



Some Properties of Demicompact Operators Related to Quasi-Fredholm Operators

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ABSTRACT: In this paper, we examine the relationship between demicompact operators and quasi-Fredholm operators. Additionally, we explore how demicompact linear operators relate to the upper B-Browder, Drazin, and B-quasi-Fredholm spectra.

Key Words: Demicompact operator, Drazin invertible operators and B -quasi-Fredholm spectra.

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1. Introduction

Let \mathcal{X} and \mathcal{Y} be two Banach spaces. We denote by $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ (respectively, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$) the set of all closed (respectively, bounded) linear operators from \mathcal{X} to \mathcal{Y} . The subset of compact operators in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is denoted by $\mathcal{K}(\mathcal{X}, \mathcal{Y})$. For $T \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ we denote by: $\mathcal{DO}(T)$ its domain, $\gamma(T)$ is the dimension of its kernel $\mathcal{NO}(T)$ and $\delta(T)$ is the codimension of its range $\mathcal{IM}(T)$ in \mathcal{Y} and

$$\mathcal{IM}^\infty(T) = \bigcap_{n=0}^{\infty} \mathcal{IM}(T^n).$$

Let $T \in \mathcal{L}(\mathcal{X})$ and let E is a subspace of \mathcal{X} . If $T(E) \subset E$, then E is said to be T -invariant. Recall that the resolvent set of $T \in \mathcal{C}(\mathcal{X})$ is defined by:

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ has a bounded inverse}\},$$

while the spectrum of T is given by:

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

The classes of upper semi-Fredholm, lower semi-Fredholm, Fredholm, and semi-Fredholm operators from \mathcal{X} to \mathcal{Y} are respectively defined as follows:

$$\begin{aligned} \mathcal{FD}_+(\mathcal{X}, \mathcal{Y}) &:= \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \gamma(T) < \infty \text{ and } \mathcal{IM}(T) \text{ closed in } \mathcal{Y}\}, \\ \mathcal{FD}_-(\mathcal{X}, \mathcal{Y}) &:= \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \delta(T) < \infty \text{ and } \mathcal{IM}(T) \text{ closed in } \mathcal{Y}\}, \end{aligned}$$

$$\mathcal{FD}(\mathcal{X}, \mathcal{Y}) := \mathcal{FD}_-(\mathcal{X}, \mathcal{Y}) \cap \mathcal{FD}_+(\mathcal{X}, \mathcal{Y}),$$

and the index of T is defined by $i(T) = \gamma(T) - \delta(T)$.

The class of semi-Weyl operators is defined by:

$$\begin{aligned} \mathcal{R}_+(\mathcal{X}) &:= \{T \in \mathcal{FD}_+(\mathcal{X}) : i(T) \leq 0\}, \\ &:= \mathcal{R}_1(\mathcal{X}) \cup \mathcal{R}_2(\mathcal{X}), \end{aligned}$$

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with $\mathcal{R}_1(\mathcal{X}) = \{T \in \mathcal{FD}_+(\mathcal{X}) : i(T) = 0\}$ and $\mathcal{R}_2(\mathcal{X}) = \{T \in \mathcal{FD}_+(\mathcal{X}) : i(T) < 0\}$. The Gustafson and semi-Weyl spectra of T , are respectively defined by:

$$\begin{aligned}\sigma_{Gu}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{FD}_+(\mathcal{X})\}, \\ \sigma_{uw}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_+(\mathcal{X})\}, \\ &= \sigma_1(T) \cap \sigma_2(T),\end{aligned}$$

with $\sigma_1(\mathcal{X}) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_1(\mathcal{X})\}$ and $\sigma_2(\mathcal{X}) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{R}_2(\mathcal{X})\}$.

In this paper, we focus on the essential spectrum $\sigma_{ew}(T)$ of an operator T , which is defined as:

$$\sigma_{ew}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X})} \sigma(T + K).$$

The idea of demicompact operators was essential to Fredholm theory in the few years prior (see [2,9,10,12,15,16,17,20]). A nonlinear operator $T : \mathcal{DO}(T) \subset \mathcal{X} \rightarrow \mathcal{X}$ (where \mathcal{X} is a Banach space), is demicompact if every bounded sequence $\{x_n : n \in \mathbb{N}\}$ in $\mathcal{DO}(T)$ such that $\{x_n - Tx_n : n \in \mathbb{N}\}$ converges in \mathcal{X} , has a convergent subsequence. $\mathcal{DC}(\mathcal{X})$ represents this class, and $\mathcal{K}(\mathcal{X}) \subset \mathcal{DC}(\mathcal{X})$. B. Krichen first proposed the idea of relative demicompactness for a closed linear operator in [15] in 2014. More specifically, T is considered S_0 -demicompact (or relative demicompact with respect to S_0) if every bounded sequence $\{x_n : n \in \mathbb{N}\}$ in $\mathcal{DO}(T)$ such that $\{S_0x_n - Tx_n : n \in \mathbb{N}\}$ converges in \mathcal{X} has a convergent subsequence, and $T : \mathcal{DO}(T) \subset \mathcal{X} \rightarrow \mathcal{X}$ and $S_0 : \mathcal{DO}(S_0) \subset \mathcal{X} \rightarrow \mathcal{X}$ are two linear operators with $\mathcal{DO}(T) \subset \mathcal{DO}(S_0)$. This class is denoted by $\mathcal{DC}_{S_0}(\mathcal{X})$. Now, two other classical quantities in operator theory, which are the ascent $\mathcal{A}_{\mathcal{SC}}(T)$ and the descent $\mathcal{D}_{\mathcal{ES}}(T)$ of T , are provided by $\mathcal{A}_{\mathcal{SC}}(T) = \inf \{n \in \mathbb{N} : \mathcal{NO}(T^n) = \mathcal{NO}(T^{n+1})\}$ (if there is no such integer, we set $\mathcal{A}_{\mathcal{SC}}(T) = \infty$), $\mathcal{D}_{\mathcal{ES}}(T) = \inf \{n \in \mathbb{N} : \mathcal{IM}(T^n) = \mathcal{IM}(T^{n+1})\}$ (if there is no such integer, we set $\mathcal{D}_{\mathcal{ES}}(T) = \infty$), where $\inf(\{\emptyset\}) = \infty$. The recognized fact is that when both $\mathcal{A}_{\mathcal{SC}}(T)$ and $\mathcal{D}_{\mathcal{ES}}(T)$ are finite, the equality $\mathcal{A}_{\mathcal{SC}}(T) = \mathcal{D}_{\mathcal{ES}}(T)$ holds. Also, an operator $T \in \mathcal{L}(\mathcal{X})$ is said to be semi-regular if $\mathcal{IM}(T)$ is closed and $\mathcal{NO}(T) \subset \mathcal{IM}(T^n)$ for every $n \in \mathbb{N}$.

