 1,2,...,n)$ are defined by the ordinary differential systems

$$\begin{aligned} & \langle |u_t^n|^p u_{tt}^n, w_j\rangle_\Omega + \langle \Delta u^n, \Delta w_j\rangle_\Omega + \langle \Delta u_{tt}^n(x,t), \Delta w_j\rangle_\Omega \\ & -\int_0^t g(t-s)\langle \Delta u^n(s), \Delta w_j\rangle_\Omega\, ds + \mu_1 \langle u_t^n, w_j\rangle_\Omega \\ & +\mu_2 \int_0^t \langle z^n(x,1,t)), w_j\rangle_\Omega\, ds \\ = \ & k\langle u^n \ln|u^n|, w_j\rangle_\Omega \qquad (3.3)\\ z^n(x,0,t) = \ & u_t^n(x,t) \qquad (3.4) \end{aligned}$$

$$u^n(0) = u_0^n = \sum_{j=1}^{n}(u_0, w_j)w_j \longrightarrow u_0, \text{ in } H_0^2(\Omega) \text{ as } n \longrightarrow +\infty \tag{3.5}$$

$$u_t^n(0) = u_1^n = \sum_{j=1}^{n}(u_1, w_j)w_j \longrightarrow u_1, \text{ in } H_0^2(\Omega) \text{ as } n \longrightarrow +\infty \tag{3.6}$$

and

$$(\tau z_t^n + z_\rho^n, \varphi_j) = 0, 1 \le j \le n \tag{3.7}$$

$$z^n(\rho,0) = z_0^n = \sum_{j=1}^{n}(f_0,\varphi_j)\varphi_j \longrightarrow f_0 \text{ in } H_0^2(\Omega, H^2(0,1)) \text{ as } n \longrightarrow +\infty \tag{3.8}$$

Noticing that from the generalized Hölder inequality, we note that $\frac{p}{2(p+1)} + \frac{1}{2(p+1)} = \frac{1}{2}$, the nonlinear term $(|u_t^n|^p u_{tt}^n, w_j)$ in (3.3) makes sense. Taking into account, the Sobolev embedding, that $H_0^2(\Omega) \hookrightarrow L^{2(p+1)}(\Omega)$ for $0 < p \le \frac{2}{n-3}$ if $n \ge 3$. System (3.3)-(3.8) possesses a local solution in $[0, t_n)$, $0 < t_n < T$ for an arbitrary $T > 0$. Next, we have to prove that $t_n = T$, $\forall n \ge 1$ and that the local solution is uniformly bounded independently of n and t.

First estimate. By Lemma 2.16, since the sequences u_0^n, u_1^n and z_0^n converge, we can find a positive constant C_1 that is independent of n and satisfies

$$\begin{aligned}
& E^n(t)-E^n(0) \leq -\left(\mu_1-\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t \|u_t^n(s)\|^2\,ds \\
& -\left(\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t\int_\Omega |z^n(x,1,t|^2\,dxds \\
& -\frac{1}{2}\int_0^t g(t)\,\|\Delta u^n(s)\|^2\,ds+\frac{1}{2}\int_0^t (g'\circ\Delta u^n)(s)ds \\
\leq\ & -\left(\mu_1-\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t \|u_t^n(s)\|^2\,ds \\
& -\left(\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t\int_\Omega |z^n(x,1,t|^2\,dxds \qquad (3.9)
\end{aligned}$$

As g is a positive non increasing function, so we obtain

$$\begin{aligned}
& E^n(t)+\left(\mu_1-\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t \|u_t^n(s)\|^2\,ds \qquad (3.10) \\
& +\left(\frac{\epsilon}{2\tau}-\frac{|\mu_2|}{2}\right)\int_0^t \|z^n(x,1,t\|^2\,ds \leq E^n(0)\leq C_1
\end{aligned}$$

where

$$\begin{aligned}
E^n(t) &= \frac{1}{p+2}\|u_t^n(t)\|_{p+2}^{p+2}+\frac{1}{2}\left(\|\Delta u_t^n(t)\|_2^2+\left(1-\int_0^t g(s)\,ds\right)\|\Delta u^n(t)\|_2^2\right. \\
&\left. +(g\circ\Delta u^n)(t)\right)-\frac{k}{2}\int_\Omega |u^n|^2\ln|u|^n dx+\frac{k}{4}||u^n||_2^2 \\
&+\frac{\epsilon}{2}\int_\Omega\int_0^1 |z(x,\rho,t)|^2\,d\rho dx
\end{aligned}$$

By utilizing the Logarithmic Sobolev inequality, (3.10) follows.

$$\begin{aligned}
& \|u_t^n(t)\|_{p+2}^{p+2}+\left(l-\frac{ka^2}{2\pi}\right)\|\Delta u^n(t)\|_2^2+\left(\frac{k}{2}+k(1+\ln a)\right)\|u^n(t)\|_2^2 \\
& +(g\circ\Delta u^n)(t)+\|\Delta u_t^n(t)\|_2^2+\int_\Omega\int_0^1 |z^n(x,\rho,t)|^2\,d\rho dx+\int_0^t \|z^n(x,1,t\|^2\,ds \\
\leq\ & C_2+\|u^n\|^2\ln\|u^n\|^2,
\end{aligned}$$

C_2 is a positive constant. Picking a such that

$$e^{-3/2} < a < \sqrt{\frac{2\pi l}{k}}, \qquad (3.11)$$

then

$$l-\frac{ka^2}{2\pi} > 0, \qquad (3.12)$$

and

$$\frac{k}{2}+k(1+\ln a) > 0. \qquad (3.13)$$

Thanks to (A4), this passage is now possible. As a result, we obtain the following:

$$\begin{aligned}
& \|u_t^n(t)\|_{p+2}^{p+2}+\|\Delta u^n(t)\|_2^2+\|u^n(t)\|_2^2+(g\circ\Delta u^n)(t) \\
& +\|\Delta u_t^n(t)\|_2^2+\int_\Omega\int_0^t |z(x,\rho,t)|^2\,d\rho dx+\int_0^t \|z^n(x,1,t\|^2\,ds \\
\leq\ & C\left(1+\|u^n\|^2\ln\|u^n\|^2\right) \qquad (3.14)
\end{aligned}$$

Noticing that

$$u^n(t) = u^n(0) + \int_0^t u_s^n(s)ds$$

Applying Cauchy-Schwarz's, we get

$$\begin{aligned} \|u^n\|^2 &\leq \|u^n(0)\|^2 + 2\left\|\int_0^t u_s^n(s)ds\right\|^2 \\ &\leq \|u^n(0)\|^2 + 2T\int_0^t \|u_s^n(s)\|^2\, ds \end{aligned} \tag{3.15}$$

From (3.14), we get

$$\|u^n\|^2 \leq C\left(1 + \int_0^t \|u^n\|^2 \ln \|u^n\|^2\, ds\right) \tag{3.16}$$

which $C = \max\left\{2TC, 2\|u^n(0)\|^2\right\}$. Logarithmic Gronwall inequality yields

$$\|u^n\| \leq Ce^{CT}. \tag{3.17}$$

Thus, from (3.14), we obtain

$$\begin{aligned} & \|u_t^n(t)\|_{p+2}^{p+2} + \|\Delta u^n(t)\|_2^2 + \|u^n(t)\|_2^2 + (g \circ \Delta u^n)(t) \\ & + \|\Delta u_t^n(t)\|_2^2 + \int_\Omega \int_0^1 |z^n(x,\rho,t)|^2\, d\rho dx + \int_0^t \|z^n(x,1,t\|\, ds \\ \leq\ & C\left(1 + Ce^{CT}\ln(Ce^{CT})\right) = A_1 \end{aligned} \tag{3.18}$$

