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Existence of Weak Solutions for an Abstract Cauchy Problem Involving the Fractional Laplacian

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ABSTRACT: In this paper, we construct a theory of existence and uniqueness of solutions for an abstract Cauchy problem $u'(t) + Au(t) = h(t)$, $t \in (0,T)$ and $u(0) = f$, where A is the fractional Laplacian operator acting on a bounded domain with smooth boundary. The construction of the solution is based on the method of temporal discretization.

Keywords: Fractional Laplacian operator, fractional Sobolev space, accretive operators, Cauchy problem abstract.

Contents

1. Introduction

We study the following fractional equation

$$\begin{cases} u_t + (-\Delta)^{\sigma/2} u = h(x,t), & \text{in} \quad \Omega, \quad t > 0 \\ u = 0, & \text{on} \quad \partial\Omega, \quad t > 0 \\ u(x,0) = f(x), & \text{on} \quad x \in \Omega, \quad t = 0, \end{cases} \tag{1.1}$$

where $0 < \sigma < 2$, $f \in L^1(\Omega)$, $\Omega \subset \mathbb{R}^N$ and $(-\Delta)^{\sigma/2}$ is the fractional laplacian.

There is wide interest in porous medium equations, which replace the nonlocal diffusion operator $(-\Delta)^{\sigma/2}$ by the classical Laplacian $-\Delta$, for instance see [11] and [9]. We showed the existence of a weak solution, transforming the problem (1.1) to an equivalent problem using what they call a $\sigma-$ extension and guarantees the existence of a weak solution of this new formulation using the temporal discretization method.

In previuos problems the fractional laplacian operator $(-\Delta)^{\sigma/2}$ in a bounded domain is defined by a spectral decomposition (see [6]). Let (φ_j, λ_j) be the eigenfunctions and eigenvalues of $-\Delta$ in Ω (with Dirichlet boundary data). Then, $(\varphi_j, \lambda_j^{\sigma/2})$ are the eigenfunctions and eigenvalues of $(-\Delta)^{\sigma/2}$ in Ω with the same boundary conditions. The operator $(-\Delta)^{\sigma/2}$ is defined for any $u \in C_0^\infty(\Omega)$, $u = \sum a_j^2 \lambda_j^{\sigma/2}$, by

$$(-\Delta)^{\sigma/2} u = \sum a_j \lambda_j^{\sigma/2} \varphi_j.$$

This operator can be extended by density for u in the Hilbert space

$$H_0^{\sigma/2}(\Omega) = \left\{ u = \sum a_j \varphi_j \in L^2(\Omega) : \ \sum a_j^2 \lambda_j^{\sigma/2} < \infty \right\},$$

equipped with the norm

$$\|u\|^2_{H_0^{\sigma/2}(\Omega)} = \sum a_j^2 \lambda_j^{\sigma/2} = \left\| (-\Delta)^{\sigma/4} u \right\|_2^2.$$

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The fractional laplacian can be also defined by σ harmonic extension which was introduced by Caffarelli and Silvestre for the case of the whole space in [7], and extended to bounded domains in [1]. If $u = u(x)$ is a regular function in Ω, we define its σ-harmonic extension to the cylinder $C_\Omega = \Omega \times (0, \infty)$ with $\partial_L C_\Omega$ lateral boundary $\partial\Omega \times (0, +\infty)$, $E_\sigma(u) = U(x, y)$, is the unique smooth function and bounded, what is the solution to the problem

$$\begin{cases} L_\sigma U = 0, & \text{in } \ C_\Omega, \\ U = 0, & \text{on } \ \partial_L C_\Omega, \\ Tr(U) = u(x), & \text{on } \ \Omega, \end{cases}$$

where

$$L_\sigma U \equiv div(y^{1-\sigma} \nabla U) \quad \text{and} \quad Tr(U) = U(x, 0).$$

By Caffarelli and Silvestre (see [7]), we have

$$(-\Delta)^{\sigma/2} u(x) = -\mu_\sigma \lim_{y \longrightarrow 0^+} y^{1-\sigma} \frac{\partial U}{\partial y}(x, y) \equiv -\frac{\partial U}{\partial y^\sigma},$$

where μ_σ is a constant.

