Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78446

A Note on Sumsets and Difference Sets in Groups of Order 12

Neetu, Anupriya Shetty and B. R. Shankar*

ABSTRACT: A subset A of a group G is referred to as a balanced set when $|A+A| = |A-A|$, MSTD (more sums than differences) when $|A+A| > |A-A|$, and MDTS (more differences than sums) when $|A-A| > |A+A|$. In this paper, we present a comparative study of MSTD and MDTS sets in groups of order 12 up to isomorphism. Additionally, we have completely categorized such sets in these groups and have provided a set A with the current highest value of $ln(|A+A|)/ln(|A-A|)$.

Key Words: MSTD sets, MDTS sets, balanced sets.

Contents

1. Introduction

Given a subset A of a group G, we define the sumset and difference set of A as follows:

$$A + A = \{a_i + a_j : a_i, a_j \in A\},$$

$$A - A = \{a_i - a_j : a_i, a_j \in A\}.$$

The subgroup generated by the set A is denoted by $\langle A \rangle$. A set $A \subseteq G$ is said to be abelian if $\langle A \rangle$ is an abelian subgroup of G, and $A \subseteq G$ is said to be nonabelian if it is not an abelian set. The cardinality of the set A is denoted by $|A|$.

If we consider G to be a group of integers, that is $\mathbb{Z}$ then for any two different elements a and b addition is commutative but subtraction is not. So for any two different elements, we always get one new sum $a+b$ but two differences $a-b$ and $b-a$. This usually makes the difference set $A-A$ larger than the sumset $A+A$, so we often have $|A+A| \leq |A-A|$. However, for a number $a \in A$, the pair (a, a) gives a new sum $a + a = 2a$, but the difference is always zero. So, diagonal pairs contribute more to the sum set than to the difference set. Because of this, it is possible to find sets A for which the number of sums is greater than the number of differences. We define these types of sets as follows.

Definition 1.1 *If $|A+A| > |A-A|$, we say A is a* More Sums Than Differences (MSTD) *set or a* sum-dominated *set , while if $|A+A| = |A-A|$ we say A is* balanced*, and if $|A+A| < |A-A|$ then A is a* More Differences Than Sums(MDTS) *set or a* difference-dominated *set.*

---

* Corresponding author.
2010 *Mathematics Subject Classification:* 11B75, 11P70, 11P99.
Submitted August 16, 2025. Published October 02, 2025

Though MSTD sets are rare among all finite subsets of integers, they do exist. Conway in the 1960s found the first example of MSTD set $\{0, 2, 3, 4, 7, 11, 12, 14\}$ of cardinality 8. Other examples of this type can be found [9] and [6]. Almost all previous research on MSTD sets focused exclusively on subsets of the integers, see [3,4,11]. However, the phenomenon in finite groups has received some attention, notably in [1,5,7,8,13,16], the majority of them are on finite abelian groups. Miller and Vissuet [10] were the first to examine the problem for arbitrary finite groups G, also with the size of the group going to infinity, and proved that as the size of the finite group grows, almost all subsets are balanced. Recently, Ascoli et al. [2] and Neetu and Shankar [12] have studied the comparison of MSTD and MDTS sets in non-abelian groups.

Since the conventional notation for the operation of a finite group is multiplication, we match the notation from prior work (specifically from [10]) and define the sumset and difference set for a subset $A \subseteq G$ as

$$A + A = \{a_1a_2 : a_1, a_2 \in A\}$$

and

$$A - A = \{a_1a_2^{-1} : a_1, a_2 \in A\}.$$

As the smallest order of an abelian group containing an MSTD set is 12, in this paper, we have focused on sumsets and difference sets in groups of order 12. Up to isomorphism, there are 5 groups of order 12, two of them are abelian, and the remaining are nonabelian. The abelian groups are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_{12}$. The nonabelian groups are the dihedral group D_6, the alternating group A_4, and the dicyclic group Dic_3.

The paper is organized as follows: In Section 2, we present some general results related to finite groups and in particular for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_{12}$. In Section 3, we prove that Dic_3 has more MDTS sets than MSTD sets, and we conjecture that Dic_n has more MDTS sets than MSTD sets. Conversely, in Sections 4 and 5, we prove that A_4 and D_6 have a higher number of MSTD sets as compared to MDTS sets, respectively. In general, we observe that A_n and D_{2n} have more MSTD sets than MDTS sets.

2. Some Important Results and Abelian Groups of Order 12

Proposition 2.1 *Let G be a group, and let A be a finite subset of G such that for each $\alpha \in A$, $\alpha^{-1} \in A$. Then A is balanced.*

Proof: Since for each $\alpha \in A$, $\alpha^{-1} \in A$, and therefore $A + A = A - A$. Hence A is balanced. □

Corollary 2.1 *Let G be a group, and A be a finite subset of G consisting of elements of order 1 or 2. Then A is balanced.*

Proof: If α is of order 1 or 2 then $\alpha^{-1} = \alpha$. Then it follows from the above proposition that A is balanced. □

Theorem 2.1 *Let G be a finite group, and let A be a finite subset of G such that $|A| > |G|/2$. Then $A + A = A - A = G$.*

Proof: Let G be a finite group with $|G| = n$. Let $A \subset G$ with $|A| > \frac{n}{2}$. Suppose $A + A \neq G$. Then there exists $x \in G$ such that $x \notin A+A$. Define a map $f : G \to G$ by $f(g) = xg^{-1}$. Then f is bijective. Therefore, $|f(A)| = |A| > \frac{n}{2}$. If $a' \in A \cap f(A)$, then $a' = f(a) = xa^{-1}$, for some $a \in A$. This implies $x = a'a \in A+A$, a contradiction. Therefore, A and $f(A)$ are disjoint. So that $|A \cup f(A)| = |A| + |f(A)| > n$, a contradiction. Hence $A + A = G$. Similarly, we can show that $A - A = G$. □

This theorem implies that in a group of order 12, any set whose cardinality is at least 7 is a balanced set. Therefore, we look at sets with cardinality at most 6.

It was shown by Penman and Wells [14] that the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ does not possess any MSTD set. Since this is an abelian group, it was observed that in finite abelian groups MDTS sets are more as compared to MSTD sets. Consider the following example.