Then the solution (u^n, z^n) can be extended to $[0,T)$ and we obtain

$$u^n \text{ is bounded in } L^\infty_{loc}(0,\infty, H_0^2(\Omega)) \tag{3.19}$$

$$u_t^n \text{ is bounded in } L^\infty_{loc}(0,\infty, H_0^2(\Omega)) \tag{3.20}$$

$$z^n(x,\rho,t) \text{ is bounded in } L^\infty_{loc}(0,\infty\ ;\ L^2(\Omega \times (0,1)) \tag{3.21}$$

$$z^n(x,1,t) \text{ is bounded in } L^2(\Omega \times (0,T\ )) \tag{3.22}$$

Second estimate: Replacing w_j by $-\Delta w_j$ in (3.3), multiplying by c_t^{jn} and summing over j from 1 to n, it follows that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}[\|\nabla\Delta u^n\|^2 + \|\nabla\Delta u_t^n\|^2] - \int_\Omega |u_t^n|^p\, u_{tt}^n \Delta u_t^n dx \\ & - \int_0^t g(t-s)\int_\Omega \nabla\Delta u^n(t)\nabla\Delta u_t^n(s)dxds \\ & + \int_\Omega \nabla z^n(x,1,t)\nabla u_t^n dx + \mu_1\|\nabla u_t^n\|^2 \\ =\ & -k\int_\Omega \Delta u_t^n u^n \ln|u^n|\, dx \end{aligned} \tag{3.23}$$

The Green's formula gives

$$\begin{aligned} & -\int_\Omega |u_t^n|^p\, u_{tt}^n\ \Delta u_t^n dx \\ =\ & \frac{d}{dt}\int_\Omega |u_t^n|^p\, |\nabla u_t^n|^2\, dx - (p+1)\int_\Omega |u_t^n|^p\, \nabla u_{tt}^n \nabla u_t^n dx \end{aligned} \tag{3.24}$$

Substituting φ_j by $-\Delta\varphi_j$ in (3.7), multiplying by d^{jn} and summing up over j from 1 to n, it follows that

$$\tau\int_\Omega \nabla z_t^n \nabla z^n dx + \int_\Omega \nabla z_\rho^n \nabla z^n dx = 0 \tag{3.25}$$

Then, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\left\|\nabla z^n\right\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\left\|\nabla z^n\right\|_2^2 = 0 \tag{3.26}$$

We integrate over $(0,1)$ to find that

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 \left\|\nabla z^n(x,\rho,t)\right\|_2^2 d\rho + \frac{1}{2}\left\|\nabla z^n(x,1,t)\right\|_2^2 - \frac{1}{2}\left\|\nabla u_t^n(t)\right\|_2^2 = 0 \tag{3.27}$$

Considering Lemma 2.2, by combining (3.23) and (3.27), we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\Big[\left(1-\int_0^t g(s)\,ds\right)\left\|\nabla\Delta u^n\right\|^2 + \left\|\nabla\Delta u_t^n\right\|^2 + (g\circ\nabla\Delta u^n) \\ & + \tau\int_0^1 \left\|\nabla z^n\right\|^2 d\rho + 2\int_\Omega \left|u_t^n(t)\right|^p \left|\nabla u_t^n\right|^2 dx\Big] + \frac{1}{2}\left\|\nabla z^n(x,1,t)\right\|_2^2 \\ = \; & (p+1)\int_\Omega \left|u_t^n\right|^p \nabla u_{tt}^n \nabla u_t^n dx - \mu_2\int_\Omega \nabla z^n(x,1,t)\nabla u_t^n dx - \mu_1\left\|\nabla u_t^n\right\|^2 \\ & + \frac{1}{2}\left\|\nabla u_t^n(x,t)\right\|^2 - \frac{1}{2}g(t)\left\|\nabla\Delta u^n\right\|^2 \\ & + \frac{1}{2}(g'\circ\nabla\Delta u^n) - k\int_\Omega \Delta u_t^n u^n \ln\left|u^n\right| dx \end{aligned} \tag{3.28}$$

By applying Young's inequality and the initial estimate of (3.18), we obtain for $\eta > 0$

$$\begin{aligned} (p+1)\int_\Omega \left|u_t^n\right|^p \nabla u_{tt}^n \nabla u_t^n dx & \leq (p+1)C_2^{p/(p+2)+1/2}\left\|\nabla u_{tt}^n\right\|_2 \\ & \leq \eta\left\|\nabla u_{tt}^n\right\|^2 + \frac{(p+1)^2 C_2^{2p/(p+2)+1}}{4\eta} \end{aligned} \tag{3.29}$$

Utilizing Young's inequality, we have

$$\begin{aligned} \int_\Omega \nabla z^n(x,1,t)\nabla u_t^n dx & \leq \frac{1}{4\eta}\int_\Omega \left|\nabla u_t\right|^2 dx + \eta\int_\Omega \left|\nabla z^n(x,1,t)\right|^2 dx \\ & \leq c(\eta) + \eta\int_\Omega \left|\nabla z^n(x,1,t)\right|^2 dx \end{aligned} \tag{3.30}$$

Using Lemma (2.5), we can estimate the last term on the right-hand side of (3.28) with $\epsilon_0 = 1/2$ and the embedding inequalities, as well as Young's and Cauchy-Schwarz's inequalities, as follows

$$\begin{aligned} \left|k\int_\Omega \Delta u_t^n u^n \ln\left|u^n\right| dx\right| & \leq k\int_\Omega \left|\Delta u_t^n\right|\left(\left|u^n\right|^2 + d_{\epsilon_0}\sqrt{\left|u^n\right|}\right)dx \\ & \leq k\left(\eta\int_\Omega \left|\Delta u_t^n\right|^2 dx + \frac{1}{4\eta}\int_\Omega \left(\left|u^n\right|^2 + d_{\epsilon_0}\sqrt{\left|u^n\right|}\right)^2 dx\right) \\ & \leq k\eta\int_\Omega \left|\Delta u_t^n\right|^2 dx + \frac{c}{4\eta}\left(\int_\Omega \left|u^n\right|^4 dx + \int_\Omega \left|u^n\right| dx\right) \\ & \leq k\eta\left\|\Delta u_t^n\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^n\right\|^4 + \left\|u^n\right\|\right), \quad \eta > 0. \end{aligned} \tag{3.31}$$

