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Connectedness via $(1,2)S_\beta$- open sets in Bitopological spaces

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ABSTRACT: Topology is a fundamental branch of mathematics that provides a framework for studying spaces and their properties. Central to topology is the concept of open sets, which are subsets of a given space defined by specific characteristics inherent to the space's structure. These open sets form the foundation for various topological properties, allowing for a deeper understanding of connectivity within the space. In this work, we introduce a class of open sets in bitopological spaces namely (1, 2)S_β - open set by involving (1, 2) semi-open set and (1, 2)β-closed set. In addition, we present the essential properties of this class and study its relationship with the other classes of open sets with the help of counter examples. Further, we introduce (1, 2)S_β - separated sets, (1, 2)S_β - connected sets and study their properties in bitopological spaces. Also, we prove some results related to (1, 2)S_β - continuous functions.

Key Words: (1, 2)semi-open set, (1, 2)semi-connected set, (1, 2)S_β-open set, (1, 2)S_β-separated set, (1, 2)S_β-connected, (1, 2)β-closed set.

Contents

1. Introduction

In the year 1963, Kelly[8] initiated the systematic study of bitopology denoted by (X, τ, σ), where X is a non-empty set together with two distinct topologies τ, σ on X. Bitopological spaces offer a richer, more flexible, and more powerful framework for studying topological structures and their applications, especially when considering systems with multiple related topological structures. In the same year, Levine[9] introduced semi-open sets and studied their properties. After two decades, Monsef [1] defined the notion of β-open sets in topological spaces. In 2013, Khalaf and Ahmed [3] introduced a new class of semi-open set called S_β-open set. A semi-open subset A of a topological space $(X,\ \tau)$ is said to be S_β-open if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. Research on connectedness in bitopological spaces is a dynamic area with ongoing investigations into various forms of connectedness and their relationship with other concepts. In 1967, Pervin[13] defined the connectedness and continuous function in bitopological context and have been further studied in [1]. A topological space X is said to be connected if it cannot be represented as the union of two non-empty, disjoint, open subsets. Levine[9] introduced and studied the concept of S_β-connected set in general topological space. The results obtained by Levine in [9] motivates us to define and study (1, 2)S_β-connected set in a more general setting, that is bitopological spaces. We organize our paper as follows: In section 1, we give a brief history about the development of various concepts in bitopological spaces over a period of time. In section 2, we recall the required definitions in order to obtain the main results. In section 3, we introduce the concepts of τ_1- locally indiscrete, τ_1- hyperconnected and $(1,2)S_\beta$-separated set, $(1,2)S_\beta$-connected set, $(1,2)S_\beta$-continuous function in bitopological spaces and study their fundamental properties. Moreover, we present some basic results related to $(1,2)S_\beta$-continuous functions. In the conclusion, we summarize the results we have obtained in the paper and give a scope for further study.

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2. Preliminaries

In this section, we give some preliminary definitions which are used in the subsequent sequel.

Definition 2.1 *[9]* Let A be a subset of a bitopologial space $(X,\, \tau_1,\, \tau_2)$. Then A is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))), where τ_1-Int(A) is the interior of A with respect to the topology τ_1 and $\tau_1\tau_2$-Cl(A) is the intersection of all $\tau_1\tau_2$-closed sets containing A.

(iv) (1, 2)β-Int(A) is the union of all (1, 2)β-open sets contained in A.

(v) (1, 2)β-Cl(A) is the intersection of all (1, 2)β-closed sets containing A.

Definition 2.2 *[9]* A subset A of X is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))).

The set of all (1, 2)semi-open, (1, 2)regular-open and (1, 2)β-open are denoted by (1, 2)SO(X, τ_1, τ_2), (1, 2)RO(X, τ_1, τ_2), (1, 2)βO(X, τ_1, τ_2) or simply, (1, 2)SO(X), (1, 2)RO(X), (1, 2)β-O(X) respectively.

Definition 2.3 *[9]* A subset A of X is said to be

(i) (1, 2)semi-closed if $\tau_1\tau_2$-Int(τ_1-Cl(A)) $\subseteq A$.

(ii) (1, 2)regular-closed if $A = \tau_1$-Cl($\tau_1\tau_2$-Int(A)).

(iii) (1, 2)β- closed if $\tau_1\tau_2$- Int(τ_1-Cl($\tau_1\tau_2$-Int(A))) $\subseteq A$.

The set of all (1, 2)semi-closed, (1, 2)regular-closed and (1, 2)β-closed are denoted by (1, 2)SCL(X, τ_1, τ_2), (1, 2)RCL(X, τ_1, τ_2), (1, 2)βCL(X, τ_1, τ_2) or simply, (1, 2)SCL(X), (1, 2)RCL(X), (1, 2)β-CL(X) respectively.

