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Existence of Unique Stable Solutions for a system of the BVPs Involving the Caputo-Type Modification of Erdélyi-Kober Fractional Derivative

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ABSTRACT: In this article, we investigate a coupled system of fractional differential equations involving the Caputo-type modification of the Erdélyi-Kober fractional derivative. Using the Banach contraction principle, we establish the uniqueness of solutions. Furthermore, under appropriate assumptions, we apply the Leray-Schauder alternative fixed point theorem to prove the existence of at least one solution for the system. Additionally, we derive several results concerning Hyers-Ulam (H-U) stability. Finally, we provide an illustrative example to validate our findings.

Key Words: Existence theory, Erdélyi-Kober fractional derivative, analysis of stability, fixed point.

Contents

1. Introduction

Fractional differential equations have garnered significant attention from mathematicians and physicists due to their remarkable ability to model a wide range of real-world dynamical systems encountered in engineering, network vulnerability, and various applied fields. For instance, studies such as those in [1,2,3,10,14,15] highlight their practical relevance. The existence of solutions to differential equations under different conditions and through various methodologies has also been a focal point of investigation, with notable contributions found in [12]. In [9], the authors examined specific properties of the Caputo-type variation of the Erdélyi-Kober fractional derivative. Further insights and developments in this area can be explored in references such as [4,6,7].

In [5], Benchohra et al. studied the following terminal value problem:

$$\begin{cases} \left({}^{\rho}D_{\kappa_1}^{\delta_1,\delta_2}v\right)(\epsilon) = \mathcal{G}\left(\epsilon, v(\epsilon), \left({}^{\rho}D_{\kappa_1}^{\delta_1,\delta_2}v\right)(\epsilon)\right), \quad \epsilon \in \Lambda = [\kappa_1, \kappa_2], \\ v(\kappa_1) = c \in \mathbb{R}, \end{cases}$$

where ${}^{\rho}D_{\kappa_1}^{\delta_1,\delta_2}$ is Hilfer-Katugampola fractional derivative of order $\delta_1 \in (0,1)$ and type $\delta_2 \in [0,1]$, and $\mathcal{G} : (\kappa_1, \kappa_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function. Their analysis primarily relies on classical fixed point theorems, particularly the Banach contraction principle and Krasnoselskii's fixed point theorem.

In [8], the authors investigated the existence and uniqueness of solutions for a class of boundary value problems for nonlinear implicit fractional differential equations in Banach space:

$$\begin{cases} {}^{\rho}_{c}D_{\kappa_1}^{\delta}v(\epsilon) = \mathcal{G}\left(\epsilon, v(\epsilon), {}^{\rho}_{c}D_{\kappa_1}^{\delta}v(\epsilon)\right), \quad \epsilon \in \Lambda = [\kappa_1, \kappa_2], \\ \beta v(\kappa_1) + \varrho v(\kappa_2) = \sigma v(\eta), \end{cases}$$

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where $\beta, \varrho, \sigma \in \mathbb{R}$, $\beta+\varrho \neq \sigma$, $\eta \in \Lambda$, ${}^{\rho}_{c}D^{\delta}_{\kappa_1}$ is the Caputo-type modification of the Erdélyi-Kober fractional derivative, and $\mathcal{G} : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$. The primary results are demonstrated by combining the fixed point theorems of Darbo and Mönch with the approach of measures of noncompactness.

In [16], Ullah et al. studied a coupled system with non-homogeneous boundary conditions using the Caputo-Fabrizio-Hausdorff fractal fractional derivative, as follows:

$$\begin{cases} {}^{HFD}D^{\delta_1,\delta_2}v(\epsilon) = \mathcal{G}_1\left(\epsilon, v(\epsilon), u(\epsilon)\right), \quad \epsilon \in \Lambda = [0,T], \\ {}^{HFD}D^{\delta_1,\delta_2}u(\epsilon) = \mathcal{G}_2\left(\epsilon, v(\epsilon), u(\epsilon)\right), \\ \beta_1 v(0) + \varrho_1 v(T) = g_1(v), \\ \beta_2 u(0) + \varrho_2 u(T) = g_2(u), \end{cases}$$

where $\delta_1, \delta_2 \in (0,1]$, $\mathcal{G}_1, \mathcal{G}_2 : \Lambda \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g_1, g_2 : \Lambda \longrightarrow \mathbb{R}$, are continuous functions. To establish the necessary conditions for the existence and uniqueness of the solution to the considered problem, the authors applied Banach's and Krasnoselskii's fixed-point theorems. Furthermore, some results related to Hyers-Ulam (H-U) stability were also deduced.