The semi-regular spectrum of $T \in \mathcal{L}(\mathcal{X})$ is defined by:

$$\sigma_{sr}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

In what follows, we recall some definitions and results that needed in the sequel.

Definition 1.1 [4,7] *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$.*

(i) *T is called an B -Fredholm (resp. upper semi B -Fredholm and lower semi B -Fredholm) operator if for some integer $n \geq 0$, the range $\mathcal{IM}(T^n)$ is closed, and T_n , considered as an operator from the space $\mathcal{IM}(T^n)$ into itself, is a Fredholm (resp. upper semi-Fredholm and lower semi-Fredholm) operator.*

(ii) *T is called an Browder (resp. upper semi Browder and lower semi Browder) operator if T is a Fredholm (resp. upper semi Fredholm and lower semi Fredholm) and both $\mathcal{A}_{\mathcal{SC}}(T)$, $\mathcal{D}_{\mathcal{ES}}(T)$ are finite.*

(iii) *T is called an B -Browder (resp. upper semi B -Browder, lower semi B -Browder, B -Weyl, upper semi B -Weyl and lower semi B -Weyl) operator if for some integer $n \geq 0$, the range $\mathcal{IM}(T^n)$ is closed, and T_n , is a Browder (resp. upper semi-Browder, lower semi-Browder, Weyl, upper semi-Weyl and lower semi-Weyl) operator.*

The sets of all B -Fredholm, upper semi B -Fredholm, lower semi B -Fredholm, B -Browder, upper semi B -Browder and lower semi B -Browder operators are, respectively, denoted by: $\mathcal{BF}(\mathcal{X})$, $\mathcal{SBF}_+(\mathcal{X})$, $\mathcal{SBF}_-(\mathcal{X})$, $\mathcal{BB}(\mathcal{X})$, $\mathcal{SBB}_+(\mathcal{X})$ and $\mathcal{SBB}_-(\mathcal{X})$.

For $T \in \mathcal{L}(\mathcal{X})$, the upper B -Browder spectrum of T is defined by:

$$\sigma_{Ubb}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{SBB}_+(\mathcal{X})\}.$$

For $T \in \mathcal{L}(\mathcal{X})$, we define the difference sequence $z_n(T)$ by:

$$z_n(T) := \dim \left[\frac{(\mathcal{IM}(T^n) \cap \mathcal{NO}(T))}{\mathcal{IM}(T^{n+1}) \cap \mathcal{NO}(T)} \right] = \dim \left[\frac{(\mathcal{IM}(T) + \mathcal{NO}(T^{n+1}))}{\mathcal{IM}(T) + \mathcal{NO}(T^n)} \right].$$

It is well known that an $T \in \mathcal{L}(\mathcal{X})$ is Drazin invertible of degree k if there exists $B \in \mathcal{L}(\mathcal{X})$ such that

$$T^k BT = T^k, BTB = B, \text{ and } TB = BT,$$

(see [6]). The Drazin invertible spectrum $\sigma_D(T)$ of T is defined by:

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Drazin invertible operator}\}.$$

We denote by: $\mathcal{LD}(\mathcal{X}) := \{T \in \mathcal{L}(\mathcal{X}) : \mathcal{A}_{Sc}(T) < \infty \text{ and } \mathcal{R}(T^{\mathcal{A}_{Sc}(T)+1}) \text{ is closed}\}$ and

$$\sigma_{lD}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{LD}(\mathcal{X})\}$$

denote the left Drazin spectrum of T .

Definition 1.2 [18] Let \mathcal{X} be a Banach space, $d \in \mathbb{N}$ and let $T \in \mathcal{L}(\mathcal{X})$. Then, T is said to be quasi-Fredholm of degree d if,

(i) $z_n(T) = 0$, for each $n \geq d$, and

(ii) $\mathcal{IM}(T) + \mathcal{NO}(T^d)$ and $\mathcal{IM}(T^d) \cap \mathcal{NO}(T)$ are closed in \mathcal{X} .

The set of quasi-Fredholm operators is denoted by $\mathcal{QF}(\mathcal{X})$.

The quasi-Fredholm spectrum of $T \in \mathcal{L}(\mathcal{X})$ is defined by:

$$\sigma_{QF}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{QF}(\mathcal{X})\}.$$

Definition 1.3 Let \mathcal{X} be a Banach space and $T \in \mathcal{C}(\mathcal{X})$. Then, T is said to be B -quasi-Fredholm if T is an quasi-Fredholm operator and $i(T) = 0$.

The set of quasi-Fredholm operators is denoted by $\mathcal{BQF}(\mathcal{X})$.

The B -quasi-Fredholm spectrum of $T \in \mathcal{L}(\mathcal{X})$ is defined by:

$$\sigma_{BQF}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{BQF}(\mathcal{X})\}.$$

The annihilator of a subset $E \subset \mathcal{X}$ is denoted by E^\perp , while the preannihilator of a subset $G \subset \mathcal{X}^*$ is denoted by ${}^\perp G$. It is clear that E^\perp (resp. ${}^\perp G$) forms a closed subspace of \mathcal{X}^* (resp. \mathcal{X}). In addition, if E and G are subspaces of \mathcal{X} , then $E \oplus G$ stands for their algebraic sum.

Remark 1.1 Let \mathcal{X} be a Banach space and let X_1 and X_2 be two linear subspaces of \mathcal{X} (resp. of \mathcal{X}^*). It therefore follows that:

$$(i) \overline{X_2}^\perp = X_2^\perp,$$

$$(ii) {}^\perp(X_2^\perp) = \overline{X_2} \text{ (resp. } ({}^\perp X_2)^\perp = w^*\text{-closure of } X_2), \text{ and}$$

$$(iii) X_1^\perp \cap X_2^\perp \subset (X_1 + X_2)^\perp \text{ (resp. } {}^\perp X_1 \cap {}^\perp X_2 \subset {}^\perp(X_1 + X_2)).$$

Furthermore, if X_1 and X_2 are closed linear subspaces of \mathcal{X} , it holds that

$$(iv) X_1^\perp = X_2^\perp \text{ if, and only if, } X_1 = X_2, \text{ and}$$

$$(v) (X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$

Now, as $(X_1 \cap X_2)^\perp$ is always closed but $X_1^\perp + X_2^\perp$ need not be closed, then the dual relation of (v), $X_1^\perp + X_2^\perp = (X_1 \cap X_2)^\perp$ is not always true. Nevertheless, if $X_1 + X_2$ is closed, then $X_1^\perp + X_2^\perp = (X_1 \cap X_2)^\perp$, and in particular, $X_1^\perp + X_2^\perp$ is closed, see in [[14], Lemma 4.9].

