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Multiplicative b-Generalized Derivations on Non-Commutative Rings*

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ABSTRACT: In the current investigation, our primary objective is to find the structure of b-generalized derivations on Martindale ring of quotients. Infact, we establish that if b-generalized derivations satisfy some differential identities on non-commutative prime ring, then it will be a trivial (zero) derivation. Our proof contains an interesting approach in view of the existing classical theory about ordinary derivation presented in [12].

Key Words: b-generalized derivation, prime rings, centralizers, ring of quotients.

Contents

1. Introduction

Through out the manuscript, the notation $\mathcal{Z}(\mathcal{R})$ stands for the center of an associative ring $\mathcal{R}$. The symbol $[b, d]$ specifies the commutator of $b, d \in \mathcal{R}$, which is represented by the mathematical formula $bd - db$. If $pr = 0$ implies $r = 0$ for every $r \in \mathcal{R}$ and $p > 1$ is a fixed integer, then a ring $\mathcal{R}$ is a p-torsion free ring. A ring $\mathcal{R}$ is a prime if $r\mathcal{R}t = \{0\}$ gives that either $t = 0$ or $r = 0$. It is called semiprime if it fulfills the requirement that $c\mathcal{R}c = \{0\}$ yields that $c = 0$.

In simple terms, the mapping ζ is (skew)-commuting on $\mathcal{R}$ if $\zeta(c)c + c\zeta(c) = 0$ for each of $c \in \mathcal{R}$. If $\zeta(c)c + c\zeta(c) \in \mathcal{Z}(\mathcal{R})$ for each $c \in \mathcal{R}$, then a map ζ from $\mathcal{R}$ to $\mathcal{R}$ is thought to be (skew)-centralizing on $\mathcal{R}$. If the mapping η from $\mathcal{R}$ to $\mathcal{R}$ fulfills the equation $\eta(ce) = \eta(c)e + c\eta(e)$, for each of $c, e \in \mathcal{R}$, then it is regarded as a derivation on $\mathcal{R}$. An additive mapping $F : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be generalized derivation with associated derivation d if it satisfies the condition $F(ty) = F(t)y + td(y)$ for all $t, y \in \mathcal{R}$.

Derivations and inner derivations are fundamental examples of generalized derivations. Operator algebras have been the main focus of research on generalized derivations. The first result in this direction is due to Posner [12] who proved that if a prime ring $\mathcal{R}$ admits a nonzero derivation d such that $[d(w), w] \in Z(\mathcal{R})$ for all $w \in \mathcal{R}$, then $\mathcal{R}$ is commutative. An analogous result for centralizing automorphisms on prime rings was obtained by Mayne [11]. A number of authors have extended these theorems of Posner and Mayne, they have showed that derivations, automorphisms, and some related maps cannot be centralizing on certain subsets of noncommutative prime and some other subsets of rings.

In [6], the description of all centralizing additive maps of a prime ring $\mathcal{R}$ of characteristic not 2 was given and subsequently the characterization for semiprime rings of characteristic not 2 was given. It was shown that every such map f is of the form $f(k) = \lambda k + \mu(k)$, where $\lambda \in C$, the extended centroid of $\mathcal{R}$, and μ is an additive map of $\mathcal{R}$ into C. Consequently, any research from an algebraic perspective might be intriguing, one can see [1,2,13,14] for more details.

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Commutativity in semiprime and prime rings was studied by number of authors, who discovered derivations and generalized derivations that meet the necessary algebraic criteria on relevant subsets of the rings. Bell and Martindale [5], show that a semiprime ring $\mathcal{R}$ must have a nontrivial central ideal if it admits an appropriate endomorphism or derivation which is centralizing on some nontrivial one-sided ideal.

When the generalized derivation F appears on a Lie ideal of $\mathcal{R}$, the authors in [15] looked into the previously mentioned result and proved that: Let $\mathcal{R}$ be a prime ring, L a noncentral Lie ideal of $\mathcal{R}$, F a generalized derivation with associated nonzero derivation d of $\mathcal{R}$. If $a \in \mathcal{R}$ such that $a(d(u)^{l_1}F(u)^{l_2}d(u)^{l_3}F(u)^{l_4}...F(u)^{l_k})^n = 0$ for all $u \in L$, where $l_1, l_2, ..., l_k$ are fixed non negative integers not all are zero and n is a fixed integer, then either $a = 0$ or $\mathcal{R}$ satisfies s_4, the standard identity in four variables.