Remark 2.4 For any subset A of X,

(i) τ_1-Int(A) $\subseteq \tau_1\tau_2$-Int(A) and τ_2-Int(A) $\subseteq \tau_1\tau_2$-Int(A).

(ii) $\tau_1\tau_2$-Cl(A) $\subseteq \tau_1$-Cl(A) and $\tau_1\tau_2$-Cl(A) $\subseteq \tau_2$-Cl(A).

(iii) $\tau_1\tau_2$-Cl($A \cap B$) $\subseteq \tau_1\tau_2$-Cl(A) $\cap\, \tau_1\tau_2$-Cl(B).

(iv) $\tau_1\tau_2$-Int(A) $\cup\, \tau_1\tau_2$-Int(B) $\subseteq \tau_1\tau_2$-Int($A \cup B$).

Definition 2.5 *[3]* A semi-open subset A of a topological space $(X,\, \tau)$ is said to be S_β-open if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. A subset B of a topological space $(X,\, \tau)$ is S_β-closed if $X{-}B$ is S_β-open.

3. Main Results

We begin this section with the definition of (1, 2)S_β-open set and its characterizations.

Definition 3.1 A (1, 2)semi-open subset A of a bitopological space (X, τ_1, τ_2) is said to be (1, 2)S_β-open set if for each $x \in A$ there exists a (1, 2)β-closed set F such that $x \in F \subseteq A$.
The complement of a (1, 2)S_β-open set is (1, 2)S_β-closed set and the family of all (1, 2)S_β-open ((1, 2)S_β-closed) subsets of X, is denoted by (1, 2)S_β-O(X) ((1, 2)S_β-CL(X)) respectively.

Proposition 3.2 A subset A of a bitopological space (X, τ_1, τ_2) is (1, 2)S_β-open set if and only if A is (1, 2)semi-open and it is the union of (1, 2)β-closed sets.

Proof: It is obvious from Definition 3.1. □

Definition 3.3 The subsets A and B of a bitopological space (X, τ_1, τ_2) are said to be (1, 2)S_β-separated if $A \cap$ (1, 2)S_β-Cl$(B) = \phi =$ (1, 2)S_β-Cl$(A) \cap B$.

Note 1 Obviously, two (1, 2)S_β-separated sets are disjoint. That is, if A and B are two (1, 2)S_β-separated sets in X and $C, D \in X$ such that $A \supset C \neq \phi$ and $B \supset D \neq \phi$, then C and D are also (1, 2)S_β-separated sets in X.

Example 3.4 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {d}, {a, d}} and τ_2 = {ϕ, X, {a, c, d}}. Then (1, 2)SO(X) = {ϕ, X, {a}, {d}, {a, b}, {a, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}, (1, 2)S_β-O(X) = {ϕ, X, {a}, {d}, {a, b}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}} and (1, 2)S_β-CL(X) = {ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {b, c}, {b, d}, {c, d}, {a, b, c}, {b, c, d}}. Here, the subsets of X are $(1,2)S_\beta$-separated sets.

Example 3.5 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {b}, {a, b}} and τ_2 = {ϕ, X, {a, b, d}}. Then (1, 2)SO(X) = (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}} and (1, 2)S_β-CL(X) = {ϕ, X, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}}. Here, the subsets of X are (1, 2)S_β-separated sets and (1, 2)semi-separated sets.

Definition 3.6 A subset S of a bitopological space (X, τ_1, τ_2) is said to be (1, 2)S_β-connected in X if S is not the union of two $(1,2)S_\beta$-separated sets in X.

Example 3.7 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {a, b}, {a, c, d}} and τ_2 = {ϕ, X, {d}}. Then (1, 2)SO(X) = (1, 2)S_β-O(X) = {ϕ, X, {a}, {a, b}, {a, c, d}} and (1, 2)S_β-CL(X) = {ϕ, X, {b}, {c, d}, {b, c, d}}. Here, there exists no separation. Therefore, X is a $(1,2)S_\beta$-connected space.

Theorem 3.8 A bitopological space (X, τ_1, τ_2) is (1, 2)S_β-connected if and only if X cannot be expressed as the union of two non-empty disjoint (1, 2)S_β-open subsets of X.