The given duality relationships between the kernels and ranges of a bounded linear operator T on a Banach space and its dual T^* are widely recognized. The proofs for these relationships can be found, for example, in [[13], p.135].

$$\mathcal{NO}(T) = {}^\perp \overline{T^*(\mathcal{X}^*)}, \quad {}^\perp \mathcal{NO}(T^*) = T(\mathcal{X})$$

and

$$\mathcal{NO}(T^*) = \overline{T(\mathcal{X})}^\perp = T(\mathcal{X})^\perp, \quad \overline{T^*(\mathcal{X}^*)} \subset \mathcal{NO}(T)^\perp.$$

Note that the inclusion $\overline{T^*(\mathcal{X}^*)} \subset \mathcal{NO}(T)^\perp$ is, in general, strict. Nevertheless, if T has a closed range, then $\overline{T^*(\mathcal{X}^*)} = \mathcal{NO}(T)^\perp$ (see, Chapter IV of [14]).

The structure of this paper is organized as follows: In section 2, we have presented some results related the demicompact operator and its relation to quasi-Fredholm operator. In Section 3, we present some perturbation results involving the class of demicompact linear operators, situated between the Drazin spectrum and the B-quasi-Fredholm spectrum.

2. A decomposition for the class of demicompact operators

In this section, we present a new decomposition for the class of demicompact operators and explore the relationship between demicompactness and quasi-Fredholm operators. Now, we start by the following theorem.

Lemma 2.1 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} . If $I - T \in \mathcal{DC}(\mathcal{X})$, then there exists a closed subspace F of \mathcal{X} (resp. L closed of $\mathcal{IM}(T^n)$) such that $\mathcal{X} = \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \oplus F$ (resp. $\mathcal{IM}(T^n) = \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \oplus L$).*

Proof: Since $(I - T) \in \mathcal{DC}(\mathcal{X})$, it follows that $(I - T)|_{\mathcal{IM}(T^n)}$ is a demicompact operator. Now, by [[10], Theorem 2.7], we deduce that $T|_{\mathcal{IM}(T^n)}$ is upper semi-Fredholm. Then, $\dim(\mathcal{NO}(T) \cap \mathcal{IM}(T^n)) < \infty$. Hence, there exists a closed subspace F of \mathcal{X} (resp. L closed of $\mathcal{IM}(T^n)$) such that $\mathcal{X} = \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \oplus F$ (resp. $\mathcal{IM}(T^n) = \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \oplus L$). \square

Lemma 2.2 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} . If $I - T \in \mathcal{DC}(\mathcal{X})$, then there exists a sequence $\{H_i\}_{i=0}^\infty$ of subspaces such that for all $i = 0, 1, 2, \dots$, the following conditions hold:*

- (i) $H_i \subset H_{i+1}$,
- (ii) If $x \in H_{i+1}$, then $Tx \in H_i$,
- (iii) $T(H_i) \subset H_i$,
- (iv) $H_i \subset \mathcal{NO}(T^i)$,
- (v) $H_i \cap \mathcal{IM}(T^n) = \{0\}$,
- (vi) $H_i \cap \mathcal{IM}(T) = T(H_{i+1})$,
- (vii) $\mathcal{NO}(T^i) \subset H_i + \mathcal{IM}(T^n) \cap \mathcal{NO}(T^i)$, and
- (viii) $H_i + \mathcal{IM}(T^n) = \mathcal{NO}(T^i) + \mathcal{IM}(T^n)$.

Proof: Suppose that $I - T \in \mathcal{DC}(\mathcal{X})$. As $\mathcal{IM}(T^n)$ is closed of \mathcal{X} . Then, by Lemma 2.1, we deduce that, there exists a closed subspace F of \mathcal{X} such that

$$\mathcal{X} = \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \oplus F.$$

Now, we define the subspace H_i as follows:

$$H_i = \begin{cases} \{0\}, & \text{if } i = 0, \\ T^{-1}(H_{i-1}) \cap F, & \text{if } i \geq 1. \end{cases}$$

It is obvious that H_i is a subspace closed of \mathcal{X} for each i .

(i) For $i = 0$, $H_0 = \{0\} \subset \mathcal{NO}(T) \cap F = T^{-1}(\{0\}) \cap F = H_1$. Suppose that $H_i \subset H_{i+1}$. Then, $H_{i+1} = T^{-1}(H_i) \cap F \subset T^{-1}(H_{i+1}) \cap F = H_{i+2}$. Therefore, $H_i \subset H_{i+1}$ for each i .

(ii) Let $x \in H_{i+1}$. As $H_{i+1} = T^{-1}(H_i) \cap F$. Hence,

$$\begin{aligned} \text{If } Tx \in T(H_{i+1}) &\Rightarrow Tx \in T[T^{-1}(H_i) \cap F] \\ &\Rightarrow Tx \in H_i \cap T(F) \subset H_i \\ &\Rightarrow Tx \in H_i. \end{aligned}$$

(iii) For $i = 0$, $T(H_0) = T(\{0\}) = \{0\} = H_0$. Let $i \geq 1$, then $T(H_i) = T[T^{-1}(H_{i-1}) \cap F] \subset T[T^{-1}(H_{i-1})] \subset H_{i-1} \subset H_i$. Therefore, $T(H_i) \subset H_i$ for each i .

(iv) For $i = 0$, $H_0 = \{0\} = \mathcal{NO}(T) = \mathcal{NO}(T^0)$. Suppose that $H_i \subset \mathcal{NO}(T^i)$. Then, $T^{i+1}(H_{i+1}) = T^{i+1}(T^{-1}(H_i) \cap F) \subset T^{i+1}(T^{-1}(H_i)) \subset T^i(H_i) = \{0\}$. Consequently, $H_{i+1} \subset \mathcal{NO}(T^{i+1})$ and hence $H_i \subset \mathcal{NO}(T^i)$ for each i .

