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Symmetry-Based Analysis of Volterra Integral Equations via Lie Group Method

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ABSTRACT: This study investigates integral equations, particularly Volterra equations of the second kind, and their transformation into equivalent differential equations using the Leibniz rule. Various classes of linear and nonlinear integral equations are introduced, including Fredholm, Volterra, integro-differential, and singular types. The core objective is to apply Lie symmetry methods to the resulting differential equations in order to find exact analytical solutions. Through detailed examples, the work demonstrates the effectiveness of symmetry techniques in simplifying and solving complex integral equations, offering valuable tools for applications in physics, engineering, and applied mathematics.

Key Words: Volterra equations, integral equations, Lie symmetry method.

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1. Introduction

Integral equations play a central role in mathematical modeling of various phenomena in physics, engineering, biology, and other sciences. These equations are characterized by the presence of an unknown function under the integral sign, representing systems where the current state depends on cumulative past interactions, such as in heat conduction, population dynamics, and epidemic spread.

This research focuses on the classification and analysis of linear and nonlinear integral equations, with special emphasis on Fredholm and Volterra types. Additional forms such as integro-differential equations, singular integral equations, and combined Volterra-Fredholm equations are also explored to provide a comprehensive understanding of the subject [7,8,9,10,11].

A major objective of this study is to investigate the transformation of Volterra integral equations—particularly those of the second kind—into equivalent ordinary differential equations using the Leibniz rule for differentiation under the integral sign. Once converted [7,8,9,10,11], these differential equations can be effectively analyzed and solved using Lie symmetry methods [1,2,3,4,5,6,12] .

Lie symmetry analysis offers a powerful framework for studying the invariance properties of differential equations, enabling the construction of exact solutions and reduction of the equation's order. Through detailed examples, this work demonstrates how symmetry techniques can be used to solve integral equations analytically and to understand the structure of their solution spaces [1,2,3,4,5,6,12]. By combining classical integral equation theory with modern symmetry approaches, this study provides both theoretical insights and practical techniques for solving integral equations in various applied contexts.

2. Using Lie Symmetry Method for Solving Integral Equations

To illustrate the practical application of the theoretical framework developed in this study, this section presents a series of carefully selected examples. Each example involves a Volterra integral equation of the second kind, which is systematically transformed into an equivalent differential equation using differentiation rules such as Leibniz's rule.

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Once converted, the resulting differential equations are solved using Lie symmetry analysis, allowing us to identify symmetry generators, reduce the order of the equations, and construct exact analytical solutions. The examples demonstrate the effectiveness and flexibility of symmetry methods in handling different kernel structures and equation forms.

Through these step-by-step solutions, we aim to showcase not only the mathematical techniques involved but also to highlight how symmetry-based methods can simplify and unify the treatment of integral equations across various contexts.

3. Examples

Example 3.1 *Solve the integral equation by using Lie Symmetry method*

$$u(x) = 1 + \int_0^x u(t)dt$$

The equation you provided is avolterra - typ integral equation

$$u(x) = 1 + \int_0^x u(t)dt$$

- Convert to differential equation.
  To Apply Lie Symmetry methods we first convert the integral equation into a differential equation Let's differentiate both sides.

  $$u^{'}(x) = \frac{d}{dx}\left(1 + \int_0^x u(t)dt\right) \Longrightarrow u^{'}(x) = u(x)$$

  Lets solve the differential equation

  $$u^{'}(x) = u(x)$$

  Using the Lie symmetry methods .

- Identify the ODE
  We are given the first-order ODE

  $$u^{'} = u$$

- Assume a symmetry generator as:

  $$X = \zeta(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u}$$

  Comput the first prolongation $X^{(1)}$

  $$X^{(1)} = \zeta\,\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u} + \varphi^{(1)}\frac{\partial}{\partial u'}$$

  $$\varphi^{(1)} = D_x\varphi - u^{'}D_x\zeta = \varphi_x + \varphi_u u^{'} - u^{'}\left(\zeta_x + \zeta_u u^{'}\right)$$

- Apply the condition :

  $$X^{(1)}\left(u^{'} - u\right) = 0 \; when\ ever\ u^{'} = u \Longrightarrow \varphi^{(1)} - \varphi = 0$$