Example 2.1 *Consider a set* $A \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ *of* 4 *elements, say:*

$$A = \{(0,0,0),(0,1,0),(1,0,1),(0,1,1)\}.$$

Here $|A + A| = 8$ *and* $|A - A| = 11$*, so this type of sets are MDTS.*

Thus, we have more MDTS sets than MSTD sets in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Now, consider the cyclic group $\mathbb{Z}_{12}$. It is the smallest cyclic group in which an MSTD set exists, and there are exactly 24 MSTD sets in $\mathbb{Z}_{12}$ as shown by Rafal [15]. Here is an example of MSTD set in $\mathbb{Z}_{12}$.

Example 2.2 *Let*

$$A = \{1,2,3,5,6,10\},$$

then

$$A + A = \{0,1,2,3,4,5,6,7,8,9,10,11\}$$

and

$$A - A = \{0,1,2,3,4,5,7,8,9,10,11\}.$$

Here $12 = |A + A| > |A - A| = 11$.

On the other hand, MDTS sets are easier to find. For instance, take

$$A = \{0,2,4,6,8,10\}.$$

Here, $|A + A| = 6$ and $|A - A| = 6$, so A is an balanced set. Now, if we replace just one element of A, say 0, with an element outside A, such as 1, we get a new set

$$A' = \{1,2,4,6,8,10\}$$

for which $|A' + A'| = 11$ and $|A' - A'| = 12$. In general, we can generate many MDTS sets by replacing a single element of A with another element from $\mathbb{Z}_{12} \setminus A$. Specifically, for

$$A' = (A \setminus \{a\}) \cup \{b\}, \text{ where } a \in A \text{ and } b \in \mathbb{Z}_{12} \setminus A,$$

each such one element modification produces a new set A'. Since A has 6 elements and $\mathbb{Z}_{12} \setminus A = \{1,3,5,7,9,11\}$ also contains 6 elements, there are $6 \times 6 = 36$ such sets and all are MDTS. Therefore, we see that $\mathbb{Z}_{12}$ contains more MDTS sets than MSTD sets.

3. Dicyclic Group Dic_3

The *dicyclic group* Dic_3 is defined as

$$\mathrm{Dic}_3 = \langle a, b : a^6 = 1, b^2 = a^3, bab^{-1}a = 1\rangle.$$

The elements of this group are precisely $1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b$. Suppose

$$U = \{1, a, a^2, a^3, a^4, a^5\}$$

and

$$V = \{b, ab, a^2b, a^3b, a^4b, a^5b\},$$

then $\mathrm{Dic}_3 = U \cup V$. We now study subsets of Dic_3 with various cardinalities. As singleton sets are always balanced, we start with sets of cardinality 2.

Theorem 3.1 *Let* $A_{12,2}$ *denote the collection of subsets of* Dic_3 *of size* 2*. Then* $A_{12,2}$ *has strictly more MSTD sets than MDTS sets.*

Proof: Let A be a subset of Dic_3 with cardinality 2. There are only three possible cases to consider.

Case 1. If $A = \{a^i, a^j\} \subseteq U$, where $i \neq j$ then $A + A = \{a^{2i}, a^{i+j}, a^{2j}\}$ and $A - A = \{1, a^{i-j}, a^{j-i}\}$. If all are distinct, then $|A + A| = |A - A| = 3$. Suppose $a^{2i} = a^{2j}$ this gives $a^{i-j} = a^{j-i}$. Thus, when A contains only a^i and a^j, A is always balanced.

Case 2. If $A = \{a^i, a^j b\}$ where $0 \leq i, j \leq 5$, then $A + A = \{a^{2i}, a^{i+j}b, a^{j-i}b, a^3\}$ and $A - A = \{1, a^{i+j+3}b, a^{i+j}b\}$. If $a^{2i} = a^3$, then $A + A = \{a^3, a^{i+j}b, a^{j-i}b\}$ and $A - A = \{1, a^{i+j+3}b, a^{i+j}b\}$ and if $a^{i+j}b = a^{j-i}b$, then $A + A = \{1, a^{i+j}b, a^3\}$ and $A - A = \{1, a^{i+j+3}b, a^{i+j}b\}$. So A is a balanced set. When $a^{2i} \neq a^3$ and $a^{i+j}b \neq a^{j-i}b$, A will be an MSTD set. There are 24 of these types of sets. They are

$$\{a, b\}, \{a, ab\}, \{a, a^2b\}, \{a, a^3b\}, \{a, a^4b\}, \{a, a^5b\},$$

$$\{a^2, b\}, \{a^2, ab\}, \{a^2, a^2b\}, \{a^2, a^3b\}, \{a^2, a^4b\}, \{a^2, a^5b\},$$

$$\{a^4, b\}, \{a^4, ab\}, \{a^4, a^2b\}, \{a^4, a^3b\}, \{a^4, a^4b\}, \{a^4, a^5b\},$$

$$\{a^5, b\}, \{a^5, ab\}, \{a^5, a^2b\}, \{a^5, a^3b\}, \{a^5, a^4b\}, \{a^5, a^5b\}.$$

Case 3. If $A = \{a^i b, a^j b\} \subseteq V$, then $A + A = \{a^3, a^{3+i-j}, a^{3+j-i}\}$ and $A - A = \{1, a^{i-j}, a^{j-i}\}$. This shows that A is a balanced set.

Therefore, we have a total of 24 MSTD sets and 42 balanced sets in Dic_3 of cardinality 2. □

In fact, with the help of the above theorem, we can prove in general for any dicyclic group of order $4n$. The dicyclic group of order $4n (n \geq 1)$ is defined as

$$\mathrm{Dic}_n = \langle a, b : a^{2n} = 1, a^n = b^2, ab = ba^{-1} \rangle.$$

It has exactly $4n$ distinct elements, which can be written as

$$\{a^k \mid 0 \leq k < 2n\} \cup \{a^k b \mid 0 \leq k < 2n\}.$$

Theorem 3.2 *Let $n \geq 2$ be an integer, and let $A_{4n,2}$ denote the family of all 2-element subsets of the dicyclic group Dic_n. Then, there are strictly more MSTD sets than MDTS sets in $A_{4n,2}$.*

Proof: Consider an arbitrary subset $A \subseteq \mathrm{Dic}_n$ with $|A| = 2$. Since the elements of Dic_n are either powers of a or of the form $a^k b$ for some integer k, we analyze three distinct cases:

Case 1: Both elements of A are powers of a, say $A = \{a^i, a^j\}$ with $0 \leq i, j < 2n$. The inverse set is $A^{-1} = \{a^{2n-i}, a^{2n-j}\}$. The sumset is

$$A + A = \{a^{2i}, a^{i+j}, a^{2j}\},$$

and the difference set is

$$A - A = \{1, a^{2n+i-j}, a^{2n+j-i}\}.$$

Except for the special case when $a^{2i} = a^{2j}$, both $A + A$ and $A - A$ have size 3. The equality $a^{2i} = a^{2j}$ implies $a^{i-j} = a^{j-i}$, so these subsets are balanced.