(v) For $i = 0$, $H_0 \cap \mathcal{IM}(T^n) = \{0\} \cap \mathcal{IM}(T^n) = \{0\}$. Suppose that $H_i \cap \mathcal{IM}(T^n) = \{0\}$ and let $x \in H_{i+1} \cap \mathcal{IM}(T^n)$. Then, $Tx \in H_i \cap \mathcal{IM}(T^{n+1}) \subset H_i \cap \mathcal{IM}(T^n) = \{0\}$ and also $x \in F$. Therefore, $x \in \mathcal{NO}(T) \cap \mathcal{IM}(T^n) \cap F$ and hence $x = \{0\}$. Thus, $H_{i+1} \cap \mathcal{IM}(T^n) = \{0\}$ for each i .

(vi) Clearly, $T(H_{i+1}) \subset \mathcal{IM}(T)$ and

$$T(H_{i+1}) = T[T^{-1}(H_i) \cap F] \subset T[T^{-1}(H_i)] \subset H_i,$$

then $T(H_{i+1}) \subset H_i \cap \mathcal{IM}(T)$. Conversely, let $y \in H_i \cap \mathcal{IM}(T)$. In this case, there exists $x \in \mathcal{X}$ such that $y = Tx$. If we represent $x = z + w$ with $z \in F$ and $w \in \mathcal{NO}(T) \cap \mathcal{IM}(T^n)$, then $x - w = z \in F$ and $T(x - w) = Tx = y \in H_i \cap \mathcal{IM}(T) \subset H_i$. Thus, $x - w \in T^{-1}(H_i)$ and $x - w \in F$. Now, we deduce that,

$$x - w \in T^{-1}(H_i) \cap F = H_{i+1}.$$

Hence, $y \in T(H_{i+1})$. From where

$$H_i \cap \mathcal{IM}(T) = T(H_{i+1}).$$

(vii) For $i = 0$, we have $\mathcal{NO}(T^0) = \{0\} = H_0 \subset H_0 + \mathcal{NO}(T^0) \cap \mathcal{IM}(T^n)$. Suppose that

$$\mathcal{NO}(T^i) \subset H_i + \mathcal{NO}(T^i) \cap \mathcal{IM}(T^n).$$

Let $x \in \mathcal{NO}(T^{i+1})$. As a result, $Tx \in \mathcal{NO}(T^i)$ and consequently $Tx = U_1 + U_2$ for some $U_1 \in H_i$ and

$$\begin{aligned} U_2 \in \mathcal{NO}(T^i) \cap \mathcal{IM}(T^n) &= \mathcal{NO}(T^i) \cap \mathcal{IM}(T^{n+1}) \\ &= T(\mathcal{NO}(T^{i+1}) \cap \mathcal{IM}(T^n)). \end{aligned}$$

Therefore, $U_1 \in H_i \cap \mathcal{IM}(T) = T(H_{i+1})$ and we have

$$\begin{aligned} x \in H_{i+1} + L + \mathcal{NO}(T) &= H_{i+1} + \mathcal{NO}(T^{i+1}) \cap \mathcal{IM}(T^n) + \mathcal{NO}(T) \cap F \\ &\quad + \mathcal{NO}(T) \cap \mathcal{IM}(T^n), \\ &= H_{i+1} + \mathcal{NO}(T^{i+1}) \cap \mathcal{IM}(T^n) + H_1 \\ &\quad + \mathcal{NO}(T) \cap \mathcal{IM}(T^n), \\ &= H_{i+1} + \mathcal{NO}(T^{i+1}) \cap \mathcal{IM}(T^n), \end{aligned}$$

where $L = \mathcal{NO}(T^{i+1}) \cap \mathcal{IM}(T^n)$. This demonstrates that $\mathcal{NO}(T^i) \subset H_i + \mathcal{NO}(T^i) \cap \mathcal{IM}(T^n)$ for each i .

(viii) This follows from (iv) and (vii). \square

Theorem 2.1 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that there exists $n \in \mathbb{N}$ such that*

(i) $\mathcal{IM}(T^n)$ is closed of \mathcal{X} , and

(ii) $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented.

If $I - T \in \mathcal{DC}(\mathcal{X})$, then there exist T -invariant subspaces X_1 and X_2 of \mathcal{X} such that X_2 is closed, $\mathcal{X} = X_1 \oplus X_2$, $T|_{X_2}$ is semi-regular and $T|_{X_1}$ is nilpotent.

Proof: Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented. Let $I - T \in \mathcal{DC}(\mathcal{X})$. Hence, according to Lemma 2.2, there exists a subspace X_1 ($X_1 = H_n$) such that

$$\begin{aligned} \mathcal{IM}(T|_{X_1}) &\subset X_1, \\ X_1 &\subset \mathcal{NO}(T^n), \\ X_1 \cap \mathcal{IM}(T^n) &= \{0\}, \text{ and} \\ X_1 + \mathcal{IM}(T^n) &= \mathcal{NO}(T^n) + \mathcal{IM}(T^n). \end{aligned}$$

Now, suppose that F is a closed subspace of \mathcal{X} such that $\mathcal{IM}(T) \cap \mathcal{NO}(T^n) \oplus F = \mathcal{X}$. Therefore,

$$\begin{aligned} \mathcal{NO}(T^*) \cap \mathcal{IM}[(T^*)^n] + F^\perp &= [\mathcal{IM}(T) + \mathcal{NO}(T^n)]^\perp + F^\perp \\ &= ([\mathcal{IM}(T) + \mathcal{NO}(T^n)] \cap F)^\perp \\ &= \{0\}^\perp \\ &= \mathcal{X}^*, \end{aligned}$$

and

$$\begin{aligned} \mathcal{NO}(T^*) \cap \mathcal{IM}[(T^*)^n] \cap F^\perp &= [\mathcal{IM}(T) + \mathcal{NO}(T^n)]^\perp \cap F^\perp \\ &\subset [\mathcal{IM}(T) + \mathcal{NO}(T^n) + F]^\perp \\ &= \{\mathcal{X}\}^\perp \\ &= \{0\}. \end{aligned}$$

