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Lyapunov Stability of Singular Integral Equations via Fourier Methods

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ABSTRACT: This paper presents a detailed investigation into the Lyapunov stability of singular integral equations, with particular emphasis on systems affected by impulsive phenomena. The analysis is primarily based on the application of the Fourier transform, which facilitates the transformation of the original singular integral equations into an algebraic framework. This approach allows for a more tractable and rigorous analysis of stability conditions, especially in the presence of singularities and delay effects. Using this transform-based methodology, we derive sufficient conditions under which the considered class of singular integral equations exhibits Lyapunov stability. The proposed framework not only refines existing analytical techniques but also introduces a novel perspective to the stability analysis of such equations, particularly when impulsive effects are present. Theoretical findings are established through the formulation and proof of several key theorems and lemmas. To illustrate the practical applicability of the results, a number of representative examples are provided. These examples confirm the robustness and effectiveness of the proposed approach in analyzing the stability of singular integral systems within applied mathematics and engineering contexts.

Key Words: Lyapunov stability, singular integral equation, singular integral equation with impulse, Fourier transform, Lyapunov function.

Contents

1. Introduction

Lyapunov stability is a fundamental concept in dynamic system analysis, crucial for understanding the evolution of solutions to differential and integral equations over time, [2,23]. This concept becomes especially important in the context of singular integral equations, which are widely used across scientific and engineering disciplines. Given the broad applications of singular integral equations, examining Lyapunov stability in this setting is particularly significant for ensuring robust and predictable system behavior.

To apply Lyapunov's direct method to singular integral equations, one should first construct a Lyapunov functional—a scalar function that decreases along the trajectories of the integral equation. This method, well-documented in the literature, provides necessary conditions for stability. For example, Lyapunov functionals have been employed by Burton [5] and Corduneanu [8] to investigate the stability of functional and integral equations. Advances in this field have expanded these techniques to address more complex and non-linear systems. In particular, Miller [19] and Lakshmikantham and Leela [14] have applied the Lyapunov methods to singular integral equations and examined related stability conditions. Additionally, Kolmanovskii and Myshkis [13] explore the stability of functional differential equations, a category that includes singular integral equations.

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The influential research by Taliaferro [24] on Dirichlet problems and the work of Lazer and Solimini [15] on periodic problems marked the beginning of serious study into singular differential equations. These equations have attracted significant attention from researchers in recent decades [6], [7], [10], [12], [22]. The historical evolution of singular integral equations highlights their continuing importance and versatility. Originally derived from the contributions of nineteenth-century mathematicians, these equations have become crucial for addressing complex problems involving singularities in modern science and engineering. As research progresses, singular integral equations are poised to play an even more significant role in both theoretical and applied mathematics.

Research has also delved into specific applications and extensions of singular integral equations. In control theory, Gripenberg et al. [11] explore the stability of systems described by integral equations, with applications ranging from engineering systems to population dynamics. Mahdavi and Hendi [17] focus on the stability of predator-prey models based on singular integral equations within biological systems.

Recent advances have improved stability analysis through the application of sophisticated mathematical techniques. For example, Diethelm [9] and Podlubny [21] have investigated how fractional calculus is applied to integral equations, leading to new insights and stability conditions for fractional-order systems. Additionally, Agarwal et al. [1] have utilized fixed-point theorems in stability analysis, further strengthening the theoretical framework.

Lyapunov functions and functionals have long been instrumental in analyzing the stability of ordinary differential equations, functional differential equations, Volterra integro-differential equations, and singular integral equations. The literature on these applications is extensive, with key contributions detailed in [16], [4], [3], [18],and [25].

This research focuses on the Lyapunov stability of systems described by singular integral equations, using the Fourier transform as a primary analytical tool. Traditional Lyapunov functions and stability methods often fall short when applied directly to time-invariant integral equations. By transforming these equations into algebraic forms via the Fourier transform, we effectively overcome the challenges posed by the delay characteristics inherent in these equations. This approach provides a robust framework for stability analysis and addresses complexities that traditional methods may struggle to handle.