Case 2: The set A contains one power of a and one element of the form $a^j b$, i.e., $A = \{a^i, a^j b\}$. Here, the inverse set is

$$A^{-1} = \{a^{2n-i}, a^{n+j}b\}.$$

We find

$$A + A = \{a^{2i}, a^{i+j}b, a^{j-i}b, a^n\}$$

and

$$A - A = \{1, a^{i+j+n}b, a^{i+j}b\}.$$

When $a^{2i} \neq a^n$ and $a^{i+j}b \neq a^{j-i}b$, the set A is MSTD. In the cases where either $a^{2i} = a^n$ or $a^{i+j}b = a^{j-i}b$, the sumset and difference set sizes coincide, so A is balanced.

Case 3: Both elements are of the form $a^k b$, i.e., $A = \{a^i b, a^j b\}$ for some i, j. The inverse set is

$$A^{-1} = \{a^{n+i}b, a^{n+j}b\}.$$

Then,

$$A + A = \{a^n, a^{n+i-j}, a^{n+j-i}\}$$

and

$$A - A = \{1, a^{i-j}, a^{j-i}\}.$$

In this scenario, A is always balanced.

Since only subsets in Case 2 can be MSTD (under mild conditions) and the others are balanced, it follows that $A_{4n,2}$ contains strictly more MSTD than MDTS subsets. □

However, this pattern does not hold for subsets of cardinality 3, 4 and 5. In fact, we have the following results.

Theorem 3.3 *Let $A_{12,3}$ denote the collection of subsets of* Dic_3 *of size* 3. *Then $A_{12,3}$ has strictly more MDTS sets than MSTD sets.*

Proof: Let A be a subset of Dic_3 of cardinality 3. Consider the following cases.

Case 1. Let $A = \{a^i, a^j, a^k\} \subseteq U$ where $i \neq j \neq k$. There are $\binom{6}{3} = 20$ such sets. In this case,

$$A + A = \{a^{2i}, a^{i+j}, a^{i+k}, a^{2j}, a^{2k}, a^{j+k}\}$$

and

$$A - A = \{1, a^{i-j}, a^{i-k}, a^{j-i}, a^{j-k}, a^{k-i}, a^{k-j}\}$$

which implies that $|A - A| \geq |A + A|$. Among these, 8 sets satisfy $i + j + k \equiv 0$ or $3 \pmod 6$, making them balanced sets. The remaining 12 are MDTS. Suppose we take $A = \{1, a, a^4\}$ then $|A + A| = 5$ and $|A - A| = 6$. So here $|A - A| > |A + A|$. Similarly, we can find other MDTS sets. Therefore, in total, we have 12 MDTS sets and 8 balanced sets.

Case 2. Let $A = \{a^i, a^j, a^k b\}$ for $i \neq j$,. There are $\binom{6}{2} \times 6 = 90$ such sets. Then,

$$A + A = \{a^{2i}, a^{i+j}, a^{i+k}b, a^{2j}, a^{j+k}b, a^{k-i}b, a^{k-j}b, a^3\}$$

and

$$A - A = \{1, a^{i-j}, a^{3+i+k}b, a^{j-i}, a^{3+j+k}b, a^{k+i}b, a^{k+j}b\}.$$

We further classify this case into subcases:

Subcase 2.1. If $i = 0$ or $j = 0$, then $|A - A| \geq |A + A|$. There are ${}^5C_1 \times {}^6 C_1 = 30$ such sets.

Subcase 2.2. If $i = 3$ or $j = 3$, then $|A - A| \geq |A + A|$. Because in $A + A$, a^{i+k} is equal to a^{k-i} whereas the cardinality of $A - A$ is not affected. There are 24 sets of this type, different from Subcase 2.1.

Subcase 2.3. If $j = 6-i$, then $a^{j+k}b = a^{k-i}b$ and $a^{i+k} = a^{k-j}b$ in $A+A$ which yields $|A-A| \geq |A+A|$. There are 12 sets of this type.

Subase 2.4. If $i + j = 3$, then $a^{i+j} = a^3$ which implies $|A - A| \geq |A + A|$. There are 12 sets of such types.

Thus, most such subsets are MDTS. However, MSTD subsets can occur when $j - i = 3$. If A is of the form $\{a, a^4, \alpha\}$ where $\alpha \in V$, then $|A + A| = 7$ and $|A - A| = 4$. This implies $|A + A| > |A - A|$. There are 6 sets of this type, each containing a, a^4 and α. Similarly, there are 6 sets containing a^2, a^5 and α.

Therefore, we have a total of 12 MSTD sets of this type.

Case 3. Let $A = \{a^i, a^j b, a^k b\}$ for $j \neq k$. Then we have

$$A + A = \{a^{2i}, a^{i+j}b, a^{i+k}b, a^{j-i}b, a^3, a^{3+j-k}, a^{k-i}b, a^{3+k-j}\}$$

and

$$A - A = \{1, a^{i+j}b, a^{3+i+k}b, a^{j-k}, a^{3+j+i}b, a^{k-j}, a^{k+i}\}.$$

Similar to Case 2, we can show that A is MSTD precisely when $i \neq 0, 3$ and $k = 3 + j$. There are 12 sets of this type.

$$\{a, b, a^3b\}, \{a, ab, a^4b\}, \{a, a^2b, a^3b\}, \{a^2, b, a^3b\}$$

$$\{a^2, ab, a^4b\}, \{a^2, a^2b, a^3b\}, \{a^4, b, a^3b\}, \{a^4, ab, a^4b\}$$

$$\{a^4, a^2b, a^3b\}, \{a^5, b, a^3b\}, \{a^5, ab, a^4b\}, \{a^5, a^2b, a^3b\}.$$