Taking into account (3.29)-(3.31), (3.28) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_{0}^{t} g\left(s\right)ds\right)\left\|\nabla\Delta u^{n}\right\|^{2}+\left\|\nabla\Delta u_{t}^{n}\right\|^{2}+\left(g\circ\nabla\Delta u^{n}\right)\right.\\
&\left.+\tau\int_{0}^{1}\left\|\nabla z^{n}\right\|^{2}d\rho+2\int_{\Omega}\left|u_{t}^{n}(t)\right|^{p}\left|\nabla u_{t}^{n}\right|^{2}dx\right]+\left(\frac{1}{2}-\eta\right)\left\|\nabla z^{n}(x,1,t)\right\|_{2}^{2}\\
\leq\ &\eta\left\|\nabla u_{tt}^{n}\right\|^{2}-\frac{1}{2}g(t)\left\|\nabla\Delta u^{n}\right\|^{2}+\frac{1}{2}(g'\circ\nabla\Delta u^{n})\\
&+c(\eta)+k\eta\left\|\Delta u_{t}^{n}\right\|^{2}+\frac{c}{4\eta}\left(\left\|\nabla u^{n}\right\|^{4}+\left\|u^{n}\right\|\right).
\end{aligned}\tag{3.32}$$

Multiplying (3.3) by c_{tt}^{jn} and summing over j from 1 to n, it follows that

$$\int_{\Omega}\left|u_{t}^{n}\right|^{p}\left|u_{tt}^{n}\right|^{2}dx+\left\|\Delta^{2}u_{tt}^{n}\right\|^{2}\tag{3.33}$$

$$\begin{aligned}
=\ &-\int_{\Omega}\Delta^{2}u^{n}u_{tt}^{n}dx+\int_{0}^{t}g\left(t-s\right)\int_{\Omega}\Delta u^{n}\left(s\right)\Delta u_{tt}^{n}(t)dxds\\
&+k\int_{\Omega}u_{tt}^{n}u^{n}\ln\left|u^{n}\right|dx-\mu_{1}\int_{\Omega}u_{t}^{n}u_{tt}^{n}dx-\mu_{2}\int_{\Omega}z^{n}(x,1,t)u_{tt}^{n}dx
\end{aligned}$$

Differentiating (3.7) with respect to t yields

$$(\tau z_{tt}^{n}+z_{t\rho}^{n},\varphi_{j})=0\tag{3.34}$$

Multiplying by d^{jn} and summing up over j from 1 to k, it follows that

$$\frac{\tau}{2}\frac{d}{dt}\left\|z_{t}^{n}\right\|_{2}^{2}+\frac{1}{2}\frac{d}{d\rho}\left\|z_{t}^{n}\right\|_{2}^{2}=0.\tag{3.35}$$

Integrating over $(0,1)$ with respect to ρ, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\int_{0}^{1}\left\|z_{t}^{n}\right\|_{2}^{2}d\rho+\frac{1}{2}\left\|z_{t}^{n}(x,1,t)\right\|_{2}^{2}-\frac{1}{2}\left\|u_{tt}^{n}(x,t)\right\|_{2}^{2}=0\tag{3.36}$$

Adding (3.33) and (3.36), we have

$$\begin{aligned}
&\int_{\Omega}\left|u_{t}^{n}\right|^{p}\left|u_{tt}^{n}\right|^{2}dx+\left\|\Delta^{2}u_{tt}^{n}\right\|^{2}+\frac{\tau}{2}\frac{d}{dt}\int_{0}^{1}\left\|z_{t}^{n}\right\|_{2}^{2}d\rho+\frac{1}{2}\left\|z_{t}^{n}(x,1,t)\right\|_{2}^{2}\\
=\ &-\int_{\Omega}\Delta^{2}u^{n}u_{tt}^{n}dx+\int_{0}^{t}g\left(t-s\right)\int_{\Omega}\Delta u^{n}\left(s\right)\Delta u_{tt}^{n}(t)dxds+k\int_{\Omega}u_{tt}^{n}u^{n}\ln\left|u^{n}\right|dx\\
&-\mu_{1}\int_{\Omega}u_{t}^{n}u_{tt}^{n}dx-\mu_{2}\int_{\Omega}z^{n}(x,1,t)u_{tt}^{n}dx+\frac{1}{2}\left\|u_{tt}^{n}(x,t)\right\|_{2}^{2}
\end{aligned}\tag{3.37}$$

By using Young's inequality, the right hand side of (3.37) can be estimated as follows:

$$\int_{\Omega}\Delta^{2}u^{n}u_{tt}^{n}dx\leq\eta\left\|\nabla u_{tt}^{n}\right\|^{2}+\frac{1}{4\eta}\left\|\nabla\Delta u^{n}\right\|^{2},\quad\eta>0,\tag{3.38}$$

and

$$\begin{aligned}
&\int_{0}^{t}g\left(t-s\right)\int_{\Omega}\Delta u^{n}\left(s\right)\Delta u_{tt}^{n}(t)dxds\\
=\ &-\int_{0}^{t}g\left(t-s\right)\int_{\Omega}\nabla\Delta u^{n}\left(s\right)\nabla u_{tt}^{n}(t)dxds\\
\leq\ &\eta\left\|\nabla u_{tt}^{n}\right\|^{2}+\frac{\beta^{2}}{4\eta}(1+\eta)\left|\nabla\Delta u^{n}\right|^{2}+\frac{\beta}{4\eta}(1+\frac{1}{\eta})(g\circ\nabla\Delta u^{n}).
\end{aligned}\tag{3.39}$$

Employing Young's inequality, we get

$$\begin{aligned}\mu_1\int_\Omega u_t^n u_{tt}^n dx &\leq \eta\left\|u_{tt}^n\right\|^2+\frac{\mu_1^2}{4\eta}\left\|u_t^n\right\|^2\\ &\leq \eta C_s^2\left\|\nabla u_{tt}^n\right\|^2+\frac{C_s^2\mu_1^2}{4\eta}\left\|\nabla u_t^n\right\|^2\\ &\leq \eta C_s^2\left\|\nabla u_{tt}^n\right\|^2+C(\eta), \end{aligned}\tag{3.40}$$

and

$$\int_\Omega z^n(x,1,t)u_{tt}^n dx\leq \eta C_s^2\mu_1\left\|\nabla u_{tt}^n\right\|^2+\frac{1}{4\eta}\left\|z^n(x,1,t)\right\|^2 \tag{3.41}$$

By combining (2.12) with $\epsilon_0=\frac{1}{2}$ and applying the Cauchy Schwarz, Young, and embedding inequalities to the last term on the right-hand side of (3.37), we have

$$\begin{aligned}k\int_\Omega u_{tt}^n u^n\ln\left|u^n\right|dx &\leq c\int_\Omega u_{tt}^k\,(\left|u^n\right|^2+d\sqrt{u^n})dx\\ &\leq c\left(\delta\int_\Omega\left|u_{tt}^n\right|^2dx+\frac{1}{4\delta}\int_\Omega(\left|u^n\right|^2+d\sqrt{u^n})^2dx\right)\\ &\leq c\delta\left\|\nabla u_{tt}^n\right\|^2+\frac{c}{4\delta}\left(\int_\Omega\left|u^n\right|^4dx+\int_\Omega|u^n|dx\right)\\ &\leq c\delta\;\left\|\nabla u_{tt}^n\right\|^2+\frac{c}{4\delta}\left(\left\|\Delta u^n\right\|^4+\left\|u^n\right\|\right),\quad \eta\;>\;0.\end{aligned}\tag{3.42}$$