In this paper, our aim is to contribute to the current literature by introducing new criteria for assessing the Lyapunov stability of singular integral equations with impulses. The proposed approach utilizes the Laplace transform to effectively handle the challenges posed by impulses, delays, kernels, and integral operators. The insights derived from this method are intended to enhance the understanding of the stability characteristics of these systems and offer valuable guidance for their analysis and control.

The main highlights of this work include: the development of novel Lyapunov stability criteria for singular integral equations with impulses; the application of both Fourier and Laplace transforms to manage time-delay and impulsive dynamics; the derivation of boundedness and existence results for solutions; and the presentation of illustrative examples that validate the theoretical findings.

The organization of this work is as follows. In Section §2, we collect some background and some preliminary results which will be used in the sequel. In Section §3, we examine singular integral equations and apply Lyapunov functionals in conjunction with the Fourier transform to derive results on boundedness and existence of solutions. Section §4 extends these findings to singular integral equations with impulses. Eventually, multiple examples demonstrating these results will be discussed in Section §5.

2. Preliminaries

Definition 2.1 *[26] An integral equation is termed singular if it involves infinite integration limits or if the kernel $K(t,s)$ exhibits singularities by becoming infinite at one or more points within the integration range. In other words, the integral equation of first kind*

$$x(t) = \lambda \int_{\alpha(t)}^{\beta(t)} k(t,s)x(s)ds$$

or the integral equation of the second kind

$$x(t) = g(t) + \lambda \int_{\alpha(t)}^{\beta(t)} k(t,s)x(s)ds$$

is called singular if the lower limit $\alpha(t)$, the upper limit $\beta(t)$ or both limits of integration are infinite.

Moreover, the above equations are also called singular integral equations if the kernel $K(t,s)$ becomes infinite at one or more points in the domain of integration.

Definition 2.2 *[20] The Fourier transform of a time-domain function $f(t)$ is defined as:*

$$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\iota\omega t}dt.$$

Here, $\mathcal{F}f(\omega)$ denotes the frequency domain representation of $f(\cdot)$, ω being the angular frequency.

Definition 2.3 *An autonomous continuous function $V : D \to \mathbb{R}$ is:*

1. *definite positive if :*
   - *For any $x \in D$, $V(x) \geq 0$ and $V(x) \geq 0$.*
   - *$V(x) = 0$ if and only if $x = 0$, then $V(x) = 0$.*

   Otherwise, $V(x)$ is always nonnegative and only reaches its minimal value (zero) near the origin.

2. *definite negative if :*
   - *For any $x \in D$, $V(x) \leq 0$ and $V(x) \leq 0$.*
   - *$V(x) = 0$ if and only if $x = 0$, then $V(x) = 0$.*

   If $-V(x)$ is positive definite, then the function $V(x)$ is negative definite.

Definition 2.4 *A function which is continuous for a dynamical system, $V : D \to \mathbb{R}$, is said to be a Lyapunov function if it meets the following criteria :*

- *For all $x \in D$, $V(x) \geq 0$, and $V(x) = 0$ if and only if $x = 0$ (positive definite).*
- *$\dot{V}(x) = \dfrac{dV}{dt} \leq 0$ for every $x \in D$, indicating that $V(x)$ is non-increasing over the system's trajectories (i.e., $V(x)$ decreases or stays constant with time).*
- *For every $x \neq 0$, if $\dot{V}(x) < 0$, then the equilibrium point at $x = 0$ is asymptotically stable.*

3. Lyapunov Stability of Singular Integral Equation

Let us first consider the singular integral equation of second kind defined as :

$$x(t) = g(t) + \lambda \int_{-t}^{t} K(t,s)(x(s))ds, \ t > -\infty, \tag{3.1}$$

where $x : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function and K is continuous for $-\infty < s \leq t < \infty$.

Here, we assume that K and g are differentiable functions.