Case 4. Let $A = \{a^ib, a^jb, a^kb\}$ for $i \neq j \neq k$. Then we have

$$A + A = \{a^3, a^{3+i-j}, a^{3+i-k}, a^{3+j-i}, a^{3+j-k}, a^{3+k-i}, a^{3+k-j}\}$$

and

$$A - A = \{1, a^{i-j}, a^{i-k}, a^{j-i}, a^{j-k}, a^{k-i}, a^{k-j}\}.$$

Here both $A + A$ and $A - A$ are subsets of U so their cardinalities do not exceed 6. Further, $A + A$ is a dilation of $A - A$ by a^3. Therefore, A is balanced. Therefore, we have 24 MSTD sets and 48 MDTS sets in Dic_3 of cardinality 4. □

Conjecture 3.1 *Let $n \geq 3$ be an integer. Let $A_{4n,3}$ denote the collection of subsets of Dic_n of size 3. Then $A_{4n,3}$ has strictly more MDTS sets than MSTD sets.*

Lemma 3.1 *Let $A_{12,4}$ denote the collection of subsets of Dic_3 of size 4. There are 126 sets which are MSTD in $A_{12,4}$.*

Proof: Let A be a subset of Dic_3 of cardinality 4. There are $\binom{12}{4} = 495$ such sets. There are five possible cases for A, namely, $\{a^i, a^j, a^k, a^l\}, \{a^i, a^j, a^k, a^lb\}, \{a^i, a^j, a^kb, a^lb\}, \{a^i, a^jb, a^kb, a^lb\}$ and $\{a^ib, a^jb, a^kb, a^lb\}$, where $0 \leq i, j, k, l \leq 5$. When $A \subseteq U$ or $A \subseteq V$ then we can proceed similar to Theorem 3.3, Case 1, and Case 4. Thus, we do not get any MSTD sets in these two cases. Now we consider the remaining three cases.

Case 1. Suppose $A = \{a^i, a^j, a^k, a^lb\}$ where $i \neq j \neq k$. There are ${}^6C_3 \times {}^6C_1 = 120$ such sets. We start with cases when $i = 0$ and $j = 1$ or $i = 0$ and $j = 2$ with $k = 3 + j$. We observed that sets of this type i.e. $\{1, a, a^4, \alpha\}$ and $\{1, a^2, a^5, \alpha\}$ where $\alpha \in V$ are MSTD. If we take $i + j = 6$, or $i + k = 6$, or $j + k = 6$. There are 36 sets of these types and all are either balanced or MDTS. If $i + j + k = 0 \pmod 6$, then A is either balanced or MDTS. There are 12 such sets. Now, from the remaining cases, we can find 12 MSTD sets. Suppose $A = \{a, a^3, a^4, \alpha\}$, where $\alpha \in V$, then $|A + A| = 11 > 10 = |A - A|$. There are 6 sets of this type, each containing a, a^3, a^4 and α. Similarly, there are 6 sets containing a^2, a^3, a^5 and α. Therefore, we have 24 MSTD sets in this case.

Case 2. Let $A = \{a^i, a^j, a^kb, a^lb\}$. There are ${}^6C_2 \times {}^6C_2 = 225$ sets of this type.

Subcase 2.1. Suppose $i = 0$. There are ${}^5C_1 \times {}^6C_2 = 75$ such sets out of which 12 are MSTD sets and all others are either balanced or MDTS. They are $\{1, a, b, a^3b\}, \{1, a, ab, a^4b\}, \{1, a, a^2b, a^5b\}, \{1, a^2, b, a^3b\}, \{1, a^2, ab, a^4b\}, \{1, a^2, a^2b, a^5b\}, \{1, a^4, b, a^3b\}, \{1, a^4, ab, a^4b\}, \{1, a^4, a^2, a^5b\}, \{1, a^5, b, a^3b\}, \{1, a^5, ab, a^4b\}, \{1, a^5, a^2b, a^5b\}$.

Subcase 2.2. If $i = 3$ and $l = 3 + k$, then there are 12 MSTD sets of this type. They are $\{a^3, a, b, a^3b\}, \{a^3, a, ab, a^4b\}, \{a^3, a, a^2b, a^5b\}, \{a^3, a^2, b, a^3b\}, \{a^3, a^2, ab, a^4b\}, \{a^3, a^2, a^2b, a^5b\}, \{a^3, a^4, b, a^3b\}, \{a^3, a^4, ab, a^4b\}, \{a^3, a^4, a^2b, a^5b\}, \{a^3, a^5, b, a^3b\}, \{a^3, a^5, ab, a^4b\}, \{a^3, a^5, a^2b, a^5b\}$.

Subcase 2.3. If $i + j = 3 \pmod 6$, then A is either balanced or MDTS. There are 30 such sets.

Subcase 2.4. Now, we can find 30 MSTD sets from the remaining cases. There are 15 sets of the form $\{a, a^4, \alpha, \beta\}$, where $\alpha, \beta \in V$ and $|A + A| = 7 > 4 = |A - A|$. Similarly, there are 15 sets of the form $\{a^2, a^5, \alpha, \beta\}$ which are MSTD.

Therefore, in this case, we have 54 MSTD sets.

Case 3. Let $A = \{a^i, a^jb, a^kb, a^lb\}$. There are ${}^6C_1 \times {}^6C_3 = 120$ such sets. Suppose $i = 1$ and $j = 0$. There are 6 MSTD sets of this type. They are $\{a, b, ab, a^3b\}, \{a, b, ab, a^4b\}, \{a, b, a^2b, a^3b\}, \{a, b, a^2b, a^5b\}, \{a, b, a^3b, a^4b\}, \{a, b, a^3b, a^5b\}$. Similarly, if we take $i = 1$ and $j = 1$, there are 4 MSTD sets: $\{a, ab, a^2b, a^4b\}$, $\{a, ab, a^2b, a^5b\}, \{a, ab, a^3b, a^4b\}, \{a, ab, a^4b, a^5b\}$. Finally, if we consider $i = 1$ and $j = 2$, we get these 2 MSTD sets $\{a, a^2b, a^3b, a^5b\}, \{a, a^2b, a^4b, a^5b\}$. As a result, we obtain 12 MSTD sets when $i = 1$. We can similarly find 36 MSTD sets when $i = 2, 4,$ and 5. In total, there are 48 MSTD sets in this case.