Inspired by earlier studies, we examine a coupled system under the Caputo type modification of the Erdélyi-Kober fractional derivative, as follows.

$$\begin{cases} {}^{\rho}_{c}D^{\delta}_{\kappa_1}v(\epsilon) = \mathcal{G}_1\left(\epsilon, v(\epsilon), u(\epsilon)\right), \quad \epsilon \in \Lambda = [\kappa_1, \kappa_2] \\ {}^{\rho}_{c}D^{\delta}_{\kappa_1}u(\epsilon) = \mathcal{G}_2\left(\epsilon, v(\epsilon), u(\epsilon)\right), \\ \beta_1 v(\kappa_1) + \varrho_1 v(\kappa_2) = \sigma_1 v(\eta_1), \\ \beta_2 u(\kappa_1) + \varrho_2 u(\kappa_2) = \sigma_2 u(\eta_2), \end{cases} \tag{1.1}$$

where $\beta_1, \beta_2, \varrho_1, \varrho_2, \sigma_1, \sigma_2 \in \mathbb{R}$, $\beta_1 + \varrho_1 \neq \sigma_1$, $\beta_2 + \varrho_2 \neq \sigma_2$, $\eta_1, \eta_2 \in [\kappa_1, \kappa_2]$, ${}^{\rho}_{c}D^{\delta}_{\kappa_1}$ is the Caputo type modification of the Erdélyi-Kober fractional derivative with $\delta \in (0,1)$, and $\mathcal{G}_1, \mathcal{G}_2 : \Lambda \times \mathbb{R}^2 \to \mathbb{R}$.

This paper is structured as follows. In Section 2, we provide preliminary notes, review fundamental concepts related to the Erdélyi-Kober fractional derivative, and establish key auxiliary results. Section 3, the main findings concerning existence theory have been established. Stability results are presented in Section 4. Finally, in the last section, we demonstrate the application of our main findings through an illustrative example.

2. Preliminaries

Definition 2.1 *[13] Let $\delta \in \mathbb{R}$, the generalized fractional integral of order δ of a function $\mathcal{F}$ is given by*

$${}^{\rho}I^{\delta}_{\kappa_1}\mathcal{F}(\epsilon) = \frac{\rho^{1-\delta}}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}(\epsilon^{\rho} - \vartheta^{\rho})^{\delta-1}\mathcal{F}(\vartheta)\, d\vartheta.$$

Definition 2.2 *[11] Let $\delta \in \mathbb{R}$, $\alpha = [\delta] + 1$ and $\rho > 0$. The generalized fractional derivative is defined for $0 \leq \kappa_1 < \epsilon$ as follows:*

$$\begin{aligned} {}^{\rho}D^{\delta}_{\kappa_1}\mathcal{F}(\epsilon) &= \frac{\rho^{1-\alpha+\delta}}{\Gamma(\alpha-\delta)}\big(\epsilon^{1-\rho}\frac{d}{d\epsilon}\big)^{\alpha} \int_{\kappa_1}^{\epsilon} \frac{\vartheta^{\rho-1}}{(\epsilon^{\rho} - \vartheta^{\rho})^{1-\alpha+\delta}}\mathcal{F}(\vartheta)\, d\vartheta \\ &= \nu^{\alpha}_{\rho}\, {}^{\rho}I^{\alpha-\delta}_{\kappa_1}\mathcal{F}(\epsilon), \end{aligned}$$

where $\nu^{\alpha}_{\rho} = \big(\epsilon^{1-\rho}\frac{d}{d\epsilon}\big)^{\alpha}$.