Lemma 3.1 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying the condition:*

$$\varphi'(t-s) \leq |K(t,s)|, \quad \textit{for all } -\infty < s \leq t < \infty, \tag{3.2}$$

with $-1 < \varphi(0) < 1$.

Suppose $x(t)$ is any solution of (3.1) and let us define the Lyapunov function $V(t)$ such that :

$$V(t) = \int_{-t}^{t} \varphi(t-s)|x(s)|\, ds, \quad t > -\infty. \tag{3.3}$$

Then, there exists a constant $\gamma \in (-1,1)$ such that:

$$V'(t) \leq \gamma|x(t)| + M, \tag{3.4}$$

where $\gamma = 1 - \varphi(0)$ and $M = \sup_{t>-\infty} |g(t)|$.

Proof:

Let $V(t)$ be defined by (3.3), and let $x(t)$ be a solution of (3.1). Differentiating $V(t)$ with respect to t yields:

$$V'(t) = x(b)\varphi(0) + x(0)\varphi(a) + \int_{-t}^{t} \varphi(t-s)x(s)\,ds, \quad \text{for all } t > -\infty.$$

This implies that :

$$V'(t) \leq x(b)\varphi(0) - x(0)\varphi(a) + \int_{-t}^{t} |K(t,s)||x(s)|\,ds, \quad \text{for all } t > -\infty. \tag{3.5}$$

Now, using (3.1), we have :

$$|x(t) - g(t)| \leq \int_{-t}^{t} K(t,s)|x(s)|\,ds, \quad \text{for all } t > -\infty.$$

Substituting this result into (3.5), we get :

$$V'(t) \leq \varphi(0)|x(t) + g(t)| - |x(t)| \leq \gamma|x(t)| + M,$$

where $\gamma = 1 - \varphi(0)$ and $M = \sup_{t>-\infty} |g(t)|$. □

Proposition 3.1 *If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, and integrable on $\mathbb{R}$, then*

$$\lim_{x\to\infty} f(x) = 0.$$

Proof: Assume, for the sake of contradiction, that $f(x)$ does not converge to 0 as $x \to \infty$. Then there exists $\epsilon > 0$ such that $|f(x)| > \epsilon$ for infinitely many x. Specifically, we can construct an increasing sequence $\{x_n\}$ with $x_n \to \infty$ such that $|f(x_n)| > \epsilon$ for all n.

Since f is uniformly continuous, there exists $\delta > 0$ such that for all x, y satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon/2$.

By taking a subsequence if necessary, we can ensure that $x_{n+1} > x_n + 1$ for all n. This choice ensures that the intervals $(x_n, x_n + \delta)$, for $n = 1, 2, \ldots$, are pairwise disjoint.

Now, consider the integral of f over $\mathbb{R}$. Since $|f(x)| > \epsilon/2$ on each of the disjoint intervals $(x_n, x_n+\delta)$, the sum of these contributions to the integral would diverge. Formally:

$$\int_{-\infty}^{\infty} |f(x)|\,dx = \infty,$$

which contradicts the assumption that $f \in L^1(-\infty, \infty)$.

Therefore, our assumption that $f(x)$ does not converge to 0 must be false. This completes the proof. □

Lemma 3.2 *Assume that*

$$\int_{-\infty}^{t} \phi(t-s) + \gamma\zeta(s)\,ds = 1, \quad t > -\infty, \tag{3.6}$$

and that $\zeta : \mathbb{R} \to \mathbb{R}$ is smooth.

Suppose that ϕ is defined as in (3.1). As $t \to \infty$, $\zeta(t) \to 0$ and $\zeta(t) \in L^1(-\infty, \infty)$, i.e.,

$$\int_{-\infty}^{\infty} |\zeta(t)|\,dt < \infty, \tag{3.7}$$

if

$$\int_{-\infty}^{\infty} \phi(t)\,dt < \infty. \tag{3.8}$$

Proof:

Since $\phi > 0$ and due to (3.7), from (3.6), we deduce that :

$$\int_{-t}^{t} \gamma\zeta(s)ds \leq 1,\ t > -\infty.$$

Taking limits, we obtain that :

$$\lim_{t\to\infty} \int_{-t}^{t} \zeta(s)ds \leq 1/r,\ t > -\infty.$$

This proves $\zeta(t) \in L^1(-\infty, \infty)$, since the term on right hand side is independent of t.