  We get

  $$\varphi^{(1)} = \varphi_x + \varphi_u u - u\left(\zeta_x + \zeta_u u\right)$$

  Substitute

  $$\varphi_x + \varphi_u u - u\zeta_x - u^2\zeta_x - \varphi = 0$$

- Assume general forms
  Let :

$$\zeta(x,u) = a_1x + a_2 \ , \quad \varphi(x,u) = b_1u + b_2$$
$$\varphi_x + \varphi_u u - u\zeta_x - u^2\zeta_u - \varphi = 0 \Longrightarrow 0 + b_1u - ua_1 - u^2.0 = b_1u + b_2$$
$$\Longrightarrow b_1u - a_1u = b_1u + b_2 \Longrightarrow -a_1u = b_2$$

Then compute:

$$\varphi_x = 0, \varphi_u = b_1, \zeta_x = a_1, \zeta_u = 0$$

Substitute into the condition:

$$0 + b_1u - ua_1 - 0 - (b_1u + b_2) = 0 \Longrightarrow -a_1u - b_2 = 0$$

So :

$$a_1 = 0, b_2 = 0 \Longrightarrow \zeta = a_2, \varphi = b_1u$$

- Find in variants and reduce
  Take the vector field:

$$X = \frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

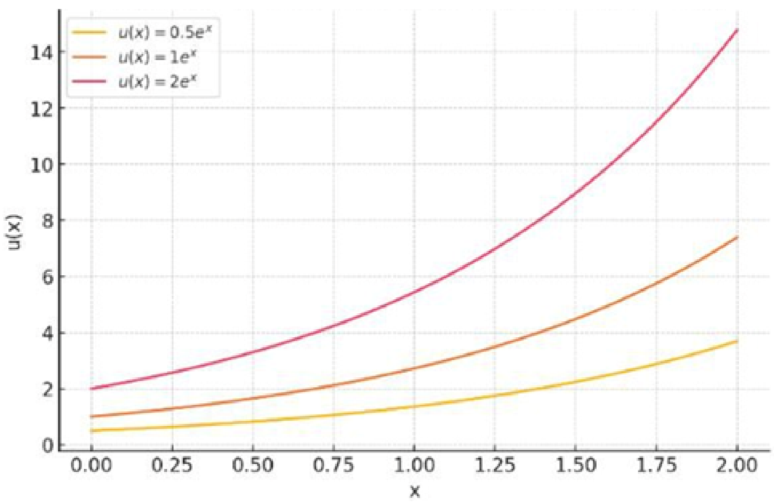
We find invariants by solving:

$$\frac{dx}{1} = \frac{du}{u} \Longrightarrow \ln u = x + c \Longrightarrow u = ce^x$$

Final solution:

$$u(x) = ce^x$$

This is the general solution to the equation $u^{'} = u$ obtained using the Lie symmetry method.
Solutions of the Differential Equation $u^{'}(x) = \ u(x)$



Here is the plot of the general Solution $u(x) = ce^x$ for three different values of the constant $C = \ 0.5, \ 1, \ 2$.

Each curve represents a different particular Solution of the differential equation

$$u^{'}(x) = u(x)$$

Solutions of the Differential Equation

$$u^{'}(x) = \ u(x)$$

Let's now use the initial integral equation $u(0) = 1$ to fined the constants $C = 1$.

- Match with integral equation

Suppose : $u(x) = ce^x$ plug into the nitegtal equation:

$$u(x) = 1 + \int_0^x u(t)dt = 1 + \int_0^x ce^t dt = 1 + c\left(e^x - 1\right)$$

$$\Longrightarrow u(x) = 1 + c\left(e^x - 1\right)$$

Compar with :

$$u(x) = ce^x \Longrightarrow ce^x = 1 + c\left(e^x - 1\right) \Longrightarrow c = 1$$

Final solution

$$u(x) = e^x$$

We will use Laplace transform method to solve the equation

$$u(x) = 1 + \int_0^x u(t)dt$$

Apply the Laplace transform
Let $U(s) = L\{u(x)\}$.
The Laplace transform of 1 is $\frac{1}{s}$ Also we know the property :

$$L\left\{\int_0^x u(t)dt\right\} = \frac{1}{s}U(s)$$

So the equation becomes :

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$$

Solve for $U(s)$

$$U(s)\left(1 - \tfrac{1}{s}\right) = \tfrac{1}{s}$$
$$U(s) = \tfrac{1}{s-1}$$

inverse Laplace transform we recall the standard result :

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$$

$$Thuse\ u(x) = e^x$$

Verification
Substitute into the original equation:

$$1 + \int_0^x e^t dt = 1 + (e^x - 1) = e^x$$

Final solution

$$u(x) = e^x$$

Example 3.2 *Solve the integral equation by using Lie Symmetry method*

$$u(x) = x + \int_0^x (x - t)u(t)\ dt$$

- The equation you provided is avolterra-typ integral equation:

$$u(x) = x + \int_0^x (x-t)u(t)\ dt$$

- Convert to differential equation
To apply Lie symmetry methods we first Convert the integral equation into adifferential equation .