Proof: Let a bitopological space (X, τ_1, τ_2) be (1, 2)S_β-connected space. Suppose A, B are two non-empty disjoint (1, 2)S_β-open subsets of X such that $X = A \cup B$. Then A and B are (1, 2)S_β-closed sets in X. Thus, $A \cap$ (1, 2)S_β-Cl$(B) = \phi =$ (1, 2)S_β-Cl$(A) \cap B$. Then X is not a (1, 2)S_β-connected space, which is a contradiction.

Conversely, suppose that $X = A \cup B$, $A \neq \phi$, $B \neq \phi$ and $A \cap$(1, 2)S_β-Cl$(B) = \phi =$ (1, 2)S_β-Cl(A) $\cap$ B. Then A and B are non-empty disjoint (1, 2)S_β-open sets, which is a contradiction. Hence, X is a $(1,2)S_\beta$-connected space. □

Theorem 3.9 For a bitopologocal space (X, τ_1, τ_2), the following are equivalent.

(i) X is (1, 2)S_β- connected.

(ii) X and ϕ are the only subsets which are both (1, 2)S_β-open and (1, 2)S_β-closed in X.

(iii) There is no non-constant onto (1, 2)S_β-continuous function from X to a discrete space that contains more than one point.

Proof: (i)⇒ (ii) from Theorem 3.8.
(ii)⇒ (iii) Let Y be a discrete space with more than one point and a function $f : X \to Y$ be onto (1, 2)S_β-continuous. Also, let $y \in Y$ and $A = \{y\}$. Since $f : X \to Y$ is onto (1, 2)S_β-continuous function, $f^{-1}(A)$ is non-empty. By hypothesis, $f^{-1}(A)$ is both (1, 2)S_β-open and (1, 2)S_β-closed which implies $f^{-1}(A) = X$. Therefore, f is constant.
(iii)⇒(i) Suppose that X is not (1, 2)S_β- connected. Let A, B be two non-empty subsets of X. If $X = A \cup B$ such that (1, 2)S_β-Cl$(A) \cap B = \phi =$ (1, 2)S_β-Cl$(B) \cap A$, then A and B are (1, 2)S_β-open sets in X. Let us consider $Y = \{0, 1\}$. We define a map f from (X, τ_1, τ_2) to (Y, σ_1, σ_2) by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Then f is non-constant onto (1, 2)S_β-continuous mapping, which is a contradiction to (iii). □

Definition 3.10 A bitopological space (X, τ_1, τ_2) is said to be τ_1-hyperconnected, if every non-empty τ_1- open subset of X is dense in X.

Lemma 3.11 If a bitopological space (X, τ_1, τ_2) is τ_1-hyperconnected, then (1, 2)S_β-O$(X) \cap$ (1, 2)S_β-Cl$(X) = \{\phi, X\}$.

Theorem 3.12 If a bitopological space (X, τ_1, τ_2) is τ_1-hyperconnected, then it is (1, 2)S_β-connected.

Proof: It is obvious from Theorem 3.9 and Lemma 3.11. □

Theorem 3.13 If a bitopological space (X, τ_1, τ_2) is (1, 2)S_β-connected space, then it is τ_1-connected.

Proof: Let a bitopological space (X, τ_1, τ_2) be (1, 2)S_β-connected space. Then X and ϕ are the only subsets which are both (1, 2)S_β-open and (1, 2)S_β-closed in X. Suppose X is not τ_1-connected, then there is a non-empty proper subset A of X which is both τ_1-open and τ_1-closed. Hence, A is both (1, 2)S_β-open and $(1, 2)S_\beta$-closed in X, which is a contradiction. □

Definition 3.14 Two non-empty subsets A and B of a bitopological space (X, τ_1, τ_2) is said to be (1, 2)semi-separated, if and only if $A \cap$ (1, 2)semi-Cl$(B) = \phi =$ (1, 2)semi-Cl$(A) \cap B$

Definition 3.15 In a bitopological space (X, τ_1, τ_2) is said to be (1, 2)semi-connected, if a set cannot be expressed as the union of two (1, 2)semi-separated sets.

Theorem 3.16 If a bitopological space (X, τ_1, τ_2) is (1, 2)semi-connected, then it is (1, 2)S_β-connected.

Proof: It follows from the fact that, (1, 2)S_β-O$(X) \subseteq$ (1, 2)SO(X). □

Remark 3.17 The following example shows that the converse is not true.

Example 3.18 Let $X = \{$a, b, c, d$\}$ with two topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. We have (1, 2)SO(X)=$\{\phi, X, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$. Then (1, 2)$S_\beta$-O$(X) = \{\phi, X\}$. Hence, X is a $(1, 2)S_\beta$-connected set but not $(1, 2)$semi-connected set.