Consequently, $\mathcal{X}^* = \mathcal{NO}(T^*) \cap \mathcal{IM}[(T^*)^n] \oplus F^\perp$. Since $\mathcal{IM}(T^{n+1})$ is closed if, and only if, the space $\mathcal{IM}((T^*)^{n+1})$ is also closed (see Theorem 1.13 of [1]) and $T \in \Phi_+(\mathcal{X})$ (see [[10], Theorem 2.7]), then there exists a closed subspace X_2^* of \mathcal{X}^* such that

$$\begin{aligned} \mathcal{IM}(T|_{X_2^*}) &\subset X_2^*, \\ X_2^* &\subset \mathcal{NO}[(T^*)^n], \\ X_2^* \cap \mathcal{IM}[(T^*)^n] &= \{0\}, \text{ and} \\ X_2^* + \mathcal{IM}[(T^*)^n] &= \mathcal{NO}[(T^*)^n] + \mathcal{IM}[(T^*)^n]. \end{aligned}$$

Let $X_2 = {}^\perp(X_2^*)$. Then,

- (1) X_2 is a closed subspace of \mathcal{X} ,
- (2) $\mathcal{IM}(T|_{X_2}) \subset X_2$,
- (3) $X_2^\perp = X_2^*$,
- (4) $\mathcal{IM}(T^n) \subset X_2$,
- (5) $X_2 + \mathcal{NO}(T^n) = \mathcal{X}$,
- (6) $X_2 \cap \mathcal{NO}(T^n) \subset \mathcal{IM}(T^n)$,
- (7) $\mathcal{X} = X_1 \oplus X_2$,
- (8) $T|_{X_2}$ is semi-regular and $T|_{X_1}$ is nilpotent.

Indeed,

- (1) This immediately follows from the definition of X_2 .
- (2) Let $x \in X_2$. Then, for each $g \in X_2^*$ $g(x) = 0$. Since $g(T(x)) = (T^*g)(x)$ and $\mathcal{IM}(T|_{X_2^*}) \subset X_2^*$, then $g(T(x)) = 0$ for each $g \in X_2^*$. Therefore, $\mathcal{IM}(T|_{X_2}) \subset X_2$.
- (3) Recall that $X_2^* = (X_2^*)_d = (T^*)^{-1}((X_2^*)_{d-1}) \cap F^\perp$.

If $d = 1$, then

$$\begin{aligned}
X_2^\perp = (\perp(X_2^*))^\perp &= [\perp(\mathcal{NO}(T^*) \cap F^\perp)]^\perp \\
&= [\perp(\overline{\mathcal{IM}(T)}^\perp \cap F^\perp)]^\perp \\
&= [\perp(\overline{(\mathcal{IM}(T) + F)^\perp})]^\perp \\
&= \overline{(\mathcal{IM}(T) + F)}^\perp \\
&= \overline{(\mathcal{IM}(T) + F)}^\perp \\
&= \overline{\mathcal{IM}(T)}^\perp \cap F^\perp \\
&= \mathcal{NO}(T^*) \cap F^\perp \\
&= X_2^*.
\end{aligned}$$

If $n > 1$, as $[\perp((X_2^*)_{n-1})]^\perp = (X_2^*)_{n-1}$, it follows that

$$\begin{aligned}
(X_2^*)_n &= (T^*)^{-1}(X_2^*)_{n-1} \cap F^\perp \\
&= (T^*)^{-1}([\perp(X_2^*)_{n-1}]^\perp) \cap F^\perp \\
&= [T(\perp(X_2^*)_{n-1})]^\perp \cap F^\perp \\
&= [\overline{T(\perp(X_2^*)_{n-1})}]^\perp \cap F^\perp \\
&= \overline{(T(\perp(X_2^*)_{n-1}) + F)}^\perp.
\end{aligned}$$

As a matter of fact,

$$\begin{aligned}
X_2^\perp = (\perp(X_2^*)_n)^\perp &= \left[\perp \left(\overline{(T(\perp(X_2^*)_{n-1}) + F)}^\perp \right) \right]^\perp \\
&= \overline{(T(\perp(X_2^*)_{n-1}) + F)}^\perp \\
&= (X_2^*)_n \\
&= X_2^*.
\end{aligned}$$

(4) Since $X_2^* \subset \mathcal{NO}[(T^*)^n]$, then

$$X_2 = {}^\perp X_2^* \supset {}^\perp \mathcal{NO}[(T^*)^n] = \overline{\mathcal{IM}(T^n)} = \mathcal{IM}(T^n).$$

(5) Relying upon [[19], Theorem A.1.13] and the fact that $X_2^* + \mathcal{IM}[(T^*)^n]$ in \mathcal{X}^* , we get

$$X_2 + \mathcal{NO}(T^n) = {}^\perp (X_2^\perp \cap \mathcal{NO}(T^n)^\perp) = {}^\perp (X_2^* \cap \mathcal{IM}[(T^*)^n]) = {}^\perp \{0\} = \mathcal{X}.$$

(6) As

$$\mathcal{NO}[(T^*)^n] \subset X_2^* + \mathcal{IM}[(T^*)^n],$$

then

$$\begin{aligned}
\mathcal{IM}(T^n) &= {}^\perp \mathcal{NO}[(T^*)^n] \\
&\supset {}^\perp (X_2^* + \mathcal{IM}[(T^*)^n]) \\
&= {}^\perp X_2^* \cap {}^\perp \mathcal{IM}[(T^*)^n] \\
&= X_2 \cap \mathcal{NO}(T^n).
\end{aligned}$$

(7) Note that:

$$\begin{aligned}
X_1 + X_2 &= X_1 + \mathcal{IM}[T^n] + X_2 \\
&= X_2 + \mathcal{IM}[T^n] + \mathcal{NO}(T^n) \\
&= X_2 + \mathcal{NO}(T^n) \\
&= \mathcal{X}.
\end{aligned}$$

Additionally,

$$X_1 \cap X_2 \subset X_1 \cap \mathcal{NO}(T^n) \cap X_2 \subset X_1 \cap \mathcal{IM}(T^n) = \{0\}.$$

Thus, we conclude that $\mathcal{X} = X_1 \oplus X_2$.