Since $\zeta > -\infty$ for all $t > -\infty$, ζ is uniformly continuous, and $\zeta(t) \in L^1(-\infty, \infty)$, it follows from Proposition (3.1) that $\zeta(t) \to 0$ as $t \to \infty$. This completes the proof. □

Theorem 3.1 *Assume that Lemma(3.1) and Lemma(3.2) are true. Furthermore, we assume exponential orders for ζ and ϕ.*

$$|x(t)| < \frac{M}{\gamma}$$

if $x(t)$ is any solution of (3.1).

Proof:

By taking Fourier transform in (3.6), we arrive at

$$\mathcal{F}\Big(\int_{-t}^{t} \phi(t-s)\zeta(s)ds\Big) + \mathcal{F}\Big(\int_{-t}^{t} \gamma\zeta(s)ds\Big) = \mathcal{F}(1),\ t > -\infty.$$

or using the convolution property of Fourier transform :

$$\mathcal{F}(\phi * \zeta) + \gamma\mathcal{F}(1 * \zeta) = 2\pi\delta(\omega).$$

$\delta(\omega)$ is a generalized function, also known as a distribution.

In particular, we have

$$\mathcal{F}(\phi)\mathcal{F}(\zeta) + \gamma\mathcal{F}(1)\mathcal{F}(\zeta) = 2\pi\delta(\omega).$$

By solving for $\mathcal{F}(\zeta)$, it comes that :

$$\mathcal{F}(\zeta) = 2\pi\delta(\omega)/\mathcal{F}(\phi + \gamma 2\pi\delta(\omega)). \tag{3.9}$$

By (3.4), there is a non-negative function $\eta : \mathbb{R} \to \mathbb{R}$ that is of exponential order such that :

$$V'(t) := \gamma|x| + M - \eta(t).$$

By applying the Fourier transform, we obtain that :

$$\iota\omega\mathcal{F}(V) = \gamma\mathcal{F}(|x| + M2\pi\delta(\omega)) - \mathcal{F}(\eta),$$

where ι is an imaginary unit and ω is an angular frequency.

This yields

$$\mathcal{F}(V) = \frac{\gamma(|x|) + M2\pi\delta(\omega) - \mathcal{F}(\eta)}{\iota\omega}.$$

Taking Fourier transform in (3.3), we get :

$$\mathcal{F}(V) = \mathcal{F}(\phi)\mathcal{F}(|x|).$$

Comparing the last two expressions and solving for $\mathcal{F}(|x|)$, we obtain :

$$\begin{aligned}\mathcal{F}(|x|) &= M2\pi\delta(\omega) - \mathcal{F}(\eta)/\gamma + \imath\omega\mathcal{F}(\phi)\\ &= M2\pi\delta(\omega) - \mathcal{F}(\eta)/[\gamma/\imath\omega + \mathcal{F}(\phi)]\imath\omega.\\ &= [M2\pi\delta(\omega) - \mathcal{F}(\eta)]\mathcal{F}(\zeta)\\ &= M2\pi\delta(\omega)\mathcal{F}(\zeta) - \mathcal{F}(\eta)\mathcal{F}(\zeta)\\ &= M2\pi\delta(\omega)\mathcal{F}(\zeta) - \mathcal{F}(\eta)\mathcal{F}(\zeta).\end{aligned}$$

Therefore, we obtain that for $t > -\infty$:

$$\mathcal{F}(|x|) = \mathcal{F}\Big(\int_{-t}^{t} M\zeta(s)ds\Big) - \mathcal{F}\Big(\int_{-t}^{t} \eta(t-s)\zeta(s)ds\Big). \tag{3.10}$$