Therefore, we obtained 24+54+48=126 MSTD sets of cardinality 4 from the above cases. □

We also found that there are 228 MDTS sets of cardinality 4. Hence, $A_{12,4}$ contains more MDTS sets than MSTD sets. Identifying these MDTS sets involves a case-by-case analysis similar to the approach used in the above lemma for MSTD sets. However, since the number of MDTS sets is larger, the number of cases increases significantly. To avoid unnecessary complications, we omit the detailed proof for the MDTS sets here.

Lemma 3.2 *Let $A_{12,5}$ denote the collection of subsets of* Dic_3 *of size* 5*. There are* 120 *sets which are MSTD in $A_{12,5}$.*

Proof: Let A be a subset of Dic_3 of cardinality 5. There are 6 possible cases to consider for A.

1. $\{a^i, a^j, a^k, a^l, a^m\}$,
2. $\{a^i, a^j, a^k, a^l, a^m b\}$,
3. $\{a^i, a^j, a^k, a^l b, a^m b\}$,
4. $\{a^i, a^j, a^k b, a^l b, a^m b\}$,
5. $\{a^i, a^j b, a^k b, a^l b, a^m b\}$,
6. $\{a^i b, a^j b, a^k b, a^l b, a^m b\}$,

where $0 \leq i, j, k, l, m \leq 5$. In Cases 1 and 6, we get either MDTS or balanced sets. Now we consider the cases where we find MSTD sets. They are following.

Case 1. Consider the sets of the type $A = \{a^i, a^j, a^k, a^l, a^m b\}$. There are 6 MSTD sets of the form $\{1, a, a^3, a^4, \alpha\}$, and 6 MSTD sets of the form $\{1, a^2, a^3, a^5, \alpha\}$, where $\alpha \in V$.

Case 2. Let $A = \{a^i, a^j, a^k, a^l b, a^m b\}$. There are 48 sets of this form, and all are MSTD.

Case 3. Let $A = \{a^i, a^j, a^k b, a^l b, a^m b\}$. In this case we find 48 MSTD sets.

Case 4. Let $A = \{a^i, a^j b, a^k b, a^l b, a^m b\}$. In this case, we identify 12 MSTD sets.

Therefore, with cardinality 5, we have 12+48+48+12=120 MSTD sets. □

Similarly, we found that $A_{12,5}$ contains 264 MDTS sets, indicating that $A_{12,5}$ has strictly more MDTS sets than MSTD sets. However, for $|A| = 6$, we obtain the following lemma.

Lemma 3.3 *Let $A_{12,6}$ denote the collection of subsets of* Dic_3 *of size* 6*. Then $A_{12,6}$ has strictly more MSTD sets than MDTS sets.*

Proof: Let A be a subset of Dic_3 with $|A| = 6$. There are seven possible cases to consider for A. Using a similar approach as in Theorem 3.3 and Lemma 3.1, we can identify MSTD and MDTS sets in this case as well. Below, we provide the number of MSTD and MDTS sets for each case based on this process.

1. Suppose $A = U$. In this case, we determine there are 12 MDTS sets and no MSTD set.
2. For the next case, let $A = \{a^i, a^j, a^k, a^l, a^m, a^n b\}$. In this scenario, we find that $|A + A| = |A - A|$.
3. Next, consider $A = \{a^i, a^j, a^k, a^l, a^m b, a^n b\}$. Here, we find 6 MSTD sets and 12 MDTS sets.
4. Now, let $A = \{a^i, a^j, a^k, a^l b, a^m b, a^n b\}$. In this case, we find 48 MSTD sets and no MDTS sets.
5. For $A = \{a^i, a^j, a^k b, a^l b, a^m b, a^n b\}$, we find 6 MSTD sets and 12 MDTS sets.
6. Let $A = \{a^i, a^j b, a^k b, a^l b, a^m b, a^n b\}$. In this case, we identify 12 MDTS sets and no MSTD sets.
7. Finally, for $A = V$, we find that $|A + A| = |A - A|$, indicating that A is balanced.

In total, we have 60 MSTD sets (6 + 48 + 6) and 48 MDTS sets (12+12+12+12) in Dic_3 with $|A| = 6$. □

Based on above Lemma 3.3, we can ask the following question.

Question 3.1 *Let $n \geq 3$ be an integer. Let $A_{4n,2n}$ denote the collection of subsets of Dic_n of size $2n$. Can we say that $A_{4n,2n}$ has strictly more MSTD sets than MDTS sets?*

By Theorem 2.1 if $|A| > 2n$, then A is a balanced set.

Question 3.2 *Let $n \geq 3$ be an integer. Let $A_{4n,m}$ denote the collection of subsets of Dic_n of size $2n$. What are the lower and upper bounds for m such that $A_{4n,m}$ has strictly more MDTS sets than MSTD sets?*

4. Alternating Group A_4

Consider the Symmetric group S_4 of all permutations on four symbols $1, 2, 3, 4$. The order of S_4 is 24 of which 12 are even permutations. The set of all even permutations in S_4 forms a subgroup of S_4 called the Alternating Group and is denoted by A_4. It consists of the identity permutation, eight $3-$cycles, and three double transpositions. Precisely,

$$A_4 = \{(1), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}.$$

We may partition A_4 into 3 sets as follows:

$$X = \{(123), (142), (134), (243)\}$$

$$Y = \{(132), (124), (143), (234)\}$$

$$Z = \{(1), (12)(34), (13)(24), (14)(23)\}$$

Notation. For a subset $X \subset G$, we denote $-X = \{x^{-1} : x \in G\}$.

Here, Z is a subgroup of A_4 in which every element is self-invertible. The product of any two elements of X is in Y and vice versa. For any $\alpha \in X$ and $\beta \in Y$, the products $\alpha\beta, \beta\alpha \in Z$. Further $-X = Y$ and $-Y = X$. So we have

$$X + X = Y, \quad X - X = X + Y = Z$$

$$Y + Y = X, \quad Y - Y = Y + X = Z$$

$$Z + Z = Z - Z = Z$$

Therefore, each of the sets X, Y and Z is balanced in A_4.