Lets differentiate both sides once :

$$u^{'}(x) = 1 + \frac{d}{dx}\left[\int_0^x (x-t)u(t)\ dt\right]$$

Apply Leibniz rule :

$$u^{'}(x) = 1 + \int_0^x \frac{d}{dx}(x-t)u(t)dt + (x-x)u(x)$$

$$\Longrightarrow u^{'}(x) = 1 + \int_0^x u(t)dt$$

Now differentiate again :

$$u^{''}(x) = \frac{d}{dx}\left[\int_0^x u(t)dt\right] = u(x)$$

- Final differential equation We arrive at the the second-order linear ODE :

$$u^{''}(x) = u(x)$$

- Apply Lie symmetry Method to $u^{''} = u$ Lets apply the Lie Point Symmetry method to $u^{''} = u$.
This is a Second-order ODE we define the Lie symmetry generator :

$$X = \zeta(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u}$$

We prolong this to second order:

$$pr^{(2)}X = X + \varphi^{(1)}\frac{\partial}{\partial u'} + \varphi^{(2)}\frac{\partial}{\partial u''}$$

Where :

$$\varphi^{(1)} = D_x(\varphi) - u^{'}D_x(\zeta)$$

$$\varphi^{(2)} = D_x\left(\varphi^{(1)}\right) - u^{''}D_x(\zeta)$$

D_x is the total derivative operator :

$$D_x = \frac{\partial}{\partial x} + u^{'}\frac{\partial}{\partial u} + u^{''}\frac{\partial}{\partial u'} + \dots$$

- Invariance condition
We require that the prolonged vector field annihilates the differential equation the Solution manifold:

$$pr^{(2)}X\left[u^{''} - u\right] = 0\ when\ ever\ u^{''} = u\ that\ gives\ \ :$$

$$\varphi^{(2)} - \varphi = 0\ on\ u^{''} = u$$

So we compute $\varphi^{(2)}$ and simplify

- Compute Polongations. Let s now compute each term step by step first compute $\varphi^{(1)}$:

$$\varphi^{(1)} = D_x(\varphi) - u^{'} D_x(\zeta)$$

Use the chain rule:

$$\varphi^{(1)} = \varphi_x + \varphi_u u^{'} - \left(\zeta_x + \zeta_u u^{'}\right) = \varphi_x + \varphi_u u^{'} - u^{'}\zeta_x - u^{'2}\zeta_u$$

Simplify :

$$\varphi^{(1)} = \varphi_x - u^{'}\zeta_x + \varphi_u u^{'} - \zeta_u u^{'2}$$

Now compute $\varphi^{(2)}$:

$$\varphi^{(2)} = D_x\left(\varphi^{(1)}\right) - u^{''} D_x(\zeta)$$

Compute $D_x\left(\varphi^{(1)}\right)$ by applying the chain rule again The expression is long so we just note that you insert $\varphi^{(1)}$ into the derivative operator :

$$D_x\left(\varphi^{(1)}\right) = \partial_x\left(\varphi^{(1)}\right) + u^{'}\partial_u\left(\varphi^{(1)}\right) + u^{''}\partial u^{'}\left(\varphi^{(1)}\right) \; and \; D_x(\zeta) = \zeta_x + \zeta_u u^{'}$$

Then plug into :

$$\varphi^{(2)} = D_x\left(\varphi^{(1)}\right) - u^{''}\left(\zeta_x + \zeta_u u^{'}\right)$$

- Plug into the invariance condition use:

$$\varphi^{(2)} - \varphi = 0 \; when \; u^{''} = u$$

This gives a determining equation involving $\zeta(x,u)$ *and* $\varphi(x,u)$ and powers of $u^{'}$ we collect coefficients of powers of $u^{'}$ and set them to zero to get a system of PDES .