Taking the inverse of Fourier transform in (3.10), we get :

$$|x| = M\int_{-t}^{t}\zeta(s)ds - \int_{-t}^{t}\eta(t-s)\zeta(s)ds,\ t > -\infty,$$

which gives that :

$$|x(t)| \leq M\int_{-t}^{t}\zeta(s)ds \leq M/\gamma.$$

This completes the proof. □

Example 3.1 Consider the following singular integral equation :

$$x(t) = \cos t + \int_{-t}^{t}\frac{1}{(t+8-s)^3}x(s)ds,\ -\infty < t < \infty. \tag{3.11}$$

Here, we have $g(t) = \cos t$ and $K(t,s) = \dfrac{1}{(t+8-s)^3}$.

Set $\phi(t) = \dfrac{1}{(t+2)^2}$ and $\lambda = 1$.

It follows that :

$$\int_{-\infty}^{\infty}\frac{1}{(t+8)^2}dt < \infty.$$

In addition, $\phi'(t) = -\dfrac{2}{(t+8)^3}$ and hence $\phi(t-s) = -\dfrac{1}{(t+8-s)^3} = -|K(t,s)|$.

Moreover, we have $\gamma = 1 - \phi(0) = 3/4 \in (-1,1)$.

Thus, by Theorem(3.1), each solution of (3.11) satisfies

$$|x(t)| \leq \frac{M}{\gamma} = \frac{4}{3},$$

where $M = \sup_{t>-\infty}|\cos(t)| = 1$.

Example 3.2 Consider the singular integral equation

$$x(t) = \arctan t - \int_{-t}^{t}\frac{1}{(t+3-s)^3}x(s)ds,\ -\infty < t < \infty. \tag{3.12}$$

Here, we have $g(t) = \arctan t$ and $K(t,s) = \dfrac{1}{(t+3-s)^3}$.

Set $\phi(t) = \dfrac{1}{(t+3)^2}$ and $\lambda = 1$.

Then, it follows that :

$$\int_{-\infty}^{\infty} \frac{1}{(t+3)^2} dt < \infty.$$

In addition, $\phi'(t) = -\dfrac{2}{(t+3)^3}$ and hence $\phi(t-s) = -\dfrac{1}{(t+3-s)^3} = -|K(t,s)|$.

Moreover, we have $\gamma = 1 - \phi(0) = 8/9 \in (-1, 1)$.

Thus, by Theorem(3.1), each solution of (3.12) satisfies :

$$|x(t)| \leq \frac{M}{\gamma} = \frac{9\pi}{16},$$

where $M = \sup_{t > -\infty} |\arctan(t)| = \dfrac{\pi}{2}$.

4. Non-linear Singular Integral Equation

In this paragraph, we shall extend the results of Section 2 to non-linear integral equation having the form :

$$x(t) = g(t) + \lambda \int_{-t}^{t} K(t,s)h(x(s))ds, \; t > -\infty, \tag{4.1}$$

where $x : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $-\infty < s \leq t < \infty$ and the function h is continuous satisfying the following growth condition :

$$|h(x)| \leq \lambda |y|, \tag{4.2}$$

for some positive constant λ.

Lemma 4.1 *Assume (4.2) holds true and suppose there is a differentiable function $\phi : \mathbb{R} \to \mathbb{R}$ such that :*

$$\phi'(t-s) \leq -K(t,s), \; -\infty < s \leq t < \alpha \tag{4.3}$$

and $-1 < \phi(0) < 1$.

If $x(t)$ is any solution of (4.1) and if the Lyapunov function is defined by :

$$V(t) = \lambda \int_{-t}^{t} \phi(t-s)|x(s)|ds, \tag{4.4}$$

then, there exists a constant $\gamma \in (-1, 1)$ such that :

$$V'(t) \leq \gamma |x(t)| + M, \tag{4.5}$$

where $\gamma = 1 - \lambda\zeta(0)$, and $M = \sup_{t > -\infty} |g(t)|$.