Considering of (3.38)-(3.42), (3.37) gives

$$\begin{aligned}&\int_\Omega\left|u_t^n\right|^p\left|u_{tt}^n\right|^2dx+\left\|\Delta^2u_{tt}^n\right\|^2+\frac{\tau}{2}\frac{d}{dt}\int_0^1\left\|z_t^n\right\|_2^2d\rho+\frac{1}{2}\left\|z_t^n(x,1,t)\right\|_2^2\\ &+(1-\eta(2+C_s^2+C_s^2\mu_1)-c\delta)\left\|\nabla u_{tt}^n\right\|^2\\ \leq\;&\frac{1}{4\eta}(1+\beta^2(1+\eta)\left\|\nabla\Delta u^n\right\|+\frac{\beta}{4\eta}(1+\frac{1}{\eta})(g\circ\nabla\Delta u^n)\\ &+\frac{1}{4\eta}\left\|z^n(x,1,t)\right\|^2+C(\eta)+\frac{c}{4\delta}(\left\|\Delta u^n\right\|^4+\left\|u^n\right\|)\end{aligned}\tag{3.43}$$

Thus, using (3.32) and (3.34), we have

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left[(1-\int_0^t g\left(s\right)ds)\left\|\nabla\Delta u^n\right\|^2+\left\|\nabla\Delta u_t^n\right\|^2+(g\circ\nabla\Delta u^n)\right.\\ \left.+\;\;\tau\int_0^1\left\|\nabla z^n(x,\rho,t\right\|^2d\rho+2\int_\Omega\left|u_t^n(t)\right|^p\left|\nabla u_t^n\right|^2dx\right]\\ +(\frac{1}{2}-\eta)\left\|\nabla z^n(x,1,t)\right\|_2^2+\big(1-\eta(3+C_s^2+C_s^2\mu_1\big)\\ -c\delta)\left\|\nabla u_{tt}^n\right\|^2+\int_\Omega\left|u_t^n\right|^p\\ \leq-\frac{1}{2}g(t)\left\|\nabla\Delta u_t^n\right\|^2+\frac{1}{2}(g'\circ\nabla\Delta u^n)+\frac{1}{4\eta}(1+\beta^2(1+\eta)\left\|\nabla\Delta u^n\right\|\\ +\frac{\beta}{4\eta}(1+\frac{1}{\eta})(g\circ\nabla\Delta u^n)+\frac{1}{4\eta}\left\|z^n(x,1,t)\right\|^2\\ +C(\eta)+\frac{c}{4\delta}\left(\left\|\Delta u^n\right\|^4+\left\|u^n\right\|\right)\end{gathered}\tag{3.44}$$

Choosing $\eta < 0$ and δ sufficiently small such that $1-\eta(3+C_s^2+C_s^2\mu_1)-c\delta>0$, and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\Big[(1-\int_0^t g\left(s\right)ds)\left\|\nabla\Delta u^n\right\|^2+\left\|\nabla\Delta u_t^n\right\|^2+\left(g\circ\nabla\Delta u^n\right)\\
&+\ \tau\int_0^1\left\|\nabla z^n(x,\rho,t\right\|^2d\rho+2\int_\Omega\left|u_t^n(t)\right|^\rho\left|\nabla u_t^n\right|^2dx\Big]\\
&+(\frac{1}{2}-\eta)\int_0^t\left\|\nabla z^n(x,1,t)\right\|_2^2ds\\
&+(1-\eta(3+C_s^2+C_s^2\mu_1)-c\delta)\int_0^t\left\|\nabla u_{tt}^n\right\|^2ds\\
&+\int_0^t\int_\Omega\left|u_t^n\right|^p\left|u_{tt}^n\right|^2dxds+\int_0^t\left\|\Delta^2u_{tt}^n\right\|^2ds\\
\leq\ &-\frac{1}{2}g(t)\int_0^t\left\|\nabla\Delta u_t^n\right\|^2ds+\frac{1}{2}\int_0^t(g'\circ\nabla\Delta u^n)ds\\
&+\frac{1}{4\eta}(1+\beta^2(1+\eta)\int_0^t\left\|\nabla\Delta u^n\right\|ds+\frac{\beta}{4\eta}(1+\frac{1}{\eta})\int_0^t(g\circ\nabla\Delta u^n)ds\\
&+\frac{1}{4\eta}\int_0^t\left\|z^n(x,1,t)\right\|^2ds+C(\eta)T+\frac{c}{4\delta}\int_0^t\left(\left\|\Delta u^n\right\|^4+\left\|u^n\right\|\right)ds
\end{aligned}$$

Taking $g_1=\min\left\{h(t)\text{ for all }t\geq t_0\right\}$ and using Gronwall's Lemma, we have

$$\begin{aligned}
&\left\|\nabla\Delta u^n\right\|^2+\left\|\nabla\Delta u_t^n\right\|^2+\left(g\circ\nabla\Delta u^n\right)+\int_0^1\left\|\nabla z^n(x,\rho,t\right\|^2d\rho\\
&+\int_0^t\left\|\nabla z^n(x,1,t)\right\|_2^2ds+\int_0^t\left\|\nabla z_t^n(x,\rho,t\right\|^2ds+\int_0^t\left\|\nabla z_t^n(x,\rho,t\right\|^2ds\\
&+\int_0^t\left\|\nabla u_{tt}^n\right\|^2ds\leq C_3
\end{aligned}\tag{3.45}$$

From (3.18) and (3.45), we have

$$u^n\text{ is uniformly bounded in }L^\infty(0,T,H_0^2\left(\Omega\right))\tag{3.46}$$

$$u_t^n\text{ is uniformly bounded in }L^\infty(0,T,L^{p+2}(\Omega))\cap L^\infty(0,T;H_0^2(\Omega))\tag{3.47}$$

$$u_{tt}^n\text{ is bounded in }L^2(0,T;H_0^2(\Omega))\tag{3.48}$$

which implies the existence of a subsequence of u^n (still denoted by u^n), such that

$$u^n\rightarrow u\text{ weakly star in }L^\infty(0,T;H^2\left(\Omega\right))\tag{3.49}$$

$$u_t^n\rightarrow u_t\text{ weakly star in }L^\infty(0,T;L^{p+2}(\Omega))\cap L^\infty(0,T;H_0^2\left(\Omega\right))\tag{3.50}$$

$$u^n\rightarrow u\text{ weakly in }L^2(0,T;H_0^2\left(\Omega\right))\tag{3.51}$$

$$u_t^n\rightarrow u_t\text{ weakly in }L^2(0,T;L^{p+2}(\Omega))\cap L^2(0,T;H_0^2\left(\Omega\right))\tag{3.52}$$

$$u_{tt}^n\rightarrow u_{tt}\text{ weakly in }L^2(0,T;H_0^2\left(\Omega\right))\tag{3.53}$$

The nonlinear terms analysis:
(1) Term $(u^n\ln|u^n|)$: From (3.45), utilizing the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)(\Omega\subset\mathbb{R}^2)$, (u^n) is bounded in $L^\infty(0,T;H_0^2(\Omega))$ which implies the boundness of (u^n) in $L^2\left(\Omega\times(0,T)\right)$. In a similar way (u_t^n) is bounded in $L^2\left(\Omega\times(0,T)\right)$. Next, by using Aubin-Lions theorem, we have a subsequence such that

$$u^n\longrightarrow u\text{ strongly in }L^2\left(\Omega\times(0,T)\right)\tag{3.54}$$

which implies

$$u^n \longrightarrow u \quad \text{a.e.} \quad \text{in } \Omega \times (0,T) \tag{3.55}$$