The operator E_σ can be extended to $H_0^{\sigma/2}(\Omega)$. we need to consider the energy space $X_0^\sigma(C_\Omega)$, given by

$$X_0^\sigma(C_\Omega) = \left\{ w \in L^2(C_\Omega) : w = 0, \ on \ \partial_L C_\Omega, \ \int_{C_\Omega} y^{1-\sigma} |\nabla w|^2 \, dxdy < \infty \right\}$$

with norm

$$\|w\|_{X_0^\sigma} = \left(\mu_\sigma \int_{C_\Omega} y^{1-\sigma} |\nabla w|^2 \, dxdy \right)^{1/2}.$$

Moreover, the operator $E_\sigma : H_0^{\sigma/2}(\Omega) \longrightarrow X_0^\sigma(C_\Omega)$ is an isometry (see [1]).

We investigate the solution of the problem (1.1), using the σ harmonic extension and the temporal discretization method, in the same way in [1].

A well-known method of construction of solutions of evolution equations, and also of generating a semigroup in a convenient functional space, is the so-called Implicit Time Discretization. given the evolution equation $v_t + A(v) = h(x, t)$, where A is a linear operator or not, bounded or not, acting on a Banach space X and given an initial data $v(0) = f \in X$, then the construction of an approximate solution of the problem in a time interval $[0, T]$ consists of dividing said interval into n subintervals of length $\epsilon = T/n$ and defining an approximate solution v_ϵ constant in each subinterval in the following sense: in each interval $(t_{k-1}, t_k]$ with $t_k = k\epsilon$, $k = 1, 2, \ldots, n$ is considered the solution $v_{\epsilon,k}$ for the discretized problem

$$\begin{cases} \frac{1}{\epsilon}(v_{\epsilon,k} - v_{\epsilon,k-1}) + A(v_{\epsilon,k}) = h_k(x) \\ v_{\epsilon,0} = f_\epsilon, \end{cases}$$

where f_ϵ is an approximation of f. Moreover, h_k is a one finite sequence of element in X such that

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|h(s) - h_k\|_X \, ds \leq \epsilon.$$

Once the existence of these solutions is guaranteed for each $k = 1, 2, \ldots, n$ the mild solution is obtained when $\epsilon \longrightarrow 0$.

We construct a theory of existence and uniqueness of general solutions for the problem (1.1). This theory provides a method or technique for solving problems related to anomalous diffusion or nonlinear diffusion equations of recent research (see [12] and [13]).

2. Main Results

In this section we will present the main results. In the same way as in the article [1] multiplying by $\varphi \in C_0^1(\Omega \times (0,T))$ and integrate by parts in the first equation (1.1), we have

$$-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} dxds + \int_0^T \int_\Omega (-\Delta)^{\sigma/4} u (-\Delta)^{\sigma/4} \varphi dxds = \int_0^T \int_\Omega h(x,t) \varphi dxds. \tag{2.1}$$

This identity will be the basis of our definition of a weak solution. The integrals in (2.1) make sense if u belong to suitable spaces.

Definition 2.1 *we say that u is a weak solution to problem (1.1) if:*

(i) $u \in C\left([0,\infty); L^1(\Omega)\right)$, $u \in L^2_{loc}\left((0,\infty); H_0^{\sigma/2}(\Omega)\right)$.

(ii) Identity (2.1) holds for every $\varphi \in C_0^1\left(\Omega \times (0,T)\right)$.