(8) Since

$$\begin{aligned}
\mathcal{IM}(T^n) &= T^n(\mathcal{X}) = T^n(X_1 + X_2) \\
&= T^n(X_1) + T^n(X_2) \\
&= T^n(X_2) \\
&= \mathcal{IM}[(T|_{X_2})^n],
\end{aligned}$$

it follows that $\mathcal{IM}[(T|_{X_2})^n]$ is closed. Now, by part (v), we have

$$\mathcal{NO}[(T|_{X_2})^n] = \mathcal{NO}(T^n) \cap X_2 \subset \mathcal{IM}(T^n) = \mathcal{IM}[(T|_{X_2})^n].$$

This reveals that $(T|_{X_2})^n$ is semi-regular. Hence, with reference to [[19], Theorem 12.7], $T|_{X_2}$ is semi-regular. \square

Theorem 2.2 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that there exist T invariant subspaces X_1 and X_2 of \mathcal{X} such that X_1 is closed, $\mathcal{X} = X_1 \oplus X_2$, $T|_{X_2}$ is upper semi-Fredholm and $T|_{X_1}$ is nilpotent, then there exist $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed and $(I - T)|_{\mathcal{IM}(T^n)}$ is a demicompact operator.*

Proof: Suppose that there exist T -invariant subspaces X_1 and X_2 of \mathcal{X} such that X_2 is closed, $T|_{X_2}$ is upper semi-Fredholm and $T|_{X_1}$ is nilpotent. Let $n \in \mathbb{N}$ such that $(T|_{X_1})^n = 0$. Hence,

$$\mathcal{IM}(T^n) = T^n(\mathcal{X}) = T^n(X_2 \oplus X_1) = T^n(X_2).$$

Therefore, $\mathcal{IM}(T^n)$ is a closed subspace. Moreover,

$$\mathcal{NO}(T) \cap \mathcal{IM}(T^n) = \mathcal{NO}(T) \cap T^n(X_2) \subset \mathcal{NO}(T) \cap X_2.$$

As $T|_{X_2}$ is upper semi-Fredholm, then

$$\dim(\mathcal{NO}(T) \cap \mathcal{IM}(T^n)) < \infty \text{ and } \mathcal{IM}(T^{n+1}) \text{ is closed.}$$

Hence, $T|_{\mathcal{IM}(T^n)}$ is upper semi-Fredholm. Now, Grounded on [[10], Theorem 2.7], we therefore have $(I - T)|_{\mathcal{IM}(T^n)}$ is a demicompact operator. \square

Theorem 2.3 *Let \mathcal{X} be a Banach space and $T, B \in \mathcal{L}(\mathcal{X})$ such that B is nilpotent and $TB = BT$. Suppose that there exists $n \in \mathbb{N}$ such that*

- (i) $\mathcal{IM}(T^n)$ is closed of \mathcal{X} , and
- (ii) $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented.

If $I - T \in \mathcal{DC}(\mathcal{X})$, then $T + B$ is a quasi-Fredholm operator.

Proof: It is sufficient from [[4], Proposition 3.2] to demonstrate that $p \in \mathbb{N}$ exists such that $\mathcal{IM}((T+B)^p)$ is closed and $(T+B)_p$ is semi-regular. Suppose that $I-T$ is a demicompact operator such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented, from Theorem 2.1, it follows that there exist T -invariant subspaces X_1 and X_2 of \mathcal{X} such that X_2 is closed, $\mathcal{X} = X_1 \oplus X_2$, $T|_{X_2}$ is semi-regular and $T|_{X_1}$ is nilpotent. Let $n, m \in \mathbb{N}$ be such that $B^n = 0$ and $(T|_{X_1})^m = 0$. If $x \in X_2$, then

$$\begin{aligned}
 (T+B)^{n+m}x &= \sum_{k=0}^{n+m} C_{p+q,k} T^{(n+m)-k} B^k x \\
 &= \sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} B^k x + \sum_{k=n+1}^{n+m} C_{n+m,k} T^{(n+m)-k} B^k x \\
 &= \sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} (x_1 + x_2) + \sum_{k=n+1}^{n+m} C_{n+m,k} T^{(n+m)-k} B^k x \\
 &\quad (\text{where } x_1 \in X_1 \text{ and } x_2 \in X_2) \\
 &= \sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} (x_1) + \sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} (x_2) \\
 &= \sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} (x_2) \in X_2.
 \end{aligned}$$

This demonstrates that $(T+B)^{(n+m)}(X_2) \subset X_2$. Likewise, we deduce that $(T+B)^{(n+m)}(X_1) = \{0\} \subset X_1$ and $\sum_{k=0}^n C_{n+m,k} T^{(n+m)-k} B^k (X_2) \subset X_2$. On the other side,

$$\begin{aligned}
 (T+B)^{n+m} &= \sum_{k=0}^{n-1} C_{p+q,k} T^{(n+m)-k} B^k \\
 &= T^{n+m} + \sum_{k=1}^{n-1} C_{n+m,k} T^{(n+m)-k} B^k \\
 &= T^{n+m} + B \left(\sum_{k=1}^{n-1} C_{n+m,k} T^{(n+m)-k} B^{k-1} \right).
 \end{aligned}$$

According to [[19], Theorem 12.7], $T^{n+m} \in \mathcal{L}(X_2)$ is semi-regular, and

$$B \left(\sum_{k=1}^{n-1} C_{n+m,k} T^{(n+m)-k} B^{k-1} \right) \text{ is nilpotent.}$$

We have $((T+B)|_{X_2})^{n+m}$ which is semi-regular since the class of semi-regular operators is invariant under perturbations by nilpotent commuting operators. Thus, there exists $d = n+m \in \mathbb{N}$ such that $((T+B)|_{X_2})^p$ is semi-regular and $((T+B)|_{X_1})^p = 0$, for all $p \geq d$. Since $\mathcal{X} = X_1 \oplus X_2$, we get

$$\mathcal{IM}((T+B)^p) = (T+B)^p(\mathcal{X}) = (T+B)^p(X_2) \oplus (T+B)^p(X_1) = (T+B)^p(X_2).$$

For every $p \geq d$, $(T+B)^p(\mathcal{X})$ is closed. Additionally, it follows that:

$$\begin{aligned}
 \mathcal{NO}((T+B)_p) &= \mathcal{NO}(T+B) \cap \mathcal{IM}((T+B)^p) \\
 &\subset \mathcal{NO}((T+B)^p) \cap (T+B)^p(X_2) \\
 &\subset \mathcal{NO}((T+B)^p) \cap X_2 \\
 &= \mathcal{NO}((T+B)|_{X_2}^p) \\
 &\subset (T+B)^q(X_2) \\
 &= \mathcal{IM}((T+B)^q), \text{ for all } q > p > d.
 \end{aligned}$$

This indicates that for all $p \geq d$, $(T + B)_p$ is semi-regular, indicating that $T + B$ is quasi-Fredholm. \square

Corollary 2.1 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Suppose that there exists $n \in \mathbb{N}$ such that*

(i) $\mathcal{IM}(T^n)$ *is closed of \mathcal{X} , and*

(ii) $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ *is complemented.*

If $I - T \in \mathcal{DC}(\mathcal{X})$, then T is a quasi-Fredholm operator.