Hvala [8] introduced the study of generalized derivations on prime (semiprime) rings. In [10], Lee proved the following essential result: every generalized derivation F on a dense left ideal of $\mathcal{R}$ can be uniquely extended to Q_m^r and assume the form

$$F(x) = ax + d(x) \text{ for all } x \in \mathcal{R}$$

for some $a \in Q_m^r$ and a derivation d on Q_m^r. Our study is motivated by the research in [9], on generalized b-derivations. By a multiplicative generalized b-derivation $\mathfrak{K}$, we mean a map (not necessarily additive) from a ring $\mathcal{R}$ to Martindale ring of quotient $Q_m^r(\mathcal{R})$ such that $\mathfrak{K}(\nu y) = \mathfrak{K}(\nu)y + b\nu d(y)$, with associated derivation d for every $y, \nu \in \mathcal{R}$ for some $0 \neq b \in \mathcal{R}$.

Example 1.1 *Let* $\mathcal{R} = \left\{ \begin{pmatrix} t_1 & 0 & v \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \mid t_1, v, t_2 \in \mathbb{Z} \right\}$, *and the mappings* $\mathfrak{K}, d : \mathcal{R} \to \mathcal{R}$ *defined by* $\mathfrak{K}\begin{pmatrix} t_1 & 0 & v \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t_1 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} t_1 & 0 & v \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t_1 & 0 & v \\ 0 & 0 & t_2 \\ 0 & 0 & 0 \end{pmatrix}$. *Setting* $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, *it can be readily shown that* $\mathfrak{K}$ *acts as a generalized* b*-derivation in relation to the derivation* d.

Example 1.2 *Consider two mappings* $F, d : \mathbb{Z} \to \mathbb{Z}$ *defined by* $F(x) = 3x$ *and* $d(x) = 0$ *for all* $x \in \mathbb{Z}$. *For any fixed element* $b \in \mathbb{Z}$, F *is a generalized* b*-derivation on* $\mathbb{Z}$.

The structure of symmetric skew n-derivations in prime and semiprime rings investigated by authors in [7]. A generalization of such structures of $(\sigma, \rho) - n-$derivation on rings obtained in [3]. Following the similar line of investigation authors obtained some commutativity results on skew commuting derivations in [16]. A thorough study of generalized derivation, we intend to prove the theorems for the case of non-commutative prime rings involved with generalized b-derivations and obtained the structure of the underlying derivations.

2. Main Results

We begin with the following results:

Lemma 2.1 *[5] Let* $\mathcal{I}$ *be a nonzero left ideal of a prime ring* $\mathcal{R}$. *If* $\mathcal{R}$ *admits a nonzero derivation* d *such that* $[d(v), v] \in \mathrm{Z}(\mathcal{R})$ *for every* $v \in \mathcal{R}$, *then* $\mathcal{R}$ *is commutative.*

Lemma 2.2 *[4] For any* $\kappa \in Q_s^r(\mathcal{R})$, *there exists a dense right ideal* J *such that* $\kappa\mathrm{J} = \{0\}$ *(or* $\mathrm{J}\kappa = \{0\}$*). Then* $\kappa = 0$.

Lemma 2.3 *[4] Let* $\mathcal{R}$ *be a prime ring,* $Z(\mathcal{R})$ *the center of* $\mathcal{R}$ *and* $a, b \in \mathcal{R}$. *If* $a \in Z(\mathcal{R})$ *and* $ab \in Z(\mathcal{R})$, *then either* $a = 0$ *or* $b \in Z(\mathcal{R})$.

Theorem 2.1 *Let a non-commutative prime ring be $\mathcal{R}$, Q_s^r the symmetric quotient ring and Q_m^r be the Martindale ring of quotients having extended centroid $\mathcal{C}$. If $\mathfrak{K}$ is a multiplicative b-generalized derivation with associated derivation d such that $\mathfrak{K}(\nu \circ w) = 0$, for every $\nu, w \in Q_m^r$, then $\mathfrak{K} = 0$.*

Proof: We split the proof in the following two cases:
Case 1 Consider the associated derivation $d = 0$. Then by given condition $\mathfrak{K}$ have the form $\mathfrak{K}(\nu w) = \mathfrak{K}(\nu)w$ for every $\nu, w \in Q_m^r$. We are given that

$$\mathfrak{K}(\nu \circ w) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.1}$$

Substitute $w\nu$ for ν in (2.1) to find

$$\mathfrak{K}(w)(\nu \circ w) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.2}$$

Making use of primeness of Q_m^r, we obtain $\mathfrak{K}(w) = 0$ for each $w \in Q_m^r$. Hence $\mathfrak{K} = 0$.