Since of the maps $s \longrightarrow ks \ln |s|$ is continuous, the following convergence hold:

$$ku^n \ln |u^n| \longrightarrow ku \ln |u| \quad \text{a.e.} \quad \text{in } \Omega \times (0,T). \tag{3.56}$$

From the embedding of $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)(\Omega \subset \mathbb{R}^2)$, we confirm that $(ku^n \ln |u^n|)$ is bounded in $L^\infty(\Omega \times (0,T))$. Now, taking into account the Lebesgue bounded convergence theorem, we get

$$ku^n \ln |u^n| \longrightarrow ku \ln |u| \text{ strongly in } L^2\left(0,T\ ; L^2(\Omega)\right) \tag{3.57}$$

(2) Term $|u_t^n|^p\, |u_t^n|$: From (3.45), (u_t^n) is uniformly bounded in $L^\infty(0,T; H^2(\Omega))$ which implies utilizing the the boundness of (u_t^n) in $L^\infty(\Omega \times (0,T)) \subset L^2(\Omega \times (0,T))$.Also, we know that (u_{tt}^n) is bounded in $L^2\left((0,T)\,; H_0^2(\Omega)\right)$, which implies that$(u_{tt}^n)$ is bounded in $L^2(\Omega \times (0,T))$.
From the first estimate in (3.18) and Lemma 2.6, we have

$$\begin{aligned}
\left\| |u_t^n|^p\, u_t^n \right\|_{L^2(0,T\ ;L^2(\Omega))} &= \int_0^T \|u_t^n\|_{2(p+1)}^{2(p+1)}\, dt \\
&\leq \left(\frac{C_s}{\sqrt{\lambda}}\right)^{2(p+1)} \int_0^T \|\Delta u_t^n\|_2^{2(p+1)}\, dt \\
&\leq \left(\frac{C_s}{\sqrt{\lambda}}\right)^{2(p+1)} C_3^{2(p+1)} T
\end{aligned} \tag{3.58}$$

Employing Aubin-Lions theorem, (Lions [10]), there exists a subsequence still denoted by $\{u^n\}$, such that

$$u_t^n \longrightarrow u_t \text{ strongly in } L^2\left(0,T\ ; L^2(\Omega)\right), \tag{3.59}$$

which implies

$$u_t^n \longrightarrow u_t \text{ a.e in } \Omega \times (0,T) \tag{3.60}$$

Thus,

$$|u_t^n|^p\, u_t^n \longrightarrow |u_t|^p\, u_t \quad \text{a.e in } \Omega \times (0,T) \tag{3.61}$$

Therefore, applying (3.59)-(3.61) and Lion's Lemma, we obtain

$$|u_t^n|^p\, u_t^n \longrightarrow |u_t|^p\, u_t \text{ weakly in } L^2\left(0,T\ ; L^2(\Omega)\right). \tag{3.62}$$

Multiplying (3.3) by $\theta(t) \in D(0,T)$ and integrate on $(0,T)$, we have

$$\begin{aligned}
& \frac{-1}{p+1} \int_0^T (|u_t^n(t)|^p\, u_t^n(t), w_j)\theta^{'}(t)dt + \int_0^T (\Delta u^n, \Delta w_j)\theta(t)dt \\
& + \int_0^T (\Delta u_{tt}^n, \Delta w_j)\theta(t)dt - \int_0^T \int_0^t g\left(t-s\right)(\Delta u^n\left(s\right), \Delta w_j)\theta(t)dsdt \\
& + \mu_1 \int_0^T (u_t^n, w_j)\theta(t)dt + \mu_2 \int_0^T \int_0^t (z^n(x,1,t), w_j)\theta(t)dsdt \\
= \ & k \int_0^T (u^n(t) \ln |u^n(t)|\,, w_j)\theta(t))dt
\end{aligned} \tag{3.63}$$

multiplying (3.7) by $\theta(t) \in D(0,T)$ and integrate on $(0,T) \times (0,1)$, to obtain

$$\int_0^T \int_0^1 (\tau z_t^n + z_\rho^n, \phi_j) d\rho dt = 0 \tag{3.64}$$

Using the convergence of (3.49)-(3.53) and (3.62) to pass to the limit in (3.63) and (3.64);

$$\begin{aligned}&\frac{-1}{p+1}\int_0^T (|u_t|^p u_t, w)\theta'(t)dt + \int_0^T (\Delta u, \Delta w)\theta(t)dt\\&+\int_0^T (\Delta u_{tt}, \Delta w)\theta(t)dt - \int_0^T\int_0^t g(t-s)(\Delta u(s), \Delta w)\theta(t)dsdt\\&+\mu_1\int_0^T (u_t, w)\theta(t)dt + \mu_2\int_0^T (z(x,1,t), w)\theta(t)dt\\=\;& k\int_0^T (u(t)\ln|u(t)|, w)\theta(t)dt,\end{aligned}$$

and

$$\int_0^T\int_0^1 (\tau z_t + z_\rho, \phi)d\rho dt = 0.$$

Integrating on $(0,T)$, we have

$$\begin{aligned}&\int_0^T \left(|u_t|^p u_{tt} + \Delta^2 u + \Delta^2 u_{tt}\right)ds - \int_0^t g(t-s)\Delta^2 u(s)ds\\&+\mu_1 u_t + z(x,1,t))\theta(t)dt\\=\;& k\int_0^T (u(t)\ln|u(t)|, w)\theta(t)dt\end{aligned}$$

The proof is completed. □

4. Global existence

In this section we show that the solution for the problem (1.1) is global. First, we define the the following functionals

$$I(t) = \left(1-\int_0^t g(s)ds\right)\|\Delta u\|^2 + \|\Delta u_t\|^2 + (g\circ\Delta u)(t) - 3k\int_\Omega u^2\ln|u|\,dx \tag{4.1}$$

$$\begin{aligned}J(t) &= \frac{1}{2}\left(\left(1-\int_0^t g(s)ds\right))\|\Delta u\|^2 + \|\Delta u_t\|^2 + (g\circ\Delta u)(t) - k\int_\Omega u^2\ln|u|\,dx\right)\\&\quad+\frac{k}{4}\|u\|^2\\&= \frac{1}{3}[(1-\int_0^t g(s)ds\;)\|\Delta u\|^2 + \|\Delta u_t\|^2 + (g\circ\Delta u)(t)] + \frac{k}{4}\|u\|^2 + \frac{1}{6}I(t).\end{aligned} \tag{4.2}$$

Noticing that

$$E(t) = J(t) + \frac{1}{p+2}\|u_t\|_{p+2}^{p+2} + \frac{\epsilon}{2}\int_0^1 \|z(x,\rho,t\|^2 d\rho. \tag{4.3}$$

Lemma 4.1 *From [2], we have the following inequalities*

$$-kd_0\sqrt{|\Omega|\,c_*^3}\,\|\Delta u\|_2^{3/2} \le k\int_\Omega u^2\ln|u|\,dx \le kc_*^3\|\Delta u\|_2^3, \quad \forall u\in H_0^2(\Omega), \tag{4.4}$$

where $d_0 = \sup_{0<s<1}\sqrt{s}\,|\ln s| = \frac{2}{e}, |\Omega|$ *is the Lebesgue measure of* Ω *and* c_* *is the smallest embedding constant;*

$$\left(\int_\Omega |u|^3 dx\right)^{1/3} \le c_*\|\Delta u\|_2, \quad \forall u\in H_0^2(\Omega), \tag{4.5}$$

c_* *exists thanks to the embedding of* $H_0^2(\Omega)\hookrightarrow L^\infty(\Omega)$ *and* $\Omega\subset\mathbb{R}^2$