Definition 2.3 *[11] The Caputo-type generalized fractional derivative ${}^{\rho}_{c}D^{\delta}_{\kappa_1}$ is expressed in the following form:*

$${}^{\rho}_{c}D^{\delta}_{\kappa_1}\mathcal{F}(\epsilon) = {}^{\rho}D^{\delta}_{\kappa_1}\big[\mathcal{F}(\epsilon) - \sum_{j=0}^{\alpha-1}\frac{\mathcal{F}^{(j)}(\kappa_1)}{j!}(\epsilon - \kappa_1)^{j}\big]. \tag{2.1}$$

Lemma 2.1 *Let $\rho, \delta \in \mathbb{R}^+$, then*

$$\left({}^{\rho}I^{\delta}_{\kappa_1}\,{}^{\rho}_{c}D^{\delta}_{\kappa_1}\mathcal{F}\right)(\epsilon) = \mathcal{F}(\epsilon) - \sum_{j=0}^{\alpha-1} \gamma_j \Big(\frac{\epsilon^{\rho} - \kappa_1^{\rho}}{\rho}\Big)^{j},$$

for some $\gamma_j \in \mathbb{R}$, $\alpha = [\delta] + 1$.

3. Findings related to existence theory

Lemma 3.1 *If $\epsilon \in \Lambda$, $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{L}(\Lambda)$, $\delta \in (0, 1]$, then the solution of*

$$\begin{cases} {}^{\rho}_{c}D^{\delta}_{\kappa_1} v(\epsilon) = \mathcal{G}_1(\epsilon), \\ {}^{\rho}_{c}D^{\delta}_{\kappa_1} u(\epsilon) = \mathcal{G}_2(\epsilon), \\ \beta_1 v(\kappa_1) + \varrho_1 v(\kappa_2) = \sigma_1 v(\eta_1), \\ \beta_2 u(\kappa_1) + \varrho_2 u(\kappa_2) = \sigma_2 u(\eta_2), \end{cases} \tag{3.1}$$

is given by

$$\begin{cases} v(\epsilon) = \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1} \left(\frac{\epsilon^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta \\ \qquad - \frac{\varrho_1}{\beta_1 + \varrho_1 - \sigma_1} \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1} \left(\frac{\kappa_2^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta \\ \qquad + \frac{\sigma_1}{\beta_1 + \varrho_1 - \sigma_1} \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1} \left(\frac{\eta_1^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta, \\ u(\epsilon) = \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1} \left(\frac{\epsilon^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta \\ \qquad - \frac{\varrho_2}{\beta_2 + \varrho_2 - \sigma_2} \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1} \left(\frac{\kappa_2^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta \\ \qquad + \frac{\sigma_2}{\beta_2 + \varrho_2 - \sigma_2} \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\eta_2} \vartheta^{\rho-1} \left(\frac{\eta_2^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta. \end{cases} \tag{3.2}$$

Proof 3.1 *By using the integration of formula (3.1), we obtain*

$$\begin{cases} v(\epsilon) = \alpha_1 + \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1} \left(\frac{\epsilon^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta, \\ u(\epsilon) = \alpha_2 + \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1} \left(\frac{\epsilon^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta. \end{cases} \tag{3.3}$$

We use the conditions

$$\beta_1 v(\kappa_1) + \varrho_1 v(\kappa_2) = \sigma_1 v(\eta_1) \ \text{and} \ \beta_2 u(\kappa_1) + \varrho_2 u(\kappa_2) = \sigma_2 u(\eta_2),$$

to compute the constants α_1 and α_2. Thus, we obtain

$$\begin{cases} \beta_1 v(\kappa_1) = \beta_1 \alpha_1, \\ \beta_2 u(\kappa_1) = \beta_2 \alpha_2, \end{cases}$$

and

$$\begin{cases} \varrho_1 v(\kappa_2) = \varrho_1 \alpha_1 + \frac{\varrho_1}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1} \left(\frac{\kappa_2^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta, \\ \varrho_2 u(\kappa_2) = \varrho_2 \alpha_2 + \frac{\varrho_2}{\Gamma(\delta)} \int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1} \left(\frac{\kappa_2^{\rho} - \vartheta^{\rho}}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta, \end{cases}$$

and we have,

$$\begin{cases} \sigma_1 v(\eta_1) = \sigma_1\alpha_1 + \frac{\sigma_1}{\Gamma(\delta)} \displaystyle\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta, \\ \sigma_2 u(\epsilon) = \sigma_2\alpha_2 + \frac{\sigma_2}{\Gamma(\delta)} \displaystyle\int_{\kappa_1}^{\eta_2} \vartheta^{\rho-1}\left(\frac{\eta_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta, \end{cases}$$

with

$$\beta_1 v(\kappa_1) + \varrho_1 v(\kappa_2) = \sigma_1 v(\eta_1) \text{ and } \beta_2 u(\kappa_1) + \varrho_2 u(\kappa_2) = \sigma_2 u(\eta_2).$$