Case 2 Consider the associated derivation $d \neq 0$. Then by given condition $\mathfrak{K}$ have the form $\mathfrak{K}(\nu w) = \mathfrak{K}(\nu)w + b\nu d(w)$ for every $\nu, w \in Q_m^r$. We are given that

$$\mathfrak{K}(\nu \circ w) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.3}$$

Substitute $w\nu$ for w in (2.3) to find

$$\mathfrak{K}(\nu \circ w)\nu + b(\nu \circ w)d(\nu) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.4}$$

Encounter (2.3)and (2.4) together, we obtain

$$b(\nu \circ w)d(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.5}$$

On simplification, we have

$$b\nu w d(\nu) + bw\nu d(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.6}$$

This implies that

$$b\nu w d(\nu) = -bw\nu d(\nu) \text{ for every } \nu, w \in Q_m^r. \tag{2.7}$$

Put bw in place of w in (2.6) and use (2.7) to find

$$b[\nu, b]wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.8}$$

Also, we have

$$b[\nu, b]\mathcal{R}wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.9}$$

By primeness of Q_m^r, we intend to find either $b[\nu, b] = 0$ or $wd(\nu) = 0$ for every $w, \nu \in Q_m^r$. Next, construct the two sets

$$\mathfrak{L_1}^+ = \{b[\nu, b] = 0 \mid \nu \in Q_m^r\}$$

and

$$\mathfrak{L_2}^+ = \{wd(\nu) = 0 \mid \nu \in Q_m^r\}.$$

Notice that $\mathfrak{L_1}^+ \cup \mathfrak{L_2}^+$ forms $(Q_m^r, +)$. That yields a contradiction to the fact that an additive group can not be the union of two proper additive subgroups. This implies that either $\mathfrak{L_1}^+ = Q_m^r$ or $\mathfrak{L_2}^+ = Q_m^r$. First, we investigate the condition $\mathfrak{L_1}^+ = Q_m^r$, that is, $b[\nu, b] = 0$ for each $\nu \in Q_m^r$. Since $b \neq 0$ and in view of Lemma 2.2, we have $b \in \mathcal{C}$. It follows from (2.9) that $wd(\nu) = 0$ for all $w, \nu \in Q_m^r$. Again primeness implies that $d = 0$, a contradiction.

Similarly we reach out to the contradiction for the case $\mathfrak{L_2}^+ = Q_m^r$ from Lemma 2.1. Therefore, in both cases we arrive at the contradiction to the fact $d \neq 0$. This implies that $\mathfrak{K} = 0$. This completes the proof. □

Theorem 2.2 *Let a non-commutative prime ring be $\mathcal{R}$, Q_s^r the symmetric quotient ring and Q_m^r be the Martindale ring of quotients having extended centroid $\mathcal{C}$. If $\mathfrak{K}$ is a multiplicative b-generalized derivation with associated derivation d such that $\mathfrak{K}([\nu, w]) = 0$, for every $\nu, w \in Q_m^r$, then $\mathfrak{K} = 0$.*

Proof: We divide the proof in the following two cases:
Case 1 Consider the associated derivation $d = 0$. Then by given condition $\mathfrak{K}$ have the form $\mathfrak{K}(\nu w) = \mathfrak{K}(\nu)w$ for every $\nu, w \in Q_m^r$. We are given that

$$\mathfrak{K}([\nu, w]) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.10}$$

Substitute νw for w in (2.10) to find

$$\mathfrak{K}(\nu[\nu, w]) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.11}$$

Making use of primeness of Q_m^r, we obtain $\mathfrak{K}(\nu) = 0$ for each $\nu \in Q_m^r$ provided Q_m^r is non-commutative. Hence $\mathfrak{K} = 0$.