(iii) $u(\cdot,0) = f$ *almost everywhere of* Ω *with respect to the Lebesgue measure.*

Now, if $w(x,y,t) = E_\sigma(u)$ in the system (1.1), we obtained

$$\begin{cases} L_\sigma w = 0 & \text{in } \; C_\Omega \times (0,\infty) \\ w = 0 & \text{on } \; \partial_L C_\Omega, \quad t > 0 \\ w_t - \dfrac{\partial w}{\partial y^\sigma} = h(x,t), & \text{on } \; \Omega, \quad y = 0, \quad t > 0 \\ w = f(x), & \text{on } \; \Omega, \quad y = 0, \quad t = 0. \end{cases} \tag{2.2}$$

We will showed the existence of a weak solution of the problem (2.2). So, multiplying (2.2) by a function $\varphi \in C_0^1(\Omega \times (0,\infty) \times (0,\infty))$ and integrate on $\Omega \times \{0\} \times (0,+\infty)$, we have

$$\int_0^\infty \int_\Omega Tr(w) \frac{\partial \varphi}{\partial t} dxds - \mu_\sigma \int_0^\infty \int_{C_\Omega} y^{1-\sigma} < \nabla w, \nabla \varphi > d\overline{x}ds + \int_0^\infty \int_\Omega h(x,t) \varphi dxds = 0. \tag{2.3}$$

The functions $Tr(w)$ and w must be in an appropriate functional space.

Definition 2.2 *A pair of function (u,w) is a weak solution to problem (2.2), if:*

(i) $u = Tr(w) \in C\left((0,\infty) : L^1(\Omega)\right)$, $w \in L^2_{loc}\left((0,\infty) : X_0^\sigma(C_\Omega)\right)$.

(ii) Identity (2.3) holds for every $\varphi \in C_0^1\left(\Omega \times (0,\infty) \times (0,\infty)\right)$

(iii) $u(\cdot,0) = f$, *almost everywhere of* Ω *with respect to the Lebesgue measure.*

The two definitions of weak solutions, Definition (2.1) and Definition (2.2), are equivalent. We have the following theorem (see [1])

Theorem 2.1 *A function u is a weak solution to problem (1.1) if only if $(u, E_\sigma(u))$ is a weak solution to problem (2.2).*

By Theorem (2.1) it will be enough to guarantee the existence of the pair of weak solutions of the extended problem. We hope, although not sufficiently, that the Crandall-Liggett Theorem (see [5]) will be fundamental to guarantee it. In this case, observe that putting together the last two equations of the system (2.2), where $w = E_\sigma(u)$, we have

$$\begin{cases} w_t - \frac{\partial w}{\partial y^\sigma} = h(x,t), & \text{on } \; \Omega, \quad y = 0, \quad t > 0 \\ w = f(x), & \text{on } \; \Omega, \quad y = 0, \quad t = 0. \end{cases}$$

The first equation is equal to $u_t + A(u) = h(x,t)$ where

$$A(u) = -Tr\left(\frac{\partial E_\sigma(u)}{\partial y^\sigma}\right) = (-\Delta)^{\sigma/2}u.$$

Also, remember that the basis of the proof guarantees the existence of a weak solution using the temporal discretization method.

For our problem (2.2), remembering that $w(x,y,t) = E_\sigma(u)$ over C_Ω or also that $Tr(w) = w(x,0,t) = u(x,t)$ in Ω, we have

$$\begin{cases} L_\sigma w_{\epsilon,k} = 0 & \text{in } C_\Omega, \\ w_{\epsilon,k} = 0 & \text{on } \partial_L C_\Omega, \\ u_{\epsilon,k} - \epsilon\dfrac{\partial w_{\epsilon,k}}{\partial y^\sigma} = \epsilon h_k(x) + u_{\epsilon,k-1}, & \text{on } \Omega, \\ u_{\epsilon,0} = f(x), & \text{on } \Omega. \end{cases} \tag{2.4}$$

In each stage of the discretized problems given in (2.4) we have that $u_{\epsilon,k-1} = Tr(w_{\epsilon,k-1})$ is known while $u_{\epsilon,k}$ and $w_{\epsilon,k} = E_\sigma(u_{\epsilon,k})$ are unknown.