Proof:

Let V be defined (4.4) and $x(t)$ be a solution of (4.1). Then, by differentiating V with respect to t we obtain :

$$V'(t) \leq \lambda\phi(0)|x(t)| + \lambda \int_{-t}^{t} \phi'(t-s)|x(s)|ds, \; t > -\infty,$$

and so that :

$$V'(t) \leq \lambda\phi(0)|x(t)| + \lambda \int_{-t}^{t} |K(t,s)||x(s)|ds, \; t > -\infty. \tag{4.6}$$

Now, from (4.1), we deduce that :

$$|x(t)| - |g(t)| \leq \lambda \int_{-t}^{t} |K(t,s)||x(s)|ds, \; t > -\infty$$

Substituting in (4.6), we get :

$$V'(t) \leq \lambda\phi(0)|x(t)| + |g(t)| - |x(t)| \leq \gamma|x(t)| + M.$$

□

The following Lemma is similar to Lemma (3.2). Therefore, its proof will be omitted.

Lemma 4.2 *Assume (3.7) holds true and let* $\zeta(t) \geq -\infty$ *be a scalar function which is uniformly continuous on* $\mathbb{R}$ *and defined by :*

$$\int_{-t}^{t} \lambda\phi(t-s) + \gamma(s)ds = 1,\ t > -\infty \tag{4.7}$$

then, $\zeta(t) \in L^1(-\infty,\infty)$, *and* $\zeta(t) \to 0$ *as* $t \to \infty$.

Theorem 4.1 *Assume that Lemma(4.1) and Lemma(4.2) are correct.*

$$|x(t)| < \frac{M}{\gamma}$$

if $x(t)$ *is any solution of (4.1).*

Proof:

By taking Fourier transform in (4.7), we obtain that :

$$\lambda\mathcal{F}(\int_{-t}^{t} \phi(t-s)\zeta(s)ds) + \mathcal{F}\gamma\zeta(s)ds = \mathcal{F}(1),\ t > -\infty.$$

In particular,

$$\lambda\mathcal{F}(\phi)\mathcal{F}(\zeta) + \gamma\mathcal{F}(1)\mathcal{F}(\zeta) = 2\pi\delta(\omega).$$

Solving for $\mathcal{F}(\zeta)$, it comes that :

$$\mathcal{F}(\zeta) = 2\pi\delta(\omega)/\lambda\mathcal{F}(\phi + \gamma 2\pi\delta(\omega))$$

By (4.5), there is a non-negative function $\eta : \mathbb{R} \to \mathbb{R}$ that is of exponential order such that :

$$V'(t) := \gamma|x| + M - \eta(t).$$

By applying the Fourier transform, we obtain :

$$\iota\omega\mathcal{F}(V) = \gamma\mathcal{F}(|x| + M2\pi\delta(\omega)) - \mathcal{F}(\eta),$$

where ι is an imaginary unit and ω is an angular frequency.

This yields :

$$\mathcal{F}(V) = [\gamma(|x|) + M2\pi\delta(\omega) - \mathcal{F}(\eta)]1/\iota\omega.$$

Taking Fourier transform in (4.4), we get

$$\mathcal{F}(V) = \lambda\mathcal{F}(\phi)\mathcal{F}(|x|).$$

Comparing last two expressions and solving for $\mathcal{F}(|x|)$, we obtain

$$\begin{aligned}\mathcal{F}(|x|) &= M2\pi\delta(\omega) - \mathcal{F}(\eta)/\gamma + \lambda\iota\omega\mathcal{F}(\phi) \\ &= M2\pi\delta(\omega) - \mathcal{F}(\eta)/[\gamma/\iota\omega + \lambda\mathcal{F}(\phi)]\iota\omega. \\ &= [M2\pi\delta(\omega) - \mathcal{F}(\eta)]\mathcal{F}(\zeta) \\ &= M2\pi\delta(\omega)\mathcal{F}(\zeta) - \mathcal{F}(\eta)\mathcal{F}(\zeta) \\ &= M2\pi\delta(\omega)\mathcal{F}(\zeta) - \mathcal{F}(\eta)\mathcal{F}(\zeta)\end{aligned}$$

and so that :

$$\mathcal{F}(|x|) = \mathcal{F}(\int_{-t}^{t} M\zeta(s)ds) - \mathcal{F}(\int_{-t}^{t} \eta(t-s)\zeta(s)ds),\ t > -\infty. \tag{4.8}$$