Then,

$$\begin{aligned} \alpha_1 = &-\frac{\varrho_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta \\ &+\frac{\sigma_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta)\, d\vartheta, \\ \alpha_2 = &-\frac{\varrho_2}{\beta_2+\varrho_2-\sigma_2}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta \\ &+\frac{\sigma_2}{\beta_2+\varrho_2-\sigma_2}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_2} \vartheta^{\rho-1}\left(\frac{\eta_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta)\, d\vartheta. \end{aligned}$$

We substitute the values of α_1 and α_2 into (3.3) to obtain (3.2).

Corollary 3.1 *In view of Lemma 3.1, the system (1.1) is equivalent to*

$$\begin{cases} v(\epsilon) = \frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta \\ \quad -\frac{\varrho_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta \\ \quad +\frac{\sigma_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta, \\ u(\epsilon) = \frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta \\ \quad -\frac{\varrho_2}{\beta_2+\varrho_2-\sigma_2}\frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta \\ \quad +\frac{\sigma_2}{\beta_2+\varrho_2-\sigma_2}\frac{1}{\Gamma(\delta)}\displaystyle\int_{\kappa_1}^{\eta_2} \vartheta^{\rho-1}\left(\frac{\eta_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} \mathcal{G}_2(\vartheta, v(\vartheta), u(\vartheta))\, d\vartheta. \end{cases} \tag{3.4}$$

Let $\Omega_1 = \mathcal{C}(\Lambda)$ and $\Omega_2 = \mathcal{C}(\Lambda)$ be two Banach spaces with norms defined by $\|v\| = \max_{\epsilon\in\Lambda}|v(\epsilon)|$, $\|u\| = \max_{\epsilon\in\Lambda}|u(\epsilon)|$ respectively. Then, $\Omega_1 \times \Omega_2 = \Omega$ under the norm $\|(v,u)\| = \|v\| + \|u\|$ is also a Banach space.

Let $\Upsilon : \Omega \to \Omega$ be an operator defined as follows:

$$\begin{aligned} \Upsilon(v,u) &= (\Upsilon_1(v,u), \Upsilon_2(v,u)) \\ &= (v,u), \end{aligned} \tag{3.5}$$

such that

$$\begin{cases} \Upsilon_1(v,u) = v(t), \\ \Upsilon_2(v,u)) = u(t). \end{cases} \tag{3.6}$$

The following hypothesis hold:

$\mathcal{H}(1)$ If there exist constants $\mathbb{K}_{\mathcal{G}_1}, \mathbb{K}_{\mathcal{G}_2} > 0$ and (v, u) ,$(\bar{v}, \bar{u}) \in \Omega$, we have

$$|\mathcal{G}_1(\vartheta, v, u) - \mathcal{G}_1(\vartheta, \bar{v}, \bar{u})| \leq \mathbb{K}_{\mathcal{G}_1}[|v - \bar{v}| + |u - \bar{u}|],$$

and

$$|\mathcal{G}_2(\vartheta, v, u) - \mathcal{G}_2(\vartheta, \bar{v}, \bar{u})| \leq \mathbb{K}_{\mathcal{G}_2}[|v - \bar{v}| + |u - \bar{u}|].$$

$\mathcal{H}(2)$ Let $B \subset \Omega$ be a bounded set, then there exist $\ell_i > 0$, $i = 1, 2$ such that

$$\begin{aligned} &|\mathcal{G}_1(\epsilon, v, u)| \leq \ell_1, \\ &|\mathcal{G}_2(\epsilon, v, u)| \leq \ell_2, \quad \forall \epsilon \in \Lambda, \quad \forall v, u \in B. \end{aligned}$$