The third equation of (2.4) can be written like

$$u_{\epsilon,k} + \epsilon A(u_{\epsilon,k}) = g^\epsilon_{k-1}(x),$$

where $g^\epsilon_{k-1}(x) := \epsilon h_k(x) + u_{\epsilon,k-1}$, for $k = 1, 2, \ldots, n$. In that order of ideas, we need to establish the solving capacity of the elliptical problem associated with the system (2.2) which is given by:

$$\begin{cases} L_\sigma w = 0 & \text{in } C_\Omega, \\ w = 0 & \text{on } \partial_L C_\Omega, \\ w - \epsilon\dfrac{\partial w}{\partial y^\sigma} = h_\epsilon(x), & \text{on } \Omega, \quad y = 0, \end{cases} \tag{2.5}$$

for a given $\epsilon > 0$ and h_ϵ represents the function g^ϵ_{k-1} that appears once the temporal discretization method is applied in the problem (2.2). To guarantee the existence of this weak solution of (2.5) we are going to assume that Ω is a bounded extension domain and also that $h_\epsilon \in L^\infty(\Omega)$.

Theorem 2.2 *Let $\epsilon > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. For each $h_\epsilon \in L^\infty(\Omega)$ there is a unique weak solution $w \in X_0^\sigma(C_\Omega)$ for the problem (2.5). Also, if w and $\overline{w}$ are two solutions with respective data h_ϵ and $\overline{h}_\epsilon$, then*

$$\int_\Omega [w(x,0) - \overline{w}(x,0)]_+ dx \le \int_\Omega [h_\epsilon - \overline{h}_\epsilon]_+ dx, \tag{2.6}$$

where $[a] := \max[a, 0]$, with $a \in \mathbb{R}$.

Proof: The solution can be obtained in a standard way by minimizing the functional

$$I(w) = \epsilon\frac{\mu_\sigma}{2}\int_{C_\Omega} y^{1-\sigma}|\nabla w|^2\, dxdy - \int_\Omega h_\epsilon(x) w dx + \frac{1}{2}\int_\Omega w^2 dx,$$

It is easy to check that the functional is coercive in $X_0^\sigma(C_\Omega)$. □

The Theorem (2.2) not only does it guarantee the existence of solutions of the discretized problem, it also shows that the operator $(-\Delta)^{\sigma/2}$ is m-accretive in

$$D(A) = \left\{v \in L^1(\Omega) \cap L^\infty(\Omega) : A(v) \in L^1(\Omega) \cap L^\infty(\Omega), \quad \|v\|_\infty \le \|f\|_\infty\right\}.$$

Next we establish the existence of a weak solution for Cauchy's abstract problem.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any $f \in L^1(\Omega) \cap L^\infty(\Omega)$ there is a weak solution (u, w) for the problem (2.2). Moreover, if (u, w) and $(\overline{u}, \overline{w})$ are the weak solutions corresponding to initial data f and $\overline{f}$, then*

$$\int_\Omega [u(x,t) - \overline{u}(x,t)]_+ dx \le \int_\Omega [f(x) - \overline{f}(x)]_+ dx. \tag{2.7}$$

3. Proof of Theorem (2.3)

Proof: Recall that the Theorem (2.2) only proves in principle the existence of a so-called mild solution. However, we will show that such a mild solution is indeed a weak solution:

Taking $\epsilon = T/n$, we build a couple of functions (u_ϵ, w_ϵ) (with $E_\sigma(u_\epsilon) = w_\epsilon$ or equivalent $Tr(w_\epsilon) = u_\epsilon$) which are constant on $[0,T]$ in the following case:

$$u_\epsilon(x,t) = \begin{cases} u_{\epsilon,0}(x) & \text{if } \ t = 0 \\ u_{\epsilon,1}(x) & \text{if } \ t \in (t_0, t_1] \\ \vdots & \\ u_{\epsilon,n}(x) & \text{if } \ t \in (t_{n-1}, t_n] \end{cases}$$

and $$w_\epsilon(x,t) = \begin{cases} w_{\epsilon,1}(x) & \text{if } \ t \in (t_0, t_1] \\ \vdots & \\ w_{\epsilon,n}(x) & \text{if } \ t \in (t_{n-1}, t_n], \end{cases}$$ where $t_k = k\epsilon$, for $k = 1, 2 \ldots, n$ and moreover these constant functions are solutions of the discretized problem (2.4) given by