Definition 3.19 A bitopological space (X, τ_1, τ_2) is τ_1-locally indiscrete, if every τ_1 - open subset of X is τ_1-closed.

Theorem 3.20 If a bitopological space (X, τ_1, τ_2) is τ_1-locally indiscrete, then (1, 2)S_β-O$(X) =$ (1, 2)SO$(X) = \tau_1$.

Proof: It is sufficient to show that every (1, 2)semi-open set is τ_1-open in X. Let A be a (1, 2)semi-open set in X. Then $A \subseteq \tau_1$ τ_2-cl(τ_1-int(A)). Since X is τ_1-locally indiscrete, τ_1 τ_2-cl(τ_1-int(A)) $\subseteq \tau_1$-int(A). □

Corollary 3.21 A τ_1-locally indiscrete bitopological space (X, τ_1, τ_2) is τ_1-connected space if and only if X is (1, 2)S_β-connected.

Theorem 3.22 If a bitopological space (X, τ_1, τ_2) is τ_1-locally indiscrete (1, 2)S_β-connected space, then it has indiscrete topology.

Corollary 3.23 A τ_1-locally indiscrete bitopological space (X, τ_1, τ_2) is a (1, 2)S_β-connected space if and only if it is τ_1-hyperconnected.

Theorem 3.24 Let (X, τ_1, τ_2) be a bitopological space. If A is a (1, 2)S_β-connected set and U, V are (1, 2)S_β-separated sets of X such that $A \subseteq U \cap V$, then there exists either $A \subseteq U$ or $A \subseteq V$.

Proof: Since $A = (A \cap U) \cup (A \cap V)$, we have $(A \cap U) \cap$ (1, 2)S_β-Cl($A \cap V$) $\subseteq U \cap$ (1, 2)S_β-Cl(V) $= \phi$. If $A \cap U$ and $A \cap V$ are non-empty, then A is not (1, 2)S_β-connected, which is a contradiction. Therefore, there exists either $A \subseteq U$ or $A \subseteq V$. □

Theorem 3.25 If A is a (1, 2)S_β-connected set of a bitopological space (X, τ_1, τ_2) and $A \subseteq N \subseteq$ (1, 2)S_β-Cl(A), then N is (1, 2)S_β-connected.

Proof: Assume that N is not a (1, 2)S_β-connected set. Then there exist two (1, 2)S_β-separated sets U and V such that $N = U \cup V$. By Theorem 3.8, $A \subseteq N$ is either $A \subseteq U$ or $A \subseteq V$. If $A \subseteq U$, then $(1,2)S_\beta$-Cl(A) $\cap V = \phi$, which is a contradiction. Hence the proof. □

Corollary 3.26 If A is a (1, 2)S_β-connected subset of a bitopological space, then (1, 2)S_β-Cl(A) is (1, 2)S_β-connected set.

Theorem 3.27 Let A and B be two non-empty subsets of a bitopological space (X, τ_1, τ_2). If A and B are (1, 2)S_β-connected, then $A \cup B$ is (1, 2)S_β-connected.

Proof: Suppose that $A \cup B$ is not (1, 2)S_β-connected. Then there are (1, 2)S_β-separated sets G and H in X such that $A \cup B = G \cup H$. By Theorem 3.8, we get either $A \subseteq G$ or $A \subseteq H$ and $B \subseteq G$ or $B \subseteq H$. If $A \subseteq G$ or $B \subseteq G$, then $(A \cup B) \subseteq G$ and $H = \phi$, which is a contradiction. If $A \subseteq G$ and $B \subseteq H$, then A and B are $(1,2)S_\beta$-separated sets in X, which is also a contradiction. □

Theorem 3.28 Let (X, τ_1, τ_2) be a bitopological space and $\{x\}$ be τ_1- closed in X for each $x \in X$. Then the space X is (1, 2)S_β-connected if and only if X is (1, 2)semi-connected.

Definition 3.29 A bitopological space (X, τ_1, τ_2) is τ_1-Hausdorff space if for any two distinct points x, y ∈ X, then there exist a τ_1 - open sets U, V ∈ X such that $x \in U$, $y \in V$ and $U \cap V$.

Corollary 3.30 Let (X, τ_1, τ_2) be a τ_1- Hausdorff space. Then the space X is (1, 2)S_β-connected if and only if X is (1, 2) semi-connected.

4. Conclusion

In this work, we define and study some properties of connectedness and compare it with (1, 2) semi-connected by using (1, 2)S_β-open sets in bitopological spaces. This work will lead to study the components of connected sets such as pairwise connected, weak connected, path connected, locally path connected and their properties in bitopological spaces.

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