Lemma 4.2 *Suppose that* ***(A1)-(A3)*** *. Let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *such that*

$$I(0) \ > 0 \ \text{ and } \sqrt{54}c_*^3 \left(\frac{E(0)}{l}\right)^{1/2} \ < l \tag{4.6}$$

Then

$$I(t) \ > \ 0, \ \forall t \in [0, T). \tag{4.7}$$

Proof: The proof is similar to [2], [12], hence, we omit it. Consequently, we completed the proof of Theorem 2.1 □

5. Asymptotic Behavior

This section presents the asymptotic behavior result for our problem by constructing a suitable Lyapunov functional. Firstly, we define the functional L as follows

$$L(t) = NE(t) + N_1\psi(t) + \chi(t) + N_2\phi(t), \tag{5.1}$$

where N, N_1 and N_2 are positive real numbers. Besides, let define:

$$\psi(t) = \frac{1}{p+1}\int_\Omega |u_t|^p u_t u dx + \int_\Omega \Delta u \Delta u_t dx, \tag{5.2}$$

$$\chi(t) = -\int_\Omega \left(\Delta^2 u_t + \frac{1}{p+1}|u_t|^p u_t\right)\int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{5.3}$$

$$\phi(t) = \int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx. \tag{5.4}$$

The following lemmas are required in order to obtain our primary result:

Lemma 5.1 *Assuming that* ***(A1)-(A3)*** *and* (4.6) *hold, let* $\epsilon_0 \in (0,1)$ *and*

$$0 < E(0) < \frac{el\pi}{4}. \tag{5.5}$$

Then, the functional $L(t)$*, for* N *sufficiently large, satisfies*

$$\lambda_0 E(t) \le L(t) \le \lambda_1 E(t), \qquad \forall t \ge 0, \tag{5.6}$$

where λ_0 *and* λ_1 *are positive constants depending on* N_1*,* N_2 *and* N*. Hence,* $L \sim E$ *and for any* $t_0 > 0$*, there exists a positive constant* m_1*, such that*

$$L'(t) \le -m_1E(t) + c^*(g \circ \Delta u)(t) + c_{\epsilon_0}(g\circ\Delta u)^{1/(1+\epsilon_0)}(t), \qquad \forall t \ge t_{\ 0} \tag{5.7}$$

Proof: The proof is similar to [2], [12], so we leave it here. □

Lemma 5.2 *For* $\eta > 0$*, the functional* $\psi(t)$ *satisfies*

$$\begin{aligned}\psi'(t) \le{}& \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} - \left(1 - \beta - \eta - \frac{2\eta C_s^2}{\lambda}\mu_1\right)\|\Delta u\|^2 \\ &+\frac{\beta}{4\eta}(g\circ\Delta u) + \left(\frac{\mu_1 C_s^2}{4\eta} + 1\right)\|\Delta u_t\|^2 + \frac{1}{4\eta}\|z(x,1,t\|^2 \\ &+k\int_\Omega u^2 \ln|u|\,dx\end{aligned} \tag{5.8}$$

for any solution (u, z) *of problem* (2.15).

Proof: Taking the derivative of $\psi(t)$ from equation (2.15) and integrating by parts, we obtain

$$\begin{aligned}\psi^{'}(t) \quad = \quad & \frac{1}{p+1}\left\|u_t\right\|_{p+2}^{p+2}+\left\|\Delta u_t\right\|^2-\left\|\Delta u\right\|^2+\int_\Omega \Delta u\int_0^t g(t-s)\Delta u(s)dsdx \\ & -\mu_1\int_\Omega uu_tdx-\mu_1\int_\Omega z(x,1,t)udx+k\int_\Omega u^2\ln\left|u\right|dx \qquad (5.9)\end{aligned}$$

Using Sobolev embedding and Young's inequality, from (5.9) we get

$$\left|\int_\Omega \Delta u\int_0^t g(t-s)\Delta u(s)dsdx\right| \leq (\beta+\eta)\left\|\Delta u\right\|^2+\frac{\beta}{4\eta}\left(g\circ\Delta u\right), \qquad (5.10)$$

Since

$$\left|\mu_1\int_\Omega uu_tdx\right| \leq \eta\frac{\mu_1C_s^2}{\lambda}\left\|\Delta u\right\|^2+\frac{C_s^2}{4\eta}\left\|\Delta u_t\right\|^2, \qquad (5.11)$$

and

$$\left|\mu_1\int_\Omega z(x,1,t)udx\right| \leq \eta\frac{\mu_1C_s^2}{\lambda}\left\|\Delta u\right\|^2+\frac{1}{4\eta}\left\|z(x,\rho,t)\right\|^2. \qquad (5.12)$$

The estimate of (5.8) was obtained by substituting (5.10)-(5.12) into (5.9). Therefore, we have completed the proof.. ☐

Lemma 5.3 *The function,$\chi(t)$ satisfies*

$$\begin{aligned}&\chi'(t)\leq\delta(2\beta^2+1+\frac{1}{4})\left\|\Delta u\right\|^2+\left(\delta+\frac{\delta a_0}{p+1}-g_0\right)\left\|\Delta u_t\right\|^2 \\ &-\frac{1}{p+1}g_0\left\|u_t\right\|_{p+2}^{p+2}+\beta\left(2\delta+\frac{1}{4\delta}+\frac{\mu_1C_s^2}{2\delta\lambda}+\frac{c}{\delta\beta}\right)\left(g\circ\Delta u\right)(t) \\ &-\frac{g(0)}{4\delta\lambda}\left(1+\frac{C_s^2}{p+1}\right)\left(g'\circ\Delta u\right)(t)+\mu_1\delta\left\|u_t\right\|^2+\mu_1\delta\left\|z(x,1,t\right\|^2 \\ &+c_{\epsilon_0,\delta}\left(g\circ\Delta u\right)^{1/(1+\epsilon_0)}(t)\end{aligned}$$

for any solution (u,z) to problem (2.15), and any $\delta>0$, where $\int_0^t g(s)ds\geq\int_0^{t_0}g(s)ds=g_0$, $\forall t\geq t_0$.