- Solve the determining system
Solving the determining equations (a known result for this ODE) we obtain: The Symmetry algebra is spanned by the following vector fields:

1. $X_1 = \frac{\partial}{\partial x}$
2. $X_2 = \frac{\partial}{\partial u}$
3. $X_3 = u\frac{\partial}{\partial u}$
4. $X_4 = e^x \; \frac{\partial}{\partial u}$
5. $X_5 = e^{-x} \; \frac{\partial}{\partial u}$
6. $X_6 = x\frac{\partial}{\partial x}$

This rich structure reflects that the equation is linear and in variant under translations Scalings and exponatial trans for mations .

- Reduce the order (optional)

Pick a symmetry to reduce the equation Let's choose :

$$X_4 = e^x \frac{\partial}{\partial u}$$

We look for an invariant under this symmetry The characteristic equations are:

$$\frac{dx}{0} = \frac{du}{e^x} \Longrightarrow X = const \; , \; Z = u - e^x$$

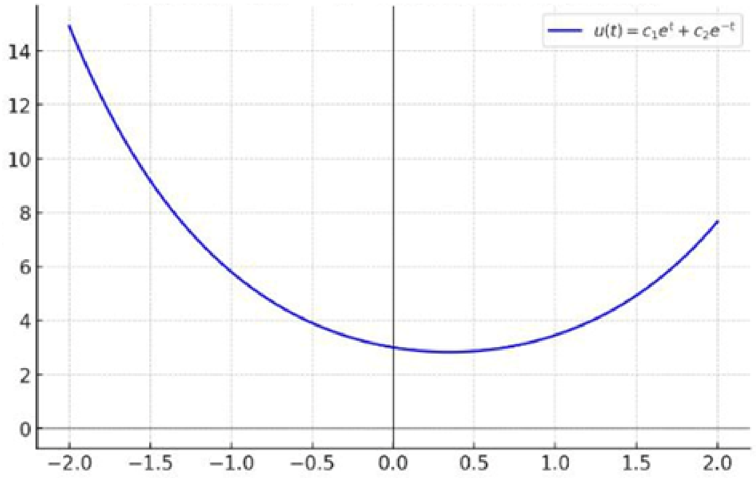
So an invariant Solution would be of the form :

$$u(x) = e^x.w(x)$$

Try substituting this into the original equation to reduce it.
But generally since this is a linear second-order constant coefficient ODE the final solution is : final General solution

$$u(x) = c_1 e^x + c_2 e^{-x}$$

pott solution $u^{''} - u = 0$ using lie symmetry



Here is the plot of the general solution to the equation $u^{''} - u = 0$ using the form :

$$u(x) = c_1 e^x + c_2 e^{-x}$$

This Solution satisfies both :
The integral equation and the differential equation $u^{''} = u$ derived Lie symmetry.
Let s now use the initial integral equation

$$u(0) = 0, u^{'}(0) = 1$$

$$u(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$u^{'}(0) = c_1 e^x - c_2 e^{-x}, u^{'}(0) = c_1 - c_2 = 1 \Rightarrow c_1 - (-c_1) = 1$$

$$\Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$$

$$\text{Thuse } u(x) = \frac{1}{2}\left(e^x - e^{-x}\right) = sinh(x)$$

compute the integral

$$\int_0^x (x - t)\sinh(t)dt = sinhx - x$$

using integration by parts

$$u(x) = x + (sinhx - x)$$

Final solution

$$u(x) = sinhx$$

We will use Laplace transform method to solve the equation

$$u(x) = x + \int_0^x (x - t)u(t)dt$$

Apply the Laplace transform

$$\text{Let } U(s) = L\{u(x)\}(s)$$

we know $L\{x\} = \frac{1}{s^2}$ The integral term is a convolution :

$$\int_0^x (x-t)u(t)dt = (k * u)(x), k(x) = x$$

$$So$$

$$L\left\{\int_0^x (x-t)u(t)dt\right\} = L\{k\}(s)U(s)$$

And since $L\{k(x)\} = L\{x\} = \frac{1}{s^2}$, we get $U(s) = \frac{1}{s^2} + \frac{1}{s^2}U(s)$
Solve for $U(S)$

$$U(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$
$$U(s) = \frac{\frac{1}{s^2}}{1-\frac{1}{s^2}} = \frac{1}{s^2-1}$$