Proof: Let $I - T \in \mathcal{DC}(\mathcal{X})$. It is clear that the zero operator is nilpotent operator and $0T = T0$. Using Theorem 2.3, we deduce that T is a quasi-Fredholm operator. \square

3. Drazin and B-quasi-Fredholm spectrum

In this section, for an operator $T \in \mathcal{C}(\mathcal{X})$ we consider the sets $\Delta_{\mathcal{X}}$, $\Gamma_T(\mathcal{X})$, and $\Omega_T(\mathcal{X})$ as follows:

$$\begin{aligned} \Delta_{\mathcal{X}} &:= \{J \in \mathcal{L}(\mathcal{X}) : \mu J \text{ is demicompact for every } \mu \in [0, 1]\}, \\ \Gamma_T(\mathcal{X}) &:= \{K \in \mathcal{L}(\mathcal{X}) : -(\lambda - T - K)^{-1}K \in \Delta_{\mathcal{X}}, \forall \lambda \in \rho(T + K)\}, \\ \Omega_T(\mathcal{X}) &:= \{K \text{ is T-bounded} : -K(\lambda - T - K)^{-1} \in \Delta_{\mathcal{X}}, \forall \lambda \in \rho(T + K)\}. \end{aligned}$$

We denote

$$\sigma_s(T) = \bigcap_{K \in \Gamma_T(\mathcal{X})} \sigma(T + K) \text{ and } \sigma_v(T) = \bigcap_{K \in \Omega_T(\mathcal{X})} \sigma(T + K).$$

Theorem 3.1 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented. Then,*

(i) $\sigma_{QF}(T) \subset \sigma_{Gu}(T)$.

(ii) $\sigma_{BQF}(T) \subset \sigma_s(T) \cap \sigma_1(T)$.

Proof: Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented. (i) Let $\lambda \in \sigma_{QF}(T)$, by Corollary 2.1, we deduce that $I - \lambda + T \notin \mathcal{DC}(\mathcal{X})$, then based on [[10], Theorem 2.7], we infer that $\lambda - T \notin \mathcal{FD}_+(\mathcal{X})$. Hence, $\lambda \in \sigma_{Gu}(T)$.

(ii) Let $\lambda \notin \sigma_s(T)$, then there exists $K \in \Gamma_T(\mathcal{X})$ such that $-(\lambda - T - K)^{-1}K \in \Delta_{\mathcal{X}}$, whenever λ is in the resolvent set of $T + K$. Using [[9], Theorem 3.2], we now obtain

$$[I + (\lambda - T - K)^{-1}K] \in \mathcal{FD}(\mathcal{X}) \text{ and } \text{ind } [I + (\lambda - T - K)^{-1}K] = 0.$$

Additionally, we have

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}K].$$

Then, applying [[21], Theorem 5.7], we conclude that

$$\lambda - T \in \mathcal{FD}(\mathcal{X}) \text{ and } i(\lambda - T) = 0.$$

One obtains

$$\lambda - T \in \mathcal{BF}(\mathcal{X}) \text{ and } i(\lambda - T) = 0.$$

Thus,

$$\lambda - T \in \mathcal{QF}(\mathcal{X}) \text{ and } i(\lambda - T) = 0.$$

Hence, $\lambda \notin \sigma_{BQF}(T)$. For the second inclusion, let $\lambda \in \sigma_{BQF}(T)$, then $\lambda - T \notin \mathcal{QF}(\mathcal{X})$ or $i(\lambda - T) \neq 0$. Based on Corollary 2.1, we therefore have $I - \lambda + T \notin \mathcal{DC}(\mathcal{X})$, and Grounded on [[10], Theorem 2.7], we have $\lambda - T \notin \mathcal{R}_+(\mathcal{X})$. Hence, $\lambda \in \sigma_1(T)$. We obtain that, $\sigma_{BQF}(T) \subset \sigma_s(T)$ and $\sigma_{BQF}(T) \subset \sigma_1(T)$. So that, $\sigma_{BQF}(T) \subset \sigma_s(T) \cap \sigma_1(T)$. \square

Corollary 3.1 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented. Then,*

$$(i) \sigma_{BQF}(T) \subset \sigma_v(T).$$

$$(ii) \sigma_{BQF}(T) \subset \bigcap_{K \in \mathcal{S}(\mathcal{X})} \sigma(T + K), \text{ where } \mathcal{S}(\mathcal{X}) \text{ is a subset of } \Gamma_T(\mathcal{X}) \text{ containing } \mathcal{K}(\mathcal{X}).$$

$$(iii) \sigma_{BQF}(T) \subset \sigma_s(T) \cap \sigma_D(T).$$

Proof: (i) From Theorem 3.1 and [[9], Theorem 4.1], we obtain $\sigma_{BQF}(T) \subset \sigma_s(T)$ and $\sigma_s(T) = \sigma_v(T)$. Thus, $\sigma_{BQF}(T) \subset \sigma_v(T)$.

(ii) By [[9], Corollary 4.3 and Theorem 4.1], we obtain $\sigma_{ew}(T) = \bigcap_{K \in \mathcal{S}(\mathcal{X})} \sigma(T + K)$ where $\mathcal{S}(\mathcal{X})$ is a subset of $\Gamma_T(\mathcal{X})$ containing $\mathcal{K}(\mathcal{X})$ and $\sigma_{ew}(T) = \sigma_s(T)$. Now, using Theorem 3.1, one obtains

$$\sigma_{BQF}(T) \subset \bigcap_{K \in \mathcal{S}(\mathcal{X})} \sigma(T + K).$$

(iii) Based on Theorem 3.1, we deduce that $\sigma_{BQF}(T) \subset \sigma_s(T)$ and we get $\sigma_{BQF}(T) \subset \sigma_D(T)$ (see [3]). Thus, $\sigma_{BQF}(T) \subset \sigma_s(T) \cap \sigma_D(T)$. \square