Proof: The first equation of (2.15) together the Leibnitz formula, gives

$$\begin{aligned}
\chi'(t) &= -\int_{\Omega}\left(\int_0^t g(t-s)\Delta u(s))ds\right)\left(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s)ds\right)dx \\
&\quad +\int_{\Omega}\Delta u(t)\left(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s)ds\right)dx \\
&\quad +\mu_1\int_{\Omega}u_t(t)\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dx \\
&\quad +\int_{\Omega}z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\quad -\int_{\Omega}\Delta u_t(t)\left(\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))ds\right)dx \\
&\quad -\frac{1}{p+1}\int_{\Omega}|u_t|^p u_t\int_0^t g^{'}(t-s)(u(t)-u(s)dsdx \\
&\quad -k\int_{\Omega}u\ln|u|\int_0^t g(t-s)(u(t)-u(s)dsdx \\
&\quad -\int_0^t g(s)ds\,\|\Delta u_t\|^2-\frac{1}{p+1}\int_0^t g(s)ds\,\|u_t\|_{p+2}^{p+2} \\
&= I_1+...+I_7-\int_0^t g(s)ds\,\|\Delta u_t\|^2-\frac{1}{p+1}\int_0^t g(s)ds\,\|u_t\|_{p+2}^{p+2}
\end{aligned}$$

Now ,we will estimate $I_1,...,I_7$ Hence, for $\delta>0$, we obtain

$$\begin{aligned}
|I_1| &\le \delta\int_{\Omega}(\int_0^t g(t-s)\Delta u(s)ds)^2dx \\
&\quad +\frac{1}{4\delta}\int_{\Omega}(\int_0^t g(t-s)\left|\Delta u(t)-\Delta u(s)\right|ds)^2dx \\
&\le 2\delta(\int_0^t g(s)ds)^2\|\Delta u\|^2+(2\delta+\frac{1}{4\delta})\int_0^t g(s)ds(g\circ\Delta u)(t) \\
&\le 2\delta\beta^2\|\Delta u\|^2+\beta(2\delta+\frac{1}{4\delta})(g\circ\Delta u)(t).
\end{aligned}$$

In a similar way

$$|I_2|\le\delta\|\Delta u\|^2+\frac{\beta}{4\delta}(g\circ\Delta u)(t),$$

$$|I_3|\le\delta\mu_1C_s^2\|\Delta u_t\|^2+\frac{\beta\mu_1C_s^2}{4\delta\lambda}(g\circ\Delta u)(t),$$

$$|I_4|\le\delta\|z(x,1,t\|^2+\frac{\beta\mu_1C_s^2}{4\delta}(g\circ\Delta u)(t),$$

$$|I_5|\le\delta\|\Delta u_t\|^2-\frac{g(0)}{4\delta\lambda}(g'\circ\Delta u)(t),$$

$$|I_6|\le\frac{\delta a_0}{p+1}\|\Delta u_t\|^2-\frac{g(0)C_s^2}{4\delta\lambda(p+1)}(g'\circ\Delta u)(t),$$

where $a_0=C_s^{2(p+1)}(2E(0))^p$.

Applying (2.12) for $s = |u|$ and employing the embedding $H_0^2(\Omega) \hookrightarrow L^\infty(\Omega)$, for any $\delta_* > 0$ and any $\epsilon_0 \in (0,1)$, we estimate I_7 as follows

$$\begin{aligned}
|I_7| &\leq -k\int_\Omega u\ln|u|\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\leq k\int_\Omega \left(u^2 + d_{\epsilon_0}|u|^{1-\epsilon_0}\right)\left|\int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\
&\leq c\int_\Omega |u|^2\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|dx + \delta_*\int_\Omega u^2dx \\
&\quad +c_{\epsilon_0,\delta_*}\int_\Omega \left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^{2/(1+\epsilon_0)}dx \\
&\leq c\delta_*\|\Delta u\|^2 + \frac{c}{\delta_*}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2 dx \\
&\quad +c_{\epsilon_0,\delta_*}\int_\Omega \left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^{2/(1+\epsilon_0)}dx,
\end{aligned}$$

Putting $\delta/4 = c\delta_*$ and utilizing Hölder's inequality, from Lemma (2.6) we have

$$\begin{aligned}
&-k\int_\Omega u\ln|u|\int_0^t g(t-s)(u(t)-u(s)dsdx \\
&\leq \frac{\delta}{4}\|\Delta u\|^2 + \frac{c}{\delta}(g\circ\Delta u)(t) + c_{\epsilon_0,\delta}(g\circ\Delta u)^{1/(1+\epsilon_0)}(t) \qquad (5.13)
\end{aligned}$$

□

Lemma 5.4 *The functional $\phi(t)$ satisfies*

$$\phi'(t) \leq \frac{-c}{\tau}\int_\Omega z^2(x,1,t)dx + \frac{1}{\tau}\int_\Omega u_t^2(x,t)dx - 2\phi(t) \qquad (5.14)$$

where c is a positive constant.

The proof of Theorem (2.2) is given as follows

Proof: Using (2.17), (5.1), (5.8), (5.13) and (5.14), we reached the following estimation

$$\begin{aligned}
L'(t) &\leq -N\left(\mu_1 - \frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\|u_t\|^2 - \frac{1}{p+1}(g_0 - N_1)\|u_t\|_{p+2}^{p+2} \\
&\quad -\left(\frac{cN_2}{\tau} - \frac{N_1}{4\tau} - \mu_1\delta + N\left(\frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\right)\|z(x,1,t)\|^2 \\
&\quad -\left(\begin{array}{c} \frac{N}{2}g(t) + N_1\left(1-\beta-\eta-\frac{2\eta C_s^2}{\lambda}\mu_1\right) \\ -\delta\left(2\beta^2+1+\frac{1}{4}\right)\end{array}\right)\|\Delta u\|^2 \\
&\quad -\left(g_0 - \delta - \frac{\delta a_0}{p+1} - N_1 - \frac{N_1C_s^2\mu_1}{4\eta} + N_2\frac{C_s^2}{\tau}\right)\|\Delta u_t\|^2 \\
&\quad +\left(\frac{N_1\beta}{4\eta} + \beta\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)\right)(g\circ\Delta u)(t) \\
&\quad +\left(\frac{N}{2} - \frac{g(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{(p+1)}\right)\right)(g'\circ\Delta u)(t) - 2N_2\phi(t) \\
&\quad +kN_1\int_\Omega u^2\ln|u|\,dx + c_{\epsilon_0,\delta}(g\circ\Delta u)^{1/(1+\epsilon_0)}(t)
\end{aligned}$$

The definition of $E(t)$ gives

$$
\begin{aligned}
L'(t) \leq & -mE(t) - N\left(\mu_1 - \frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\|u_t\|^2 \\
& - \left(\frac{g_0 - N_1}{p+1} - \frac{m}{\rho+2}\right)\|u_t\|_{p+2}^{p+2} \\
& - \left(\frac{cN_2}{\tau} - \frac{N_1}{4\tau} - \mu_1\delta + N(\left(\frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\right)\|z(x,1,t)\|^2 \\
& - \left(\begin{array}{c} \frac{N}{2}g(t) + N_1\left(1 - \beta - \eta - \frac{2\eta C_S^2}{\lambda}\mu_1\right) \\ -\delta\left(2\beta^2 + 1 + \frac{1}{4}\right) - \frac{m(1-g_0)}{2} \end{array}\right)\|\Delta u\|^2 \\
& - \left(\begin{array}{c} g_0 - \delta - \frac{\delta a_0}{p+1} - N_1 - \frac{N_1 C_s^2 \mu_1}{4\eta} \\ +N_2\frac{C_s^2}{\tau} + \mu_1\delta C_s^2 - \frac{m}{2} \end{array}\right)\|\Delta u_t\|^2 \\
& + \left(\frac{N_1\beta}{4\eta} + \beta\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1 C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)\right)(g \circ \Delta u)(t) \\
& + \left(\frac{N}{2} - \frac{g(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{p+1}\right)\right)(g' \circ \Delta u)(t) + \frac{4N_2}{\epsilon}E(t) + \frac{mk}{4}\|u\|^2 \\
& + k\left(N_1 - \frac{m}{2}\right)\int_\Omega u^2 \ln|u|\,dx + \frac{m\epsilon}{2}\int_\Omega\int_0^1 z(x,\rho,t)d\rho dx \\
& + c_{\epsilon_0,\delta}(g \circ \Delta u)^{1/(1+\epsilon_0)}(t),
\end{aligned}
$$