Inverse Laplace transform

$$u(x) = L^{-1}\left\{\frac{1}{s^2-1}\right\}(x) = \sinh(x)$$

Final solution $u(x) = \sinh(x)$

Example 3.3 *Solve the integral equation by using Lie Symmetry method.*

$$u(x) = 1 + x + \int_0^x (x-t)^2 u(t)dt$$

This is avolterra integral equation of the Second kind.
Convert the integral equation to a differential equation.
We differential both sides with the respect to x

$$u^{'}(x) = 1 + \frac{d}{dx}\left[\int_0^x (x-t)^2 u(t)dt\right]$$

Use Leibinz rule for differentiation under the integral

$$\frac{d}{dx}\int_0^x (x-t)^2 u(t)dt = \int_0^x \frac{d}{dx}\left((x-t)^2\right)\ u(t)dt + (x-x)^2 u(x)$$

$$= \int_0^x 2(x-t)u(t)dt$$

$$u^{'}(x) = 1 + \int_0^x 2(x-t)u(t)dt$$

Now differentiate again

$$u^{''}(x) = \int_0^x 2u(t)dt$$

Differentiate once more

$$u^{'''}(x) = 2\ u(x)$$

Solve the differential equation

$$u^{'''}(x) = 2\ u(x)$$

This is alinear 3rd order ODE by using maple code we compute Lie symmetry method for $u^{(3)}(x) = 2u(x)$ and we get

$$u(x) = c_1 e^{2^{\frac{1}{3}x}} - c_2 e^{-\frac{2^{\frac{1}{3}x}}{2}} \sin\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right) + c_3 e^{-\frac{2^{\frac{1}{3}x}}{2}} \cos\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right)$$

General solution :

$$u(x) = c_1 e^{2^{\frac{1}{3}x}} - c_2 e^{-\frac{2^{\frac{1}{3}x}}{2}} \sin\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right) + c_3 e^{-\frac{2^{\frac{1}{3}x}}{2}} \cos\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right) \quad (3.53)$$

Lie symmetry generators :

$$liesymmetries\ \left(\frac{d^3}{dx^3}u(x) = 2u(x), u(x)\right) \quad (3.54)$$

The general Solution will be

$$u(x) = c_1 e^{\sqrt[3]{2}x} + c_2 e^{\frac{-\sqrt[3]{2}}{2x}} cos\ \left(\frac{\sqrt{3}}{2}\sqrt[3]{2}x\right) + c_3 e^{-\frac{\sqrt[3]{2}}{2x}} \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{2}x\right)$$

The lie Symmetry generators returned by maple typically include:
Translations : $\partial x, \partial u$
Scalings : $x\partial x + 3u\partial u$
Exponentials like $e^{\lambda x}\partial u$
Compute Initial conditions .
From the original equation :

$$u(x) = 1 + x + \int_0^x (x-t)^2 u(t)dt$$

At $x = 0$

$$u(0) = 1$$

$$u^{'}(x) = 1 + \int_0^x 2(x-t)u(t)dt \Longrightarrow u^{'}(0) = 1$$

$$u^{''}(x) = \int_0^x 2u(t)dt \Longrightarrow u^{''}(0) = 0$$

Initial conditions :

$$u(0) = 1, u^{'}(0) = 1, u^{''}(0) = 0$$

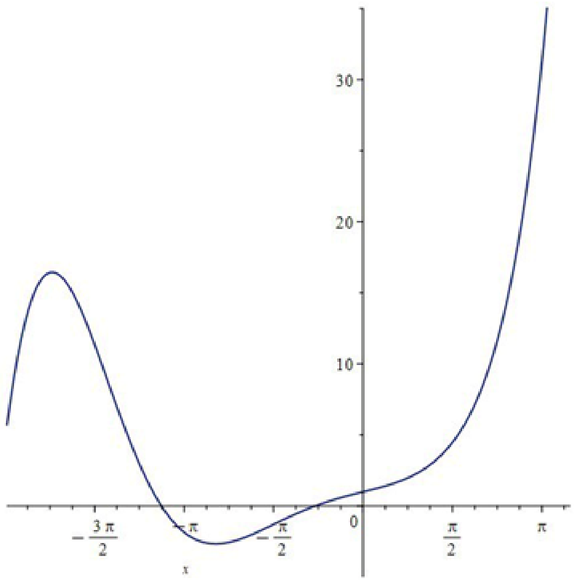
Solve ODE with initial conditions by using maple code the compute to do the full solution .
final maple code for $u^{(3)}(x) = 2u(x)$ and we get .