$$\begin{cases} L_\sigma w_{\epsilon,k} = 0 & \text{in } \ C_\Omega, \\ w_{\epsilon,k} = 0 & \text{on } \ \partial_L C_\Omega, \\ u_{\epsilon,k} - \epsilon \dfrac{\partial w_{\epsilon,k}}{\partial y^\sigma} = \epsilon h_k(x) + u_{\epsilon,k-1}, & \text{on } \ \Omega, \\ u_{\epsilon,0} = f(x), & \text{on } \ \Omega, \end{cases} \tag{3.1}$$

with $u_{\epsilon,k-1} = w_{\epsilon,k-1}(\cdot, 0)$. The mild solution is obtained where $\epsilon \longrightarrow 0^+$, which exists thanks to the Theorem (2.2).

Consider

$$\lim_{\epsilon \longrightarrow 0^+} (u_\epsilon, w_\epsilon) = (u, w).$$

Notice that by José M. Ruiz Theorem (see [4]), we have that u_ϵ converge in $L^1(\Omega)$ for some function $u \in C([0,\infty) : L^1(\Omega))$. In fact, the convergence is for $u \in C([0,\infty) : L^1(\Omega) \cap L^\infty(\Omega))$. On the other hand, the extension u coincides with $\lim\limits_{\epsilon \longrightarrow 0^+} w_\epsilon$. By construction, it conclude that $w \in L^\infty(C_\Omega \times [0,\infty))$.

Now, note that multiplying the equation (3.1) by $w_{\epsilon,k}(x,0)$, integrating on Ω and using integration by parts, we have:

$$\epsilon \int_\Omega \frac{\partial w_{\epsilon,k}}{\partial y^\sigma} \cdot w_{\epsilon,k} = -\epsilon \int_\Omega h_k(x) w_{\epsilon,k} + \int_\Omega (u_{\epsilon,k} - u_{\epsilon,k-1}) w_{\epsilon,k}$$

Equivalent,

$$\epsilon \mu_\sigma \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_{\epsilon,k}\right|^2 = \epsilon \int_\Omega h_k(x) u_{\epsilon,k} + \int_\Omega (u_{\epsilon,k-1} \cdot u_{\epsilon,k} - u_{\epsilon,k}^2).$$

Moreover, since the solutions $u_{\epsilon,k}$ are nonnegative, using the inequality of Young's numbers note that

$$\begin{aligned} \epsilon \mu_\sigma \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_{\epsilon,k}\right|^2 &= \epsilon \int_\Omega h_k(x) u_{\epsilon,k} + \int_\Omega (u_{\epsilon,k-1} \cdot u_{\epsilon,k} - u_{\epsilon,k}^2) \\ &\leq \epsilon \int_\Omega h_k(x) u_{\epsilon,k} + \int_\Omega \left(\frac{(u_{\epsilon,k-1})^2}{2} + \frac{(u_{\epsilon,k})^2}{2} - u_{\epsilon,k}^2 \right) \\ &= \epsilon \int_\Omega h_k(x) \left|u_{\epsilon,k}\right| + \frac{1}{2} \int_\Omega \left|u_{\epsilon,k-1}\right|^2 - \left|u_{\epsilon,k}\right|^2. \end{aligned}$$

This is,

$$\epsilon\mu_\sigma \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_{\epsilon,k}\right|^2 \leq \epsilon \int_\Omega h_k(x) \left|u_{\epsilon,k}\right| + \frac{1}{2}\int_\Omega \left|u_{\epsilon,k-1}\right|^2 - \left|u_{\epsilon,k}\right|^2 . \tag{3.2}$$

Now, adding from $k = 1$ to $k = n$ in (3.2) we have

$$\mu_\sigma \int_{C_\Omega} y^{1-\sigma} \sum_{k=1}^{n} \epsilon \left|\nabla w_{\epsilon,k}\right|^2 \leq \epsilon \sum_{k=1}^{n} \int_\Omega h_k(x) \left|u_{\epsilon,k}\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 .$$