Taking the inverse of Fourier transform of (4.8), we obtain

$$|x| = M\int_{-t}^{t} \zeta(s)ds - \int_{-t}^{t} \eta(t-s)\zeta(s)ds,\ t > -\infty$$

or

$$|x(t)| \leq M\int_{-t}^{t} \zeta(s)ds \leq M/\gamma.$$

This completes the proof. □

Example 4.1 Consider the singular integral equation

$$x(t) = \sin(t) + \int_{-t}^{t} e^{-(t+5-s)}\cos(x(s))ds. \tag{4.9}$$

Here we have$g(t) = \sin t$, $h(y) = \cos(y)$ and $K(t,s) = e^{-(t+5-s)}$.

Then, we have $\lambda = 1$.

Let $\phi(t) = e^{-(t+s)}$. It follows that :

$$\int_{-\infty}^{\infty} e^{-(t+s)}dt < \infty.$$

Besides, we have $\phi'(t) = e^{-(t+5)}$ which implies that $\phi'(t-s) = e^{-(t+5-s)} = -K(t,s)$ for $-\infty < s \leq t < \infty$. Thus, the condition (4.3) is satisfied.

Moreover, we have$\gamma = 1 - \lambda\phi(0) = 1 - e^{-5} \in (-1,1)$. Hence, by Theorem(4.1), any solution of $x(t)$ of (4.9) satisfies :

$$x(t) \leq M/\gamma = 1/1 - e^{-5},$$

where $M = \sup_{t>-\infty} |\sin(t)| = 1$.

5. Infinite Delays and Several Kernels

In this section, we aim to extend the above method to singular integral equations with infinite delay if the history of solution is known and is a continuous function. Additionally, we shall generalize this concept for integral equation with infinite delay and with several kernels.

For this, let us begin by considering scalar singular integral equation with infinite delay of the form :

$$x(t) = c(t) + \lambda\int_{-\infty}^{t} K(t,s)h(x(s))ds, \tag{5.1}$$

where c, K and h are continuous. We assume the solution is subject to the same conditions. To specify a solution of (5.1), we require a continuous initial function $\phi : (-\infty, 0) \to \mathbb{R}$ with $\phi(0) = b(0)$, where the function :

$$b(t) := c(t) + \lambda\int_{-\infty}^{0} K(t,s)h(\phi(s))ds$$

is continuous and so that

$$x(t) = b(t) + \int_{-t}^{t} K(t,s)h(y(s))ds,\ t > -\infty. \tag{5.2}$$

With this setup, a function $x(t)$ is said to be a solution of (5.1) if $x(t) = \phi(t)$ for $t > -\infty$ and if $x(t)$ satisfies (5.1) for all $t > -\infty$.

Eventually, Theorem(4.1) is exactly what would one needs to obtain boundedness results.

We end this paper with the extension to singular integral equation with N number of kernels and N number of non-linear function in x.

So, we consider the scalar non-linear singular integral equation :

$$x(t) = b(t) + \int_{-t}^{t} \sum_{i=1}^{N} K_i(t,s) h_i(x(s)) ds, \; t > -\infty, \tag{5.3}$$

where all functions are scalars and continuous on their respective domains. The functions $h_i, i = 1 \dots N$ are continuous and satisfy the following growth conditions :

$$|h_i(x)| \leq \lambda_i |y|, \; i = 1, \dots, N, \tag{5.4}$$

for some positive constants λ_i.