A_4 has 10 subgroups listed below:

$$\langle(1)\rangle, \langle(12)(34)\rangle, \langle(13)(24)\rangle, \langle(14)(23)\rangle, \langle(123)\rangle, \langle(142)\rangle, \langle(134)\rangle, \langle(243)\rangle, Z \text{ and } A_4.$$

For each subgroup H of a group G, $H + H = H - H = H$. So all of the above subgroups are balanced.

Lemma 4.1 *A set of the form $\{\alpha, \beta, \gamma, \delta\}$, where $\alpha \in X \cup Y, \beta, \gamma, \delta \in Z$ is MDTS.*

Proof: Let $S = \{\alpha, \beta, \gamma, \delta\}$, where $\alpha \in X, \beta, \gamma, \delta \in Z$.

Let $Z' = \{\beta, \gamma, \delta\}$. Now $S = Z' \cup \{\alpha\}$. We note that $Z' + Z' = Z' - Z' = Z$. So Z' is balanced. With $\alpha \to Z'$ the new sums obtained are

$$\{\alpha\beta, \alpha\gamma, \alpha\delta, \beta\alpha, \gamma\alpha, \delta\alpha\} \cup \{\alpha^2\} \subseteq X \cup \{\alpha^2\}$$

and the newly generated differences are

$$\{\alpha\beta, \alpha\gamma, \alpha\delta\} \subset X \quad \text{and} \quad \{\beta\alpha^{-1}, \gamma\alpha^{-1}, \delta\alpha^{-1}\} \subset Y.$$

So we get

$$|S + S| \leq |Z' + Z'| + 5 = 9 \text{ and } |S - S| = |Z' - Z'| + 6 = 10.$$

This implies

$$|S - S| - |S + S| \geq 1.$$

Hence, S is MDTS.

The case when $\alpha \in Y$ can be discussed similarly. □

Lemma 4.2 *A set of the form $\{\alpha, \beta, \gamma, \delta\}$, where $\alpha, \beta, \gamma \in X, \delta \in Z$ is MDTS.*

Proof: Let $S = \{\alpha, \beta, \gamma, \delta\}$, where $\alpha, \beta, \gamma \in X,\ \delta \in Z$.
Let $S' = \{\alpha, \beta, \gamma\}$. Then $S = S' \cup \{\delta\}$. For S' we get $S' + S' = Y$ and $S' - S' = Z$. So S' is balanced. If $\delta = (1)$ then with $\delta \to S'$ the 4 new sums obtained are

$$(1), \alpha, \beta, \gamma$$

and we get 6 new differences

$$\alpha, \beta, \gamma \in X \text{ and } \alpha^{-1}, \beta^{-1}, \gamma^{-1} \in Y.$$

Let $\delta \neq (1)$. With $\delta \to S'$ the new sums obtained are

$$\delta^2, \alpha\delta, \beta\delta, \gamma\delta, \delta\alpha, \delta\beta, \delta\gamma \text{ which constitute } X \cup \{(1)\}$$

and we get exactly 6 new differences

$$\alpha\delta, \beta\delta, \gamma\delta \in X \text{ and } \delta\alpha^{-1}, \delta\beta^{-1}, \delta\gamma^{-1} \in Y.$$

Therefore $|S + S| \leq |S' + S'| + 5 = 9$ and $|S - S| = |S' - S'| + 6 = 10$. Hence, S is MDTS. □

Similarly, the set $\{\alpha, \beta, \gamma, \delta\}$, where $\alpha, \beta, \gamma \in Y, \delta \in Z$ is MDTS.

Lemma 4.3 *Let $S = \{\alpha, \beta, \gamma, \delta, \rho\}$ be a subset of A_4 where $\alpha, \beta \in X, \gamma, \delta \in Y$, with $\gamma, \delta \notin \{\alpha^{-1}, \beta^{-1}\}$, $\rho \in Z$, then S is an MSTD set.*

Proof: Given $S = \{\alpha, \beta, \gamma, \delta, \rho\}$ where $\alpha, \beta \in X, \gamma, \delta \in Y$, with $\gamma, \delta \notin \{\alpha^{-1}, \beta^{-1}\}$, $\rho \in Z$. We claim that

$$S + S = A_4 \text{ and } S - S \neq A_4.$$

While computing the sums due to the elements from X and Y we observe that $\{\alpha^2, \beta^2, \alpha\beta, \beta\alpha\} = Y$ and $\{\gamma^2, \delta^2, \gamma\delta, \delta\gamma\} = X$. So we get all the eight 3−cycles in $S + S$. Sum of a 3−cycle in X and a 3−cycle in Y gives the elements of $Z \setminus \{(1)\}$ and $\rho^2 = (1)$. Hence $S + S = A_4$.
In $S - S$, elements of order 2 are obtained by taking the difference of a 3− cycle in X and a 3− cycle in Y, which are precisely: $\alpha\gamma^{-1}, \beta\sigma^{-1}, \gamma\alpha^{-1}, \sigma\beta^{-1}$. Here $\alpha\gamma^{-1} = \gamma\alpha^{-1}$ and $\beta\sigma^{-1} = \sigma\beta^{-1}$. Therefore $\{\alpha\gamma^{-1}, \beta\sigma^{-1}\} \subset \{(12)(34), (13)(24), (14)(23)\}$. This shows that there are only two elements of order 2 in $S - S$. So $S - S \neq A_4$. Hence S is an MSTD set. □

Lemma 4.4 *Sets of the form $X \cup \{\alpha\}$ or $Y \cup \{\alpha\}$, where $\alpha \in Z$ is always MDTS.*

Proof: Let $S = X \cup \{\alpha\}$, where $\alpha \in Z$. Now

$$S + S = (X + X) \cup (X + \alpha) \cup (\alpha + X) \cup \{(1)\} = Y \cup X \cup \{(1)\}$$

and

$$S - S = (X - X) \cup (X - \alpha) \cup (\alpha - X) = Z \cup X \cup Y = A_4.$$

So $|S + S| = 9$ and $|S - S| = 12$. Similarly, for $T = Y \cup \{\alpha\}$, where $\alpha \in Z$, we get $|T + T| = 9$ and $|T - T| = 12$. Therefore S is MDTS. □

We now study subsets of A_4 with various cardinalities. As singleton sets are always balanced, we start with sets of cardinality 2.