$$u(x) = -\frac{e^{-\frac{2^{\frac{1}{3}x}}{2}}\left(2^{\frac{2}{3}} - 4\right)\cos\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right)}{6} + \frac{2^{\frac{2}{3}}\sqrt{3}e^{-\frac{2^{\frac{1}{3}x}}{2}}\sin\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right)}{6} + \frac{e^{2^{\frac{1}{3x}}}\left(2^{\frac{2}{3}} + 2\right)}{6}$$

Solution to the integral equation :

$$u(x) = -\frac{e^{-\frac{2^{\frac{1}{3}x}}{2}}\left(2^{\frac{2}{3}} - 4\right)\cos\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right)}{6} + \frac{2^{\frac{2}{3}}\sqrt{3}e^{-\frac{2^{\frac{1}{3}x}}{2}}\sin\left(\frac{\sqrt{3}\ 2^{\frac{1}{3}}x}{2}\right)}{6} + \frac{e^{2^{\frac{1}{3x}}}\left(2^{\frac{2}{3}} + 2\right)}{6}$$

Warning, expecting only range variable x in expression $u(x)$ to be plotted but found name u.



we will use Laplace transform method to solve the equation

$$u(x) = 1 + x + \int_0^x (x-t)^2 u(t) dt$$

Apply the Laplace transform

$$\text{Let } U(s) = L\{u(x)\}(s)$$

we use known transforms :

$$L\{1\} = \frac{1}{s}, L\{x\} = \frac{1}{s^2}, L\left\{x^2\right\} = \frac{2}{s^3}$$

The integral term is a convolution :

$$\int_0^x (x-t)^2 u(t) dt = \left(x^2 * u\right)(x)$$

so its Laplace transform is

$$L\left\{x^2\right\} U(s) = \frac{2}{s^3} U(s)$$

Thuse in the Laplace domain we have:

$$U(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} U(s)$$

Solve for $U(S)$

$$U(s)\left(1 - \frac{2}{s^3}\right) = \frac{1}{s} + \frac{1}{s^2}$$

So

$$U(s) = \frac{\frac{1}{s} + \frac{1}{s^2}}{1 - \frac{2}{s^3}} = \frac{(s+1)s}{s^3 - 2}$$

Partial fraction decomposition factor the denominator:

$$s^3 - 2 = (s-a)\left(s^2 + as + a^2\right) \text{ where } a = 2^{\frac{1}{3}}$$

So we set

$$\frac{s(s+1)}{(s-a)\left(s^2+as+a^2\right)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}$$

Solving gives

$$A = \frac{a+1}{3a}, B = \frac{2a-1}{3a}, C = \frac{a+1}{3}$$

Simplify the quadratic part
The quadratic factor can be written as

$$s^2 + as + a^2 = \left(s + \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a\right)^2$$

So let $w = \frac{\sqrt{3}}{2}a$
Rewrite the numerator:

$$Bs + C = B\left(S + \frac{a}{2}\right) + \left(C - \frac{Ba}{2}\right)$$

one can check that $C - \frac{Ba}{2} = \frac{1}{2}$

Example 3.4 *Solve the integral equation by using Lie Symmetry method.*

$$u(x) = x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt,$$

This is a volterra nitegral equation of the second kind. Convert the integral equation to a Differential Equation, we Differentit att both sides withe respect to x

$$u^{'}(x) = 2x + \frac{1}{6}\frac{d}{dx}\left[\int_0^x (x-t)^2 u(t)dt\right]$$

Use Leibniz's rule for differentiation under the integral

$$\frac{d}{dx}\int_0^x (x-t)^3 u(t)dt = \int_0^x 3(x-t)^2 u(t)dt$$

So :

$$u^{'}(x) = 2x + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt$$

Second Derivatve:

$$u^{''}(x) = 2 + \frac{1}{2}\frac{d}{dx}\left[\int_0^x (x-t)^2 u(t)dt\right]$$

$$u^{''}(x) = 2 + \int_0^x (x-t)u(t)dt$$

Third Derivative:

$$u^{'''}(x) = \frac{d}{dx}\left[2 + \int_0^x (x-t)u(t)dt\right] = \int_0^x u(t)dt$$