Case 2 Consider the associated derivation $d \neq 0$. Then by given condition $\mathfrak{K}$ have the form $\mathfrak{K}(\nu w) = \mathfrak{K}(\nu)w + b\nu d(w)$ for every $\nu, w \in Q_m^r$. We are given that
Simplify (2.11) to get

$$\mathfrak{K}(\nu[\nu, w]) = \mathfrak{K}(\nu)[\nu, w] + b\nu d([\nu, w]) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.12}$$

From (2.11), (2.12) reduces to the form

$$b\nu d([\nu, w]) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.13}$$

Again replace $w\nu$ for w in (2.13) to find

$$b\nu d([\nu, w])\nu + b\nu[\nu, w]d(\nu) = 0, \text{ for every } \nu, w \in Q_m^r. \tag{2.14}$$

Making use of (2.13), (2.14) becomes

$$b\nu[\nu, w]d(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.15}$$

Putting $b\nu w$ in place of w in above equation to get

$$b\nu b\nu[\nu, w]d(\nu) + b\nu[\nu, b\nu]wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.16}$$

From (2.16) and (2.15), we conclude

$$b\nu[\nu, b\nu]wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.17}$$

This implies that

$$b\nu[\nu, b]\nu wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.18}$$

A simple calculation gives us that

$$b[\nu, b]\mathcal{R}wd(\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.19}$$

Next, arguing in the same manner and using primeness argument as in the proof of Theorem 2.1, we obtain the required result.

□

We listed the following corollaries as a consequences of our theorems. Further, it is also describe the commutative structure of ring.

Corollary 2.1 *Let a prime ring be $\mathcal{R}$, Q_s^r the symmetric quotient ring and Q_m^r be the Martindale ring of quotients having extended centroid $\mathcal{C}$. If $\mathfrak{K}$ is a multiplicative b-generalized derivation with associated derivation d such that $\mathfrak{K}(\nu \circ w) = 0$, for every $\nu, w \in Q_m^r$, then either $\mathcal{R}$ is commutative or $\mathfrak{K}$ acting as left centralizer provided $d = 0$.*

Proof: The proof is straight forward by utilizing Theorem 2.1 and Lemma 2.3. □

Corollary 2.2 *Let a prime ring be $\mathcal{R}$, Q_s^r the symmetric quotient ring and Q_m^r be the Martindale ring of quotients having extended centroid $\mathcal{C}$. If $\mathfrak{K}$ is a multiplicative b-generalized derivation with associated derivation d such that $\mathfrak{K}([\nu, w]) = 0$, for every $\nu, w \in Q_m^r$, then either $\mathcal{R}$ is commutative or $\mathfrak{K}$ acting as left centralizer provided $d = 0$.*

Corollary 2.3 *Let a prime ring be $\mathcal{R}$, Q_s^r the symmetric quotient ring and Q_m^r be the Martindale ring of quotients having extended centroid $\mathcal{C}$. If $\mathfrak{K}$ is a b-generalized derivation with associated derivation d such that $\mathfrak{K}(\nu w) = 0$, for every $\nu, w \in Q_m^r$, then either $\mathcal{R}$ is commutative or $\mathfrak{K}$ acting as left centralizer provided $d = 0$.*

Proof: Consider the given condition in the hypothesis

$$\mathfrak{K}(\nu w) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.20}$$

Interchange the role of ν and w in (2.20) to obtain

$$\mathfrak{K}(w\nu) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.21}$$

Next, adding the two expression as we discard the multiplicative condition in the hypothesis, we get

$$\mathfrak{K}(\nu \circ w) = 0 \text{ for every } \nu, w \in Q_m^r. \tag{2.22}$$

An application of Theorem 2.1, completes the proof. □

3. Conclusion and Future Scope

The present investigation endeavors to elucidate the structure of b-generalized derivations within the context of non-commutative prime rings. As a direct implication, it is demonstrated that b-generalized derivations, which satisfy certain differential identities on non-commutative prime rings, manifest as zero derivations. Exploring the implications of these mappings in alternative contexts presents a promising avenue for future research. Furthermore, we elucidate the scenario wherein $\mathcal{R}$ is commutative and $\mathfrak{K}$ functions as centralizers. Beyond the results already established, our findings hold applicability in the domains of additive mapping, generalized matrix algebra, and allied areas.

Conflict of interest: The authors declare that there are no conflicts of interest.

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