Theorem 3.2 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{IM}(T^n)$ is closed of \mathcal{X} and $\mathcal{NO}(T^n) + \mathcal{IM}(T)$ is complemented. Then,*

$$\sigma_{Ubb}(T) \subset \mathcal{S}(T) \cup [\sigma_{Gu}(T) \cap \sigma_{BQF}(T)].$$

Proof: Based on Theorem 3.1, we get $\sigma_{QF}(T) \subset \sigma_{Gu}(T)$. We clearly have that $\sigma_{QF}(T) \subset \sigma_{BQF}(T)$. So, we conclude that $\sigma_{QF}(T) \subset \sigma_{eGu}(T) \cap \sigma_{BQF}(T)$. Hence,

$$\sigma_{QF}(T) \cup \mathcal{S}(T) \subset \mathcal{S}(T) \cup [\sigma_{Gu}(T) \cap \sigma_{BQF}(T)].$$

Now, from Theorem 3.1 and [[8], Theorem 6], we deduce that

$$\sigma_{Ubb}(T) \subset \mathcal{S}(T) \cup [\sigma_{Gu}(T) \cap \sigma_{BQF}(T)].$$

\square

Lemma 3.1 *Let \mathcal{X} be a Banach space and $T, S_0 \in \mathcal{L}(\mathcal{X})$. Suppose that $S_0 - T$ is semi-regular. Then,*

$$T \in \mathcal{DC}_{S_0}(\mathcal{X}) \text{ if, and only if, } S_0 - T \in \mathcal{SBF}_+(\mathcal{X}).$$

Proof: Let $T \in \mathcal{DC}_{S_0}(\mathcal{X})$. Grounded on [[10], Theorem 2.6], we therefore have $S_0 - T \in \mathcal{FD}_+(\mathcal{X}) \subset \mathcal{SBF}_+(\mathcal{X})$. Conversely, suppose that $S_0 - T \in \mathcal{SBF}_+(\mathcal{X})$. Then, there exists $n \in \mathbb{N}$ such that $\mathcal{IM}((S_0 - T)^n)$ is closed of \mathcal{X} and $(S_0 - T)_n$ is a upper semi Fredholm operator. Since $S_0 - T$ is semi-regular, then $\mathcal{IM}(S_0 - T)$ is closed and

$$\mathcal{NO}(S_0 - T) = \mathcal{NO}(S_0 - T) \cap \mathcal{IM}((S_0 - T)^n) < \infty.$$

Hence, $S_0 - T \in \mathcal{FD}_+(\mathcal{X})$. When applying [[10], Theorem 2.6], we deduce that $T \in \mathcal{DC}_{S_0}(\mathcal{X})$. \square

Theorem 3.3 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. If $\lambda \notin \sigma_{sr}(T)$, then*

$$\sigma_{Gu}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] \subset \sigma_{Ubb}(T),$$

where $\mathcal{S}(T)$ is the set of all $\lambda \in \mathbb{C}$ such that T does not have the single-valued extension property at λ .

Proof: Let $\lambda \notin \sigma_{Ubb}(T)$. Then, we get $\lambda - T \in \mathcal{SBF}_+(\mathcal{X})$. As $\lambda \notin \sigma_{sr}(T)$, by Lemma 3.1, we have $T \in \mathcal{DC}_{\lambda T}(\mathcal{X})$. Applying [[10], Theorem 2.7], we deduce that $\lambda - T \in \mathcal{FD}_+(\mathcal{X})$. In reality, we get to the conclusion that $\lambda \notin \sigma_{Gu}(T)$. Since, $\mathcal{S}(T) \subset \sigma_{iD}(T)$ (see [3]) and by [[11], Lemma 2.1], we have $\mathcal{S}(T) \subset \sigma_{Ubb}(T)$, then $\mathcal{S}(T^*) \cap \mathcal{S}(T) \subset \sigma_{Ubb}(T)$. Hence,

$$\sigma_{Gu}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)] \subset \sigma_{Ubb}(T).$$

□

Theorem 3.4 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Then,*

$$\sigma_{BQF}(T) \subset \bigcap_{K \in \Gamma_T(\mathcal{X}) \text{ and } KT=TK} \sigma_D(T + K).$$

Proof: Let $\lambda \notin \bigcap_{K \in \Gamma_T(\mathcal{X})} \sigma_D(T + K)$, then there exists $K_1 \in \Gamma_T(\mathcal{X})$ such that $\lambda - T - K_1$ is Drazin invertible. By [[5], Theorem 4.2], $\lambda - T - K_1$ is a B -Fredholm operator and $i(\lambda - T - K_1) = 0$. On the other hand, since $-\mu(\lambda - T - K_1)^{-1}K_1$ is demicompact for every $\mu \in [0, 1]$, then using [[9], Theorem 3.2], we obtain

$$[I + (\lambda - T - K_1)^{-1}K_1] \in \mathcal{BQF}(\mathcal{X}) \text{ and } i[I + (\lambda - T - K_1)^{-1}K_1] = 0.$$

Additionally, we have

$$\lambda - T = [I + (\lambda - T - K_1)^{-1}K_1](\lambda - T - K_1).$$

Since $K_1T = TK_1$, then $K_1(\lambda - T - K_1) = (\lambda - T - K_1)K_1$, for all $\lambda \in \rho(T + K_1)$. Grounded on [[6], Lemma 3.4], we deduce that $\lambda \notin \sigma_{BQF}(T)$. □

Corollary 3.2 *Let \mathcal{X} be a Banach space and $T \in \mathcal{L}(\mathcal{X})$. Let $\mathcal{S}(\mathcal{X})$ be a subset of $\Gamma_T(\mathcal{X})$. Then,*

$$\sigma_{BQF}(T) \subset \bigcap_{K \in \mathcal{S}(\mathcal{X}) \text{ and } KT=TK} \sigma_D(T + K).$$

Proof: Since $\mathcal{S}(\mathcal{X}) \subset \Gamma_T(\mathcal{X})$, one obtains

$$\bigcap_{K \in \Gamma_T(\mathcal{X}) \text{ and } KT=TK} \sigma_D(T + K) \subset \bigcap_{K \in \mathcal{S}(\mathcal{X}) \text{ and } KT=TK} \sigma_D(T + K).$$

According to Theorem 3.4, we deduce that

$$\sigma_{BQF}(T) \subset \bigcap_{K \in \mathcal{S}(\mathcal{X}) \text{ and } KT=TK} \sigma_D(T + K).$$

□

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