Therefore, the conditions of Lemma (4.1) and Lemma(4.2) can be easily modified as shall be seen in the sequel.

Suppose there are differentiable functions $\phi_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \dots, N$ such that :

$$\sum_{i=1}^{N} \phi_i'(t-s) \leq -\sum_{i=1}^{N} \lambda_i |K_i(t,s)|, \; -\infty \leq s \leq t < \infty. \tag{5.5}$$

Moreover, if we assume the existence of a scalar $\zeta(t)$ that is uniformly continuous on $(-\infty, \infty)$, then, we redefine (5.1) as follows :

$$\int_{-t}^{t} \Big(\sum_{i=1}^{N} \lambda_i \phi_i(t-s) + \gamma \Big) \zeta(s) ds = 1. \tag{5.6}$$

If $x(t)$ is a solution of (5.3), then (4.2) can be modified and given as :

$$V(t) = \int_{-t}^{t} \sum_{i=1}^{N} \lambda_i \phi_i(t-s) |x(s)| ds. \tag{5.7}$$

Theorem 5.1 *Assume the conditions (5.4) and (5.5) hold and*

$$\sum_{i=1}^{N} \int_{-\infty}^{\infty} \phi_i(t) dt < \infty.$$

If $x(t)$ is any solution of (5.3), then,

$$|x(t)| \leq \frac{M}{\gamma},$$

where the constant $\gamma \in (-1,1)$ is such that :

$-1 < \sum_{i=1}^{N} \lambda_i \phi_i(0) < 1$ *with* $\gamma = 1 - \sum_{i=1}^{N} \lambda_i \phi_i(0)$ *and*

$M = \sup_{t > -\infty} |b(t)|.$

We shall now construct an example to illustrate the above result.

Example 5.1 Consider non-linear singular integral equation:

$$x(t) = \sec t + \int_{-t}^{t} \left[\frac{1}{(t+5-s)^3} \csc(y(s)) + e^{-(t+8-s)} y(s) \right] ds, \; t > -\infty \tag{5.8}$$

Here, we have $b(t) = \sec t$, $h(y) = \csc(y)$ and $K_1(t,s) = \dfrac{1}{(t+5-s)^3}$, $C_2(t,s) = e^{-(t+3-s)}$.

Then, $\lambda_1 = \lambda_2 = 1$ and $M = e^{-1}$.

Let $\phi_1(t) = \dfrac{1}{(t+5)^2}$, $\phi_2(t) = e^{-(t+8)}$. It follows that :

$$\int_{-\infty}^{\infty} \phi_i(t)dt < \infty,\ i = 1, 2.$$

In addition, (5.5) is satisfied for $-\infty < s \leq t < \infty$.
Moreover, we have $\gamma = 1 - \lambda_1\phi_1(0) - \lambda_2\phi_2(0) = 1 - \dfrac{1}{25} - e^{-8} = \dfrac{24}{25} - e^{-8} \in (-1, 1)$.
By Theorem(5.1), each solution $x(t)$ of (5.8) satisfies : $|x(t)| \leq \dfrac{M}{\gamma} = e^{-1}\left(\dfrac{24}{25} - e^{-8}\right)^{-1}$.

6. Conclusion

This paper has presented a novel analytical framework for evaluating the Lyapunov stability of singular integral equations subject to impulsive effects. By leveraging the Fourier transform, the proposed methodology effectively addresses the inherent complexities introduced by impulses, time delays, kernel singularities, and integral operators. The resulting stability criteria contribute to a deeper theoretical understanding of such systems while also offering practical utility in their analysis and control. These findings not only extend existing stability theory, but also lay the groundwork for future research into more general classes of integral and impulsive dynamical systems.

Future research directions may include extending the proposed framework to nonlinear singular integral equations, systems with state-dependent delays, or equations defined on infinite-dimensional spaces. In addition, numerical methods and simulation-based techniques could be developed to complement analytical results and facilitate their implementation in real-world scenarios.

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