$\mathcal{H}(3)$ There exist $\varphi_0, \varpi_0 > 0$, and $\varphi_i, \varpi_i \leq 0$, $i = 1, 2$ such that

$$\begin{aligned} &|\mathcal{G}_1(\epsilon, v(\epsilon), u(\epsilon))| \leq \varphi_0 + \varphi_1|v| + \varphi_2|u|, \\ &|\mathcal{G}_2(\epsilon, v(\epsilon), u(\epsilon))| \leq \varpi_0 + \varpi_1|v| + \varpi_2|u|, \quad \forall \epsilon \in \Lambda, \quad \forall v, u \in \mathbb{R}. \end{aligned}$$

Note: To simplify, we utilize

$$\aleph_\epsilon = \left(\frac{\epsilon^\rho - \kappa_1^\rho}{\rho}\right)^\delta, \quad \aleph_\kappa = \left(\frac{\kappa_2^\rho - \kappa_1^\rho}{\rho}\right)^\delta, \quad \aleph_{\eta_1} = \left(\frac{\eta_1^\rho - \kappa_1^\rho}{\rho}\right)^\delta, \quad \aleph_{\eta_2} = \left(\frac{\eta_2^\rho - \kappa_1^\rho}{\rho}\right)^\delta.$$

And

$$\xi = \xi_1 + \xi_2, \tag{3.7}$$

with

$$\begin{aligned} \xi_1 &= \frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_\epsilon + \frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_\kappa \\ &+ \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_{\eta_1}, \end{aligned}$$

and

$$\begin{aligned} \xi_2 &= \frac{\mathbb{K}_{\mathcal{G}_2}}{\Gamma(\delta+1)}\aleph_\epsilon + \frac{|\varrho_2|}{|\beta_2 + \varrho_2 - \sigma_2|}\frac{\mathbb{K}_{\mathcal{G}_2}}{\Gamma(\delta+1)}\aleph_\kappa \\ &+ \frac{|\sigma_2|}{|\beta_2 + \varrho_2 - \sigma_2|}\frac{\mathbb{K}_{\mathcal{G}_2}}{\Gamma(\delta+1)}\aleph_{\eta_2}. \end{aligned}$$

Theorem 3.1 *Under assumption $\mathcal{H}(1)$, and if $\xi < 1$ (where ξ is defined in (3.7)), the system (1.1) admits a unique solution.*

Proof 3.2 *Considering (v, u) and $(\bar{v}, \bar{u}) \in \Omega$, we obtain from system (3.6) that:*

$$\begin{aligned} &\|\Upsilon_1(v, u) - \Upsilon_1(\bar{v}, \bar{u})\| = |v(t) - \bar{v}(t)| \\ \leq\ & \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} |\mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta)) - \mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))|\, d\vartheta \\ +\ & \frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} |\mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta)) - \mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))|\, d\vartheta \\ +\ & \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1} |\mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta)) - \mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))|\, d\vartheta \\ \leq\ & \frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_\epsilon[|v - \bar{v}| + |u - \bar{u}|] \\ +\ & \frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_\kappa[|v - \bar{v}| + |u - \bar{u}|] \\ +\ & \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{\mathbb{K}_{\mathcal{G}_1}}{\Gamma(\delta+1)}\aleph_{\eta_1}[|v - \bar{v}| + |u - \bar{u}|]. \end{aligned}$$

Then,

$$\|\Upsilon_1(v,u)-\Upsilon_1(\bar{v},\bar{u})\| \leq \xi_1\left[\|v-\bar{v}\|+\|u-\bar{u}\|\right]. \tag{3.8}$$

Similarly, we have,

$$\|\Upsilon_2(v,u)-\Upsilon_2(\bar{v},\bar{u})\| \leq \xi_2\left[\|v-\bar{v}\|+\|u-\bar{u}\|\right]. \tag{3.9}$$

On adding Equations (3.8) and (3.9), we have

$$\|\Upsilon_1(v,u)-\Upsilon_1(\bar{v},\bar{u})\|+\|\Upsilon_2(v,u)-\Upsilon_2(\bar{v},\bar{u})\| \leq \xi\left[\|v-\bar{v}\|+\|u-\bar{u}\|\right].$$