But note that for any $k = 1, 2, \ldots, n$

$$\int_{t_{k-1}}^{t_k} dt = \epsilon \quad \text{y} \quad \int_{[0,T]} dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} dt = n\epsilon.$$

Implying that

$$\int_0^T \left|\nabla w_\epsilon\right|^2 dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|\nabla w_{\epsilon,k}\right|^2 dt = \sum_{k=1}^{n} \left(\left|\nabla w_{\epsilon,k}\right|^2 \int_{t_{k-1}}^{t_k} dt \right) = \sum_{k=1}^{n} \epsilon \left|\nabla w_{\epsilon,k}\right|^2 .$$

That is,

$$\mu_\sigma \int_0^T \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_\epsilon\right|^2 \leq \epsilon \sum_{k=1}^{n} \int_\Omega h_k(x) \left|u_{\epsilon,k}\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 . \tag{3.3}$$

Now, like u_ϵ converge to u in $L^\infty(\Omega)$, then there is convergence in norm. Moreover, h_k converges to h in $L^1(\Omega)$ for every $t > 0$ by definition the mild solution. So, using this fact in (3.3), we have

$$\begin{aligned}
\mu_\sigma \int_0^T \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_\epsilon\right|^2 &\leq \epsilon \sum_{k=1}^{n} \int_\Omega h_k(x) \left|u_{\epsilon,k}\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 \\
&\leq \int_0^T \left\|h(s) - h_k\right\|_1 \left|u_\epsilon\right| + \int_0^T \int_\Omega |h| \left|u_\epsilon\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 \\
&\leq \epsilon \left\|u_\epsilon\right\|_\infty + \int_0^T \int_\Omega |h| \left|u_\epsilon\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 .
\end{aligned}$$

This is,

$$\mu_\sigma \int_0^T \int_{C_\Omega} y^{1-\sigma} \left|\nabla w_\epsilon\right|^2 \leq \epsilon \left\|u_\epsilon\right\|_\infty + \int_0^T \int_\Omega |h| \left|u_\epsilon\right| + \frac{1}{2}\int_\Omega \left|f(x)\right|^2 . \tag{3.4}$$

Thus, Passing the limit in (3.4) we obtain the same inequality for $|\nabla w|$ and $|u|$. Showing that $w \in L^2([0,T]; X_0^\sigma(C_\Omega))$.

So far, we have that the mild solution satisfies only two conditions to make it a weak solution. It remains to verify that it also satisfies the integral identity. This is, for any $\varphi \in C_0^1(\Omega \times (0,T))$,

$$-\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t} dxds + \int_0^T \int_\Omega (-\Delta)^{\sigma/4} u (-\Delta)^{\sigma/4} \varphi dxds = \int_0^T \int_\Omega h(x,t)\varphi dxds, \tag{3.5}$$

where

$$u = \lim_{\epsilon \longrightarrow 0^+} u_\epsilon = \lim_{\epsilon \longrightarrow 0^+} Tr(w_\epsilon).$$

Before starting the proof, note that the integral identity (3.5) is equals

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}+\int_0^T\int_\Omega(-\Delta)^{\sigma/4}u(-\Delta)^{\sigma/4}\varphi=\int_0^T\int_\Omega h(x,t)\varphi+\int_\Omega f(x)\varphi(x,0,0)$$
$$-\int_\Omega u(x,T)\varphi(x,0,T) \tag{3.6}$$

first, notice that for $k=1,2,\dots,n$

$$-\int_\Omega\frac{\partial w_{\epsilon,k}}{\partial y^\sigma}\varphi(x,0,t)=\mu_\sigma\int_{C\Omega}y^{1-\sigma}<\nabla w_{\epsilon,k},\nabla\varphi>. \tag{3.7}$$

And by (3.1) we have

$$-\frac{\partial w_{\epsilon,k}}{\partial y^\sigma}=\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}+h_k(x).$$