Theorem 4.1 *Let $\mathcal{A}_{12,2}$ denote the collection of subsets of A_4 of size 2. Then $\mathcal{A}_{12,2}$ has strictly more MSTD sets than MDTS sets.*

Proof: Let A be a subset of A_4 with $|A| = 2$. If A contains the sets of type $\{\alpha, \beta\}$, where $\alpha, \beta \in Z$ then A is balanced by Corollary 2.1 and there are $\binom{4}{2} = 6$ sets of this type. Similarly, by using Corollary 2.1, we can observe that sets of the form $\{\alpha, \alpha^{-1}\}$, where $\alpha \in X$ with 4 sets and $\{1, \alpha\}$, where $\alpha \in X \cup Y$ with 8 sets are balanced. Therefore, we have total $6 + 4 + 8 = 18$ balanced sets. Next, we count MSTD sets.

1. If A contains sets of the form α, β, where $\alpha, \beta \in X \cup Y$ and $\beta \neq \alpha^{-1}$, then we get

$$A + A = \{\alpha^2, \beta^2, \alpha\beta, \beta\alpha\},$$
$$A - A = \{1, \alpha\beta^{-1}, \beta\alpha^{-1}\}.$$

   This gives $|A + A| = 4 > 3 = |A - A|$. Since sets of the form $\{\alpha, \alpha^{-1}\}$ are balanced so we are left with $\binom{8}{2} - 4 = 24$ MSTD sets.

2. If A contains sets of the form $\{\alpha, \beta\}$, where $\alpha \in X \cup Y$ and $\beta \in Z$, then $|A+A| = 4$ qnd $|A-A| = 3$. Thus, these are also MSTD sets, contributing another 24 sets.

Therefore, we have 48 MSTD sets and 18 balanced sets of size 2. This implies there are no MDTS subsets of size 2. □

Using the above theorem, we can conjecture the following.

Conjecture 4.1 *Let $\mathcal{A}_{n!/2,2}$ denote the collection of subsets of A_n of size 2. Then for all $n \geq 4$, the number of MSTD subsets in $\mathcal{A}_{n!/2,2}$ are strictly greater than the number of MDTS subsets.*

Theorem 4.2 *Let $\mathcal{A}_{12,3}$ denote the collection of subsets of A_4 of size 3. Then $\mathcal{A}_{12,3}$ has strictly more MSTD sets than MDTS sets.*

Proof: Let A be a subset of A_4 with cardinality 3. By the corollary 2.1, if A contains each of the following sets, then A is balanced.

1. $\{\alpha, \beta, \gamma\} \subseteq Z$. There are ${}^4C_3 = 4$ such sets.

2. $\{\alpha, \beta, \beta^{-1}\}$, where $\alpha \in Z, \beta \in X$. There are 16 such sets.

If A contains sets of the form $\{\alpha, \beta, \gamma\}$, where $\alpha \in X \cup Y$, β and $\gamma \in Z$ then $|A+A| = |A-A|$ and hence A is always balanced. We have 48 such sets. If A contains $\{\alpha, \beta, \gamma\} \subseteq X$ or $\{\alpha, \beta, \gamma\} \subseteq Y$ then also we get balanced sets and there are 8 sets of this type. The set $\{\alpha, \beta, \gamma\}$, where $\alpha \in X, \beta \in Y$ with $\beta \neq \alpha^{-1}$ and $\gamma \in Z$, is balanced. There are 36 sets of this form. Next, we count MSTD sets of size 3.

Consider the sets of the form $A = \{(1), \alpha, \beta\}$, where $\alpha, \beta \in X$ or $\alpha, \beta \in Y$. If $\alpha, \beta \in X$ then $\alpha^{-1}, \beta^{-1} \in Y$. So $\alpha\beta^{-1}, \beta\alpha^{-1} \in Z$, which implies $\alpha\beta^{-1} = \beta\alpha^{-1}$. This reduces the cardinality of the difference set over that of the sumset. Therefore, the set is MSTD. Similarly, when $\alpha, \beta \in Y$, we get S to be an MSTD set. There are 12 such sets. If A contains sets of the form $\{\alpha, \beta, \gamma\}$, where $\alpha, \beta \in X \cup Y, \gamma \in Z \setminus \{(1)\}$, with $\beta \neq \alpha^{-1}$, then A is MSTD and there are 48 sets of this form. Similarly, if $A = \{\alpha, \beta, \gamma\} \subseteq X \cup Y$ with at least one element each from X and Y is MSTD and we have 48 such sets. Thus, with cardinality 3, we have 112 balanced sets and 108 MSTD sets. So there are no MDTS sets of cardinality 3. □

A similar type of analysis we have done for sizes 4, 5 and 6 also and we found that A_4 has More MSTD sets than MDTS sets. Therefore, in general, we can ask the following question.

Question 4.1 *Let $\mathcal{A}_{n!/2,m}$ denote the collection of subsets of A_n of size n. Can we say $\mathcal{A}_{n!/2,m}$ has strictly more MSTD sets than MDTS sets?*

5. Dihedral Group D_6

This group, usually denoted D_6, is the dihedral group of order 12. In other words, it is the dihedral group of degree six, i.e., the group of symmetries of a regular hexagon. It is the direct product of the symmetric group of degree three and the cyclic group of order two. The usual presentation is: $D_6 = \langle r, s \mid r^6 = s^2 = (sr)^2 = 1\rangle$.
Given a set $A \subseteq D_6$, define R(resp.F) as the set of elements of A of the form r^i (resp. $r^i s$), called rotation elements (resp. flip elements). Hence, $A = R \cup F$. Then, we can write

$$A + A = (R + R) \cup (F + F) \cup (R + F) \cup (-R + F),$$

$$A - A = (R - R) \cup (F + F) \cup (R + F).$$

Here elements in $F + F$ are same as in $F - F$ because of flips ordered 2. It is shown by Ascoli et al. [2] that D_6 has no MDTS sets of size 2, 24 MSTD sets and 42 Balanced sets. They also proved that if A is a subset of D_6 with 3 elements, then A has more MSTD sets than MDTS sets. Miller and Vissuet, who studied first the dihedral groups in [10], conjectured that for $n \geq 3$, D_{2n} has more MSTD subsets than MDTS subsets. Recently, Ascoli et al. in [2] made progress towards this conjecture by partitioning subsets of D_{2n} by their size. They conjectured the following.