Pose

$$\Re_{\eta_1}=\left(\aleph_\epsilon+\frac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_\kappa+\frac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_{\eta_1}\right),$$

and

$$\Re_{\eta_2}=\left(\aleph_\epsilon+\frac{|\varrho_2|}{|\beta_2+\varrho_2-\sigma_2|}\aleph_\kappa+\frac{|\sigma_2|}{|\beta_2+\varrho_2-\sigma_2|}\aleph_{\eta_2}\right).$$

Theorem 3.2 *If $\mathcal{H}(2)$ and $\mathcal{H}(3)$ are satisfied, and if*

$$\varphi_1\Re_{\eta_1}+\varpi_1\Re_{\eta_2}<1,$$

$$\varphi_2\Re_{\eta_1}+\varpi_2\Re_{\eta_2}<1,$$

then, the fractional problem given by (1.1) has at least one solution on Λ.

Proof 3.3 *We prove that Υ is continuous and compact. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are continuous, which implies that Υ is continuous.*
By $\mathcal{H}(2)$, we have

$$\begin{aligned}
|\Upsilon_1(v,u)| \quad &\leq \tfrac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon}\vartheta^{\rho-1}\left(\frac{\epsilon^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\mathcal{G}_1(\vartheta,v,u)|\,d\vartheta\\
&+\tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2}\vartheta^{\rho-1}\left(\tfrac{\kappa_2^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\mathcal{G}_1(\vartheta,v,u)|\,d\vartheta\\
&+\tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1}\vartheta^{\rho-1}\left(\tfrac{\eta_1^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\mathcal{G}_1(\vartheta,v,u)|\,d\vartheta\\
&\leq \tfrac{\ell_1}{\Gamma(\delta+1)}\aleph_\epsilon+\tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{\ell_1}{\Gamma(\delta+1)}\aleph_\kappa\\
&+\tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{\ell_1}{\Gamma(\delta+1)}\aleph_{\eta_1},
\end{aligned}$$

similarly,

$$\begin{aligned}
|\Upsilon_2(v,u)| \quad &\leq \tfrac{\ell_2}{\Gamma(\delta+1)}\aleph_\epsilon+\tfrac{|\varrho_2|}{|\beta_2+\varrho_2-\sigma_2|}\tfrac{\ell_2}{\Gamma(\delta+1)}\aleph_\kappa\\
&+\tfrac{|\sigma_2|}{|\beta_2+\varrho_2-\sigma_2|}\tfrac{\ell_2}{\Gamma(\delta+1)}\aleph_{\eta_2}.
\end{aligned}$$

On addition, one has

$$\begin{aligned}
\|\Upsilon_1(v,u)\|+\|\Upsilon_2(v,u)\| \quad &\leq \tfrac{\ell_1}{\Gamma(\delta+1)}\left[\aleph_\epsilon+\tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_\kappa+\tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_{\eta_1}\right]\\
&+\tfrac{\ell_2}{\Gamma(\delta+1)}\left[\aleph_\epsilon+\tfrac{|\varrho_2|}{|\beta_2+\varrho_2-\sigma_2|}\aleph_\kappa+\tfrac{|\sigma_2|}{|\beta_2+\varrho_2-\sigma_2|}\aleph_{\eta_2}\right].
\end{aligned}$$

Then,

$$\begin{aligned}
\|\Upsilon(v,u)\| \quad &\leq \tfrac{(\ell_1+\ell_2)\aleph_\epsilon}{\Gamma(\delta+1)}+\left[\tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}+\tfrac{|\varrho_2|}{|\beta_2+\varrho_2-\sigma_2|}\right]\aleph_\kappa+\tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_{\eta_1}\\
&+\tfrac{|\sigma_2|}{|\beta_2+\varrho_2-\sigma_2|}\aleph_{\eta_2}.
\end{aligned}$$