the Logarthmic Sobolev inequality assure that

$$
\begin{aligned}
L'(t) \quad \leq \quad & -mE(t) - N\left(\mu_1 - \frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\|u_t\|^2 \\
& - \left(\frac{g_0 - N_1}{p+1} - \frac{m}{p+2}\right)\|u_t\|_{p+2}^{p+2} \\
& - \left(\frac{cN_2}{\tau} - \frac{N_1}{4\tau} - \mu_1\delta + N\left(\frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right)\right)\|z(x,1,t)\|^2 \\
& - \left(\begin{array}{c} \frac{N}{2}g(t) + N_1\left(1 - \beta - \eta - \frac{2\eta C_s^2}{\lambda}\mu_1\right) - \delta\left(2\beta^2 + 1 + \frac{1}{4}\right) \\ -\frac{m(1-g_0)}{2} - \left(N_1 - \frac{m}{2}\right)\frac{ka^2}{2\pi} \end{array}\right)\|\Delta u\|^2 \\
& - \left(g_0 - \delta - \frac{\delta a_0}{p+1} - N_1 - \frac{N_1 C_s^2 \mu_1}{4\eta} + N_2\frac{C_s^2}{\tau} + \mu_1\delta C_s^2 - \frac{m}{2}\right)\|\Delta u_t\|^2 \\
& + \left(\frac{N_1\beta}{4\eta} + \beta\left(2\delta + \frac{1}{4\delta} + \frac{\mu_1 C_s^2}{2\delta\lambda} + \frac{c}{\delta\beta}\right)\right)(g \circ \Delta u)(t) \\
& - \left(N_1 - \frac{m}{2}\right)\frac{k}{2}\left(2(1 + \ln a) - \ln\|u\|^2\right)\|u\|^2 \\
& + (\frac{N}{2} - \frac{g(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{p+1}\right))(g' \circ \Delta u)(t) + \frac{4N_2}{\epsilon}E(t) + \frac{mk}{4}\|u\|^2 \\
& + \frac{m\epsilon}{2}\int_\Omega\int_0^1 z(x,\rho,t)d\rho dx \;\; + c_{\epsilon_0,\delta}(g \circ \Delta u)^{1/(1+\epsilon_0)}(t)
\end{aligned}
$$

Let $0 < N_1 < g_0$, δ, λ, β, and μ_1 be sufficiently small, while N_2, and N should be sufficiently large such that

$$
\begin{aligned}
\gamma_0 &= \frac{1}{p+1}(g_0 - N_1) > 0, \\
\gamma_1 &= \frac{N}{2}g(t) + N_1\left(1 - \beta - \eta - \frac{2\eta C_S^2}{\lambda}\mu_1\right) - \delta\left(2\beta^2 + 1 + \frac{1}{4}\right) > 0,
\end{aligned}
$$

$$\gamma_2 = g_0 - \delta - \frac{\delta a_0}{p+1} - N_1 - \frac{N_1 C_s^2 \mu_1}{4\eta} - \mu_1 C_s^2 (1+\delta) > 0,$$

$$\frac{N}{2} - \frac{g(0)}{4\delta\lambda}\left(1 + \frac{C_s^2}{p+1}\right) > 0,$$

$$\begin{aligned} N\left(\mu_1 - \frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2} - \mu_1\delta - \frac{N_2}{N\tau}\right) &> 0, \\ \frac{cN_2}{\tau} - \frac{N_1}{4\tau} - \mu_1\delta + \mu\left(\frac{\epsilon}{2\tau} - \frac{|\mu_2|}{2}\right) &> 0, \end{aligned}$$

Consequently, we get

$$\begin{aligned} L'(t) \leq\ & -(m - \frac{4N_2}{\epsilon})E(t) - (\frac{g_0 - N_1}{p+1} - \frac{m}{p+2})\, \|u_t\|_{p+2}^{p+2} \\ & - \left(\gamma_1 - \frac{m(1-g_0)}{2} - \left(N_1 - \frac{m}{2}\right)\frac{ka^2}{2\pi}\right)\|\Delta u\|^2 \\ & - \left(\gamma_2 - \frac{m}{2}\right)\|\Delta u_t\|^2 + \left(c + \frac{m}{2}\right)(g \circ \Delta u)(t) + \frac{mk}{4}\|u\|^2 \\ & - \left(N_1 - \frac{m}{2}\right)\frac{k}{2}\left(2(1 + \ln a) - \ln\|u\|^2\right)\|u\|^2 \\ & + \frac{m\epsilon}{2}\int_\Omega \int_0^1 z(x,\rho,t)\,d\rho\, dx \ + c_{\epsilon_0,\delta}(g \circ \Delta u)^{1/(1+\epsilon_0)}(t) \end{aligned}$$

Finally, we choose m, k such that $m > N_2$ and $m \leq N_1$ so that

$$\frac{mk}{4} \leq \left(N_1 - \frac{m}{2}\right)\frac{k}{2},$$

$$\gamma_1 - \frac{m(1-g_0)}{2} - \left(N_1 - \frac{m}{2}\right)\frac{ka^2}{2\pi} > 0,$$

$$\gamma_2 - \frac{m}{2} > 0,$$

$$\frac{g_0 - N_1}{p+1} - \frac{m}{p+2} > 0,$$

$$m - \frac{4N_2}{\epsilon} > 0,$$

we get

$$\begin{aligned} L'(t) \leq\ & -m_1 E(t) + c_{\epsilon_0}(g \circ \Delta u)^{1/(1+\epsilon_0)}(t) + c^*(g \circ \Delta u)(t) \\ & - \left(N_1 - \frac{m}{2}\right)\frac{k}{2}\left(1 + 2\ln a - \ln\|u\|^2\right)\|u\|^2, \end{aligned} \tag{5.15}$$

using (2.16), (4.2) and (5.5), we get

$$\ln\|u\|^2 \leq \ln\left(\frac{4}{k}J(t)\right) \leq \ln\left(\frac{4}{k}E(t)\right) \leq \ln\left(\frac{4}{k}E(0)\right) \leq \ln\left(\frac{el\pi}{k}\right).$$

Picking a such that

$$\max\left\{e^{\frac{-3}{2}}, \sqrt{\frac{l\pi}{k}}\right\} < a < \sqrt{\frac{2l\pi}{k}}.$$

The assertion (3.11) is satisfied, and we provide

$$1 + 2\ln a - \ln\|u\|^2 \geq 0$$

From (2.16), (5.7) and (5.15), we get

$$L'(t) \leq -\alpha E(t), \forall t \geq 0 \tag{5.16}$$

For some $\alpha > 0$. By combining (5.6) and (5.16) satisfies

$$L'(t) \leq -k_1 L(t), \forall t \geq 0 \tag{5.17}$$

where $k_1 = \alpha/\alpha_1$. Therefore, integrating (5.17) on $(0, t)$ we produce

$$L'(t) \leq L(0)e^{-k_1 t}, \ \forall t \geq 0, \tag{5.18}$$

hence, the proof is completed □

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