So, replacing in (3.7)

$$\int_\Omega\left(\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}+h_k(x)\right)\varphi=\mu_\sigma\int_{C\Omega}y^{1-\sigma}<\nabla w_{\epsilon,k},\nabla\varphi>. \tag{3.8}$$

Now, integrating over $(t_{k-1},t_k]$ and adding from $k=1,2,\dots,n$ in (3.8) it follows that

$$\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_\Omega\left(\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}\right)\varphi+\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_\Omega h_k(x)\varphi=\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\mu_\sigma\int_{C\Omega}y^{1-\sigma}<\nabla w_{\epsilon,k},\nabla\varphi>. \tag{3.9}$$

Notice that the first integral on the left side with respect to the variable t is equivalent to

$$\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left(\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}\right)\varphi dt=\frac{1}{\epsilon}\int_0^\epsilon f(x)\varphi(x,0,t)dt+\int_0^T u_\epsilon\frac{1}{\epsilon}\left(\varphi(x,0,t+\epsilon)-\varphi(x,0,t)\right)dt$$
$$-\frac{1}{\epsilon}\int_{T-\epsilon}^T u_\epsilon\varphi(x,0,t+\epsilon)dt.$$

From that way

$$\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_\Omega\left(\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}\right)\varphi=\frac{1}{\epsilon}\int_0^\epsilon\int_\Omega f(x)\varphi(x,0,t)dxdt+\int_0^T\int_\Omega u_\epsilon\frac{1}{\epsilon}\left(\varphi(x,0,t+\epsilon)-\varphi(x,0,t)\right)dxdt$$
$$-\frac{1}{\epsilon}\int_{T-\epsilon}^T\int_\Omega u_\epsilon\varphi(x,0,t+\epsilon)dxdt.$$

While the second integral from the left side is equal to

$$\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_\Omega h_k(x)\varphi(x,0,t)=\int_0^T\int_\Omega h(x,t)\varphi(x,0,t)dxdt.$$

On the other hand, notice that

$$\mu_\sigma\int_0^T\int_{C\Omega}y^{1-\sigma}<\nabla w_\epsilon,\nabla\varphi>=\mu_\sigma\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_{C\Omega}y^{1-\sigma}<\nabla w_\epsilon,\nabla\varphi>$$
$$=\mu_\sigma\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_{C\Omega}y^{1-\sigma}<\nabla w_{\epsilon,k},\nabla\varphi>.$$

Furthermore, since E_σ is an isometric and $w_\epsilon = E_\sigma(u_\epsilon)$ we have

$$\mu_\sigma \int_{C_\Omega} y^{1-\sigma} < \nabla w_\epsilon, \nabla\varphi > = \mu_\sigma \int_{C_\Omega} y^{1-\sigma} < \nabla E_\sigma(u_\epsilon), \nabla E_\sigma(\varphi) >$$
$$= \int_\Omega (-\Delta)^{\sigma/4} u_\epsilon (-\Delta)^{\sigma/4}\varphi$$

Therefore, replacing in (3.9) it follows that

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_\Omega \left(\frac{u_{\epsilon,k-1}-u_{\epsilon,k}}{\epsilon}\right)\varphi + \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_\Omega h_k(x)\varphi = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mu_\sigma\int_{C_\Omega} y^{1-\sigma} < \nabla w_{\epsilon,k}, \nabla\varphi > .$$

Equivalent

$$\frac{1}{\epsilon}\int_0^\epsilon\int_\Omega f(x)\varphi(x,0,t)dtdx + \int_0^T\int_\Omega u_\epsilon \frac{1}{\epsilon}\left(\varphi(x,0,t+\epsilon)-\varphi(x,0,t)\right)dtdx$$
$$-\frac{1}{\epsilon}\int_{T-\epsilon}^{T}\int_\Omega u_\epsilon\varphi(x,0,t+\epsilon)dtdx + \int_0^T\int_\Omega h(x,t)\varphi(x,0,t)dxdt = \int_0^T\int_\Omega (-\Delta)^{\sigma/4}u_\epsilon(-\Delta)^{\sigma/4}\varphi dxdt.$$