Then, The operator Υ is uniformly bounded.
Next, we prove that Υ is equicontinuous, let $\epsilon_1, \epsilon_2 \in \Lambda$ with $\epsilon_1 < \epsilon_2$, $i = 1, 2$, we have

$$\begin{aligned}
|\Upsilon_1(v,u)(\epsilon_2) - \Upsilon_1(v,u)(\epsilon_1)| &\leq \frac{1}{\Gamma(\delta)} \int_{\kappa_1}^{\epsilon_1} \vartheta^{\rho-1} \left[\left(\frac{\epsilon_2^\rho - \vartheta^\rho}{\rho} \right)^{\delta-1} - \left(\frac{\epsilon_1^\rho - \vartheta^\rho}{\rho} \right)^{\delta-1} \right] |\mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta))| \, d\vartheta \\
&+ \frac{1}{\Gamma(\delta)} \int_{\epsilon_1}^{\epsilon_2} \vartheta^{\rho-1} \left(\frac{\epsilon_2^\rho - \vartheta^\rho}{\rho} \right)^{\delta-1} |\mathcal{G}_1(\vartheta, v(\vartheta), u(\vartheta))| \, d\vartheta \\
&\leq \frac{\ell_1}{\Gamma(\delta+1)} \left[2 \left(\frac{\epsilon_2^\rho - \epsilon_1^\rho}{\rho} \right)^\delta + \left(\frac{\epsilon_1^\rho - \kappa_1^\rho}{\rho} \right)^\delta - \left(\frac{\epsilon_2^\rho - \kappa_1^\rho}{\rho} \right)^\delta \right].
\end{aligned}$$

As $\epsilon_1 \to \epsilon_2$, the right hand side of the above inequality tends to zero. Similarly, As $\epsilon_1 \to \epsilon_2$, we have

$$|\Upsilon_2(v,u)(\epsilon_2) - \Upsilon_2(v,u)(\epsilon_1)| \to 0.$$

Then, Υ is equicontinuous.
Finally, we let $P = \{(v,u) \in \Omega : (v,u) = \lambda \Upsilon(v,u), \lambda \in [0,1]\}$, for all $\epsilon \in [0,1]$, we obtain

$$v(\epsilon) = \lambda \Upsilon_1(v,u)(\epsilon),$$

and

$$u(\epsilon) = \lambda \Upsilon_2(v,u)(\epsilon).$$

By $\mathcal{H}(3)$, we obtain

$$\|v\| \leq \frac{(\varphi_0 + \varphi_1\|v\| + \varphi_2\|u\|)}{\Gamma(\delta+1)} \left(\aleph_\epsilon + \frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|} \aleph_\kappa + \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|} \aleph_{\eta_1} \right), \tag{3.10}$$

and

$$\|u\| \leq \frac{(\varpi_0 + \varpi_1\|v\| + \varpi_2\|u\|)}{\Gamma(\delta+1)} \left(\aleph_\epsilon + \frac{|\varrho_2|}{|\beta_2 + \varrho_2 - \sigma_2|} \aleph_\kappa + \frac{|\sigma_2|}{|\beta_2 + \varrho_2 - \sigma_2|} \aleph_{\eta_2} \right). \tag{3.11}$$

Adding (3.10) and (3.11), we obtain

$$\|v\| + \|u\| \leq \varphi_0 \Re_{\eta_1} + \varpi_0 \Re_{\eta_2} + (\varphi_1 \Re_{\eta_1} + \varpi_1 \Re_{\eta_2})\|v\| + (\varphi_2 \Re_{\eta_1} + \varpi_2 \Re_{\eta_2})\|u\|. \tag{3.12}$$

Equation (3.12) can be rewritten as

$$\|(v,u)\| \leq \frac{\varphi_0 \Re_{\eta_1} + \varpi_0 \Re_{\eta_2}}{min(1 - \varphi_1 \Re_{\eta_1} + \varpi_1 \Re_{\eta_2}, 1 - \varphi_2 \Re_{\eta_1} + \varpi_2 \Re_{\eta_2})}. \tag{3.13}$$

This result shows that P is bounded. Consequently, by using the Leray-Schauder alternative, the fractional problem (1.1) has at least one solution defined on Λ.