Fourth Derivative:

$$u^{(4)}(x) = u(x)$$

Final Differential Equation.
We arrive at the 4th-order linear ODE :

$$u^{(4)}(x) - u(x) = 0$$

With initial conditions from the original eanation

$$u(0) = 0^2 + \frac{1}{6}.0 = 0$$

$$u^{'}(0) = 2.0 + \frac{1}{2}.0 = 0$$

$$u^{''}(0) = 2 + 0 = 2$$

$$u^{'''}(0) = 0$$

Solve the ODE

$$u^{(4)}(x) - u(x) = 0$$

Characteristic equation

$$r^4 - 1 = 0 \Longrightarrow r = \pm 1, \pm i$$

General Solution

$$u(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Apply initial conditions :

$$u(0) = c_1 + c_2 + c_3 = 0$$

$$u^{'}(x) = c_1 e^x - c_2 e^{-x} - c_3 \cos x + c_4 \sin x$$

$$\Longrightarrow u^{'}(0) = c_1 - c_2 + c_4 = 0$$

$$u^{''}(x) = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$$

$$\Longrightarrow u^{''}(0) = c_1 + c_2 - c_3 = 2$$

$$u^{'''}(x) = c_1 e^x - c_2 e^{-x} + c_3 \sin x - c_4 \cos x$$

$$\Longrightarrow u^{'''}(0) = c_1 - c_2 - c_4 = 0$$

Solving this system gives :

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = 0, \quad c_4 = 0$$

So the solutions :

$$u(x) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x$$

Lie Symmetry Perspective

The differential equation :

$$u^{(4)} - u = 0$$

is linear and admits an -8 dimensional Lie Symmetry algebra with symmetries related to :
$x-$ translations
$u-$ Scalings
Exponential solutions forming avector space of exponentials and trigonometric functions (real and imaginary parts of exponentials)
By using maple code we compute Lie Symmetries for $u^{(4)} - u = 0$ and we get

$$ODE := \frac{d^4}{dx^4} u(x) = u(x)$$

$$u(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \sin(x) + c_4 \cos(x)$$

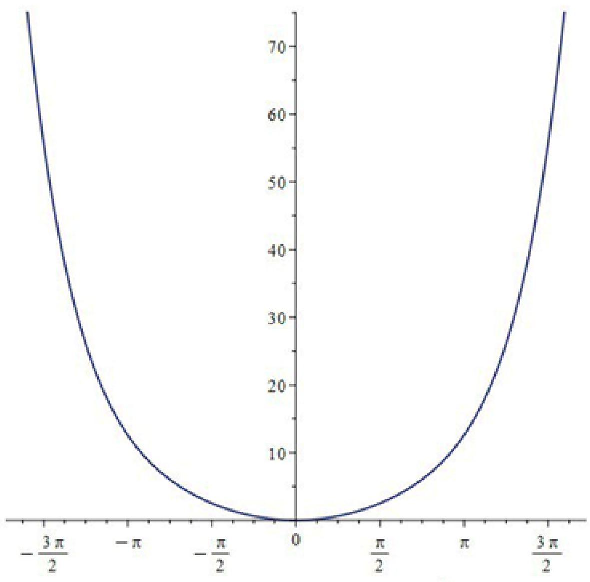
$$ICs := u(0) = 0, D(u)(0) = 0, D^{(2)}(u)(0) = 2, D^{(3)}(u)(0) = 0$$

$$sol \ := u(x) = \frac{e^{-x}}{2} + \frac{e^{x}}{2} - \cos(x)$$

Solution to the ODE :

$$u(x) = \frac{e^{-x}}{2} + \frac{e^{x}}{2} - \cos(x)$$

Warning, expecting only range variable x in expression $u(x)$ to be plotted but found name u



Lie Symmetry Generators :

$$\text{lie symmetries } \left(\frac{d^4}{dx^4} u(x) = u(x), u(x) \right)$$

Warning, expecting only range variable x in expression liesymmetries $(diff\ (diff\ (diff\ (u(x), x), x), x), x) = u(x),\ u(x))$ to be plotted but found names [u, liesymmetries]
The solution to the differential equation:

$$u(x) \;=\; cosh(x)$$

The Lie Symmetry generators which may include infinitesimal transformations like:

$$\begin{aligned} X &= \; \partial x \; (translation \; in \; x) \\ X &= \; u\partial u \; (scaling \; in \; u) \\ X &= e^{x}\partial x \\ X &= \; e^{-x}\partial x, \; etc. \end{aligned}$$

These generators span the symmetry algebra of the linear ODE $u(4) = u$ which has an 8- dimensional Lie algebra (since its Linear and of order 4) .