This is,

$$-\int_0^T\int_\Omega u_\epsilon\frac{1}{\epsilon}\left(\varphi(x,0,t+\epsilon)-\varphi(x,0,t)\right)dtdx + \int_0^T\int_\Omega (-\Delta)^{\sigma/4}u_\epsilon(-\Delta)^{\sigma/4}\varphi dxdt =$$
$$\int_0^T\int_\Omega h(x,t)\varphi(x,0,t)dxdt + \frac{1}{\epsilon}\int_0^\epsilon\int_\Omega f(x)\varphi(x,0,t)dtdx - \frac{1}{\epsilon}\int_{T-\epsilon}^{T}\int_\Omega u_\epsilon\varphi(x,0,t+\epsilon)dtdx$$

Thus, doing $\epsilon \longrightarrow 0^+$ follows (3.6).

Finally, for $k = 1, 2, \dots, n$ suppose that $w_{\epsilon,k}$ and $\overline{w}_{\epsilon,k}$ are two different approximations of weak solutions w and $\overline{w}$, respectively from problem (3.1). Then, applying the elliptic inequality of Theorem (2.2) with $w_{\epsilon,k}$ and $\overline{w}_{\epsilon,k}$, we have

$$\int_\Omega \left(w_{\epsilon,k}(x,0) - \overline{w}_{\epsilon,k}(x,0)\right)_+ \le \int_\Omega \left(h_\epsilon(x) - \overline{h}_\epsilon(x)\right)_+ .$$

This is, for each $k = 1, 2, \dots, n$

$$\int_\Omega \left(u_{\epsilon,k}(x) - \overline{u}_{\epsilon,k}(x)\right)_+ \le \int_\Omega \left(\epsilon h(x) + u_{\epsilon,k-1}(x) - \epsilon h(x) - \overline{u}_{\epsilon,k-1}(x)\right)_+$$
$$= \int_\Omega \left(u_{\epsilon,k-1}(x) - \overline{u}_{\epsilon,k-1}(x)\right)_+$$

From where we can conclude that

$$\int_\Omega \left(u_\epsilon - \overline{u}_{\epsilon,}\right)_+ \le \int_\Omega \left(f(x) - \overline{f}(x)\right)_+ . \tag{3.10}$$

Thus, passing the limit in (3.10) we conclude that

$$\int_\Omega \left(u(x,t) - \overline{u}(x,t)\right)_+ \le \int_\Omega \left(f(x) - \overline{f}(x)\right)_+ .$$

□

4. Conclusion

We construct a theory of existence and uniqueness of solutions for an abstract Cauchy problem. Using Duhamel's formula the problem (1.1) has an explicit solution given by a Green's function,

$$u(x,t) = \int_\Omega f(y)G(x,y,t)dy + \int_0^t \int_\Omega h(y,s)G(x,y,t-s)dyds, \tag{4.1}$$

where

$$G(x,y,t) = \sum_{j=1}^{\infty} \varphi_j(x)\varphi_j(y)e^{-\lambda_j^{\sigma/2}t}.$$

This solution (4.1) given by the classical Duhamel formula is not necessarily a strong solution of problem (1.1). Consequently, in this article, we guarantee the existence of a more general solution that solves the abstract Cauchy problem (see [4]).

We can extend problem (1.1) to use the Perturbation Theorem (see [10]). In that sense, we will have the system given by

$$\begin{cases} u_t + (-\Delta)^{\sigma/2}u + G(u) = h(x,t), & \text{in } \ \Omega \times \mathbb{R}_+ \\ u = 0, & \text{on } \ \partial\Omega \times \mathbb{R}_+ \\ u(x,0) = f(x), & \text{on } \ x \in \Omega, \end{cases} \tag{4.2}$$

where G is an accretive and continuous operator in the same fractional Laplacian domain is $m-$ accretive. This system given in (4.2) is in fact an extension of problem (1.1) since even the operator G can be nonlinear.

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