Conjecture 5.1 ([2]) *Let G be an abelian group with at least one element of order 3 or greater, and let $D = \mathbb{Z}_2 \ltimes G$ be the corresponding generalized dihedral group. Then, there are more MSTD subsets of D than MDTS subsets of D.*

They also proved the following:

Lemma 5.1 ([2]) *Let $n \geq 3$ be an integer and let $\mathcal{S}_{2n,2}$ denote the collection of subsets of D_{2n} of size 2. Then $\mathcal{S}_{2n,2}$ has strictly more MSTD sets than MDTS sets.*

Lemma 5.2 ([2]) *Let $n \geq 3$ be an integer and let $\mathcal{S}_{2n,3}$ denote the collection of subsets of D_{2n} of size 3. Then, $\mathcal{S}_{2n,3}$ has strictly more MSTD sets than MDTS sets.*

Theorem 5.1 ([2]) *Let $D = \mathbb{Z}_2 \ltimes G$ be a generalized dihedral group of size $2n$. Let $S_{D,m}$ denote the collection of all subsets of D of size m, and let j denote the number of elements in G with the order at most 2. If $6 \leq m \leq c_j\sqrt{n}$, where $c_j = 1.3229\sqrt{111 + 5j}$, then there are more MSTD sets than MDTS sets in $S_{D,m}$.*

6. Concluding Remarks and Future Problems

In this paper, we explored and compared MSTD (More Sums Than Differences) and MDTS (More Differences Than Sums) sets within groups of order 12. We have provided various explicit constructions of sum-dominant, difference-dominant, and balanced sets in groups of order 12. We found that non-abelian groups A_4 and D_6 have more MSTD sets and fewer MDTS sets. In contrast, the abelian groups and the dicyclic group Dic_3 contain more MDTS sets and fewer MSTD sets. One can generalize these results to non-abelian groups of higher order. In fact one can prove that A_n has for MSTD sets than MDTS sets for $n \geq 3$, similarly for D_n and Dic_n. Some first steps in considering analogous questions have recently been taken by Neetu and Shankar [12] and Ascoli et al. [2] for generalized quaternion group Q_{4n} and dihedral group D_{2n}, respectively. It can also be tried in alternating groups.

There has been a lot of interest in finding sets A with large values of the ratio $ln|A+A|/ln|A-A|$. In [1,7,13] authors addressed the issue of finding finite sets $A \subseteq \mathbb{Z}$ for which $f(A) = ln(|A+A|)/ln(|A-A|)$ is large, obtaining a new record high value of this function. The current highest is about 1.03059 found by Penman and Wells in [13]. It was not immediately obvious whether it would be easier or harder to find large values of the functions analogous to f when A is taken from a finite group rather than the integers: Penman and Wells in [14] obtained a new high value for a subset of the finite abelian group and the value is about 1.041334216. We tried to look at the function f in the case of finite non-abelian groups. For the set $A = \{(123), (124), (12)(34)\}$ in A_4, $|A + A| = 9$, and $|A - A| = 3$ then $f(A) = ln(9)/ln(3) = 2$. This is higher than the previous results.

Acknowledgements

The first author expresses sincere gratitude for the financial support provided by the National Institute of Technology Karnataka (NITK), Surathkal.

References

1. Asada, M., Manski, S., Miller, S. and Suh, H., *Fringe pairs in generalized sum-dominant sets*, Int. J. Number Theory 13, 2653–2675, (2017).
2. Ascoli, R., Cheigh, J., Dantas e Moura, G. Z., Jeong, R., Keisling, A., Lilly, A., Miller, S., Ngamlamai, P. and Phang, M., *Sum and difference sets in generalized dihedral groups*, arXiv preprint, arXiv:2210.00669, (2022).
3. Chu, H., Luntzlara, N., Miller, S. and Shao, L., *Generalizations of a curious family of sum-dominant sets hidden by interior blocks*, Integers 20A, (2020).
4. Chu, H., King, D., Luntzlara, N., Martinez, T., Miller, S., Shao, L., Sun, C. and Xu, V., *Generalizing the distribution of missing sums in sumsets*, J. Number Theory 239, 402–444, (2022).
5. Do, T., Kulkarni, A., Miller, S., Moon, D. and Wellens, J., *Sums and differences of correlated random sets*, J. Number Theory 147, 44–68, (2015).
6. Freiman, G. and Pigarev, V., *The relation between the invariants R and T*, Kalinin. Gos. Univ. Moscow (Russian), 172–174, (1973).
7. Hegarty, P., *Some explicit constructions of sets with more sums than differences*, Acta Arith. 130, 61–77, (2007).
8. Hemmady, A., Lott, A. and Miller, S., *When almost all sets are difference-dominated in $\mathbb{Z}_n$*, Integers 15, A54, (2020).
9. Marica, J., *On a conjecture of Conway*, Canad. Math. Bull. 12, 233–234, (1969).
10. Miller, S. and Vissuet, K., *Most subsets are balanced in finite groups*, in Combinatorial and Additive Number Theory, 101, 147–157, (2014).
11. Nathanson, M., *Sets with more sums than differences*, Integers: Electron. J. Combin. Number Theory 7, A5 (24pp), (2007).
12. Neetu and Shankar, B. R., *Sum and difference sets in generalized quaternion groups*, Proc. Jangjeon Math. Soc. 27, no. 4, 773–780, (2024).
13. Penman, D. and Wells, M., *On sets with more restricted sums than differences*, Integers 13, A57, (2013).
14. Penman, D. and Wells, M., *Sum-dominant sets and restricted-sum-dominant sets in finite abelian groups*, Acta Arith. 165, 361–383, (2014).
15. Zduńczyk, R., *Sets with small differences and big sums in finite groups*, in Selected Problems on Experimental Mathematics, 672, 149–166, (2017).
16. Zhao, Y., *Counting sum-dominant sets in finite abelian groups*, J. Number Theory 130, 2308–2322, (2010).

Neetu and B. R. Shankar,
Department of Mathematical and Computational Sciences,
National Institute of Technology Surathkal,
Karnataka, 575025, India.
E-mail address: neetu.207ma004@nitk.edu.in, shankarbr@nitk.edu.in

and

Anupriya Shetty,
Department of Mathematics,
St Aloysius (Deemed to be University),
Mangaluru, Karnataka, 575003, India.
E-mail address: anupriyashetty.207ma500@nitk.edu.in