We will use Laplace transform method to solve the equation

$$u(x) = x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt$$

we apply the Laplace transform L define

$$U(s) = L\{u(x)\}(s)$$

we know : $L\left\{x^2\right\} = \frac{2}{s^3}, L\{(k*u)\} = k(s)U(s)$
Here the kernal is $k(x) = x^3$, so

$$k(s) = L\left\{x^3\right\} = \frac{6}{s^4}$$

Taking Laplace transforms of both sides :

$$U(s) = \frac{2}{s^3} + \frac{1}{2}k(s)U(s) = \frac{2}{s^3} + \frac{1}{2}\cdot\frac{6}{s^4}U(s) = \frac{2}{s^3} + \frac{3}{s^4}U(s)$$

Solve for $U(s)$:

$$\left(1 - \frac{3}{s^4}\right)U(s) = \frac{2}{s^3} \Rightarrow U(s) = \frac{\frac{2}{s^3}}{1-\frac{3}{s^4}} = \frac{2s}{s^4-3}$$

Partial fraction decomposition
Let $a = \sqrt{3}, r = 3^{\frac{1}{4}}$ then $r^2 = a$

$$\frac{2s}{s^4-3} = \frac{2s}{(s^2-a)(s^2+a)} = \frac{1}{a}\frac{s}{s^2-a} - \frac{1}{a}\frac{s}{s^2+a}$$

Inverse Laplace transform Now we invert term by term

$$\frac{A}{s-a} \longleftrightarrow Ae^{ax}$$

$$\frac{B\left(s+\frac{a}{2}\right)}{\left(s+\frac{a}{2}\right)^2 + w^2} \longleftrightarrow Be^{-\frac{ax}{2}}\cos(wx)$$

$$\frac{1/2}{\left(s+\frac{a}{2}\right)^2 + w^2} \longleftrightarrow \frac{1}{2w}e^{-\frac{ax}{2}}sin(wx)$$

final solution

$$u(x) = \frac{a+1}{3a}e^{ax} + e^{-\frac{a}{2}x}\left(\frac{2a-1}{3a}\cos\left(\frac{\sqrt{3}}{2}ax\right) + \frac{1}{2\cdot\left(\frac{\sqrt{3}}{2}a\right)}\sin\left(\frac{\sqrt{3}}{2}ax\right)\right)$$

Inverse Laplace transforms we use standard formulas :

$$L^{-1}\left\{\frac{s}{s^2-a}\right\} = \cosh(rx)$$

$$L^{-1}\left\{\frac{s}{s^2+a}\right\} = \cos\ (rx)$$

$$where\ r = \sqrt{a} = 3^{\frac{1}{4}}$$

So,

$$u(x) = \frac{1}{a}\left(\cosh(rx) - \cos(rx)\right) = \frac{1}{\sqrt{3}}\left(\cosh\left(3^{\frac{1}{4}}x\right) - \cos\left(3^{\frac{1}{4}}x\right)\right)$$

Final Solution

$$u(x) = \frac{1}{\sqrt{3}}\left(\cosh\left(3^{\frac{1}{4}}x\right) - \cos\left(3^{\frac{1}{4}}x\right)\right)$$

4. Conclusion

In this work, we have explored the transformation of Volterra integral equations of the second kind into equivalent differential equations by applying differentiation rules such as Leibniz's rule. Once converted, Lie symmetry methods were successfully employed to analyze and solve the resulting differential equations. Through several detailed examples, the study demonstrated the effectiveness of symmetry analysis in simplifying complex problems, reducing the order of differential equations, and constructing exact analytical solutions.

The results confirm that Lie symmetry provides a powerful and systematic approach to solving various classes of integral equations, offering both theoretical insights and practical tools for applications in physics, engineering, and applied mathematics. This integration of classical integral equation theory with modern symmetry techniques highlights the potential of symmetry methods as a unifying framework for tackling challenging mathematical problems, and opens the door for further extensions to more general nonlinear and multi-dimensional cases.

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