4. Findings related to stability theory

In this section, we establish stability outcomes by employing the H-U concept. We recall the operator $\Upsilon : \Omega \longrightarrow \Omega$, which is defined as follows:

$$\Upsilon(v,u) = (v,u). \tag{4.1}$$

Definition 4.1 *The solution of (v,u) of (4.1) is H-U be stable if for every $\varepsilon > 0$, and any solution $(\bar{v}, \bar{u}) \in \Omega$ of the inequality*

$$|\Upsilon(v,u) - (\bar{v}, \bar{u})| \leq \varepsilon. \tag{4.2}$$

There exist a constant $C > 0$ and unique solution (v,u) of (4.1) in Ω, the given inequality satisfies

$$\|(\bar{v}, \bar{u}) - (v,u)\| \leq C\varepsilon.$$

Remark 4.1 *$(\bar{v}, \bar{u})$ satisfies the inequality (4.2) if and only if there are functions $\psi_1, \psi_2 : \Lambda \to \mathbb{R}$, which are independent of solution (v, u), such that for every $\epsilon \in \Lambda$, the following condition is met:*
(i)$|\psi_1(\epsilon)| \leq \varepsilon$, $|\psi_2(\epsilon)| \leq \varepsilon$;
(ii) ${}^{\rho}_{c}D^{\delta}_{\kappa_1}\bar{v}(\epsilon) = \mathcal{G}_1(\epsilon, \bar{v}(\epsilon), \bar{u}(\epsilon)) + \psi_1(\epsilon)$, ${}^{\rho}_{c}D^{\delta}_{\kappa_1}\bar{u}(\epsilon) = \mathcal{G}_2(\epsilon, \bar{v}(\epsilon), \bar{u}(\epsilon)) + \psi_2(\epsilon)$.

By applying Remark (4.1), we obtain the following problem involving the specified perturbation functions $\psi_1(\epsilon)$ and $\psi_2(\epsilon)$.

$$\begin{cases} {}^{\rho}_{c}D^{\delta}_{\kappa_1}\bar{v}(\epsilon) = \mathcal{G}_1(\epsilon, \bar{v}(\epsilon), \bar{u}(\epsilon)) + \psi_1(\epsilon), \quad \epsilon \in \Lambda = [\kappa_1, \kappa_2], \\ {}^{\rho}_{c}D^{\delta}_{\kappa_1}\bar{u}(\epsilon) = \mathcal{G}_2(\epsilon, \bar{v}(\epsilon), \bar{u}(\epsilon)) + \psi_2(\epsilon), \\ \beta_1\bar{v}(\kappa_1) + \varrho_1\bar{v}(\kappa_2) = \sigma_1\bar{v}(\eta_1), \\ \beta_2\bar{u}(\kappa_1) + \varrho_2\bar{u}(\kappa_2) = \sigma_2\bar{u}(\eta_2). \end{cases} \tag{4.3}$$

Lemma 4.1 *The solution to system (4.3) is given by*

$$\begin{cases} \bar{v}(\epsilon) = \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon}\vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad - \frac{\varrho_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2}\vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad + \frac{\sigma_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1}\vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_1(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad + \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon}\vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta \\ \quad - \frac{\varrho_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2}\vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta \\ \quad + \frac{\sigma_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1}\vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta, \\ \bar{u}(\epsilon) = \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon}\vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_2(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad - \frac{\varrho_2}{\beta_2 + \varrho_2 - \sigma_2}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2}\vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_2(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad + \frac{\sigma_2}{\beta_2 + \varrho_2 - \sigma_2}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_2}\vartheta^{\rho-1}\left(\frac{\eta_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\mathcal{G}_2(\vartheta, \bar{v}(\vartheta), \bar{u}(\vartheta))\,d\vartheta \\ \quad + \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon}\vartheta^{\rho-1}\left(\frac{\epsilon^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta \\ \quad - \frac{\varrho_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2}\vartheta^{\rho-1}\left(\frac{\kappa_2^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta \\ \quad + \frac{\sigma_1}{\beta_1 + \varrho_1 - \sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1}\vartheta^{\rho-1}\left(\frac{\eta_1^\rho - \vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta. \end{cases} \tag{4.4}$$

Proof 4.1 *The solution is obtained via Corollary 3.1.*

Lemma 4.2 *The following inequalities are satisfied.*

$$\begin{cases} |\bar{v}(\epsilon) - \Upsilon_1(\bar{v}, \bar{u})| \leq \omega_1\varepsilon, \\ |\bar{u}(\epsilon) - \Upsilon_2(\bar{v}, \bar{u})| \leq \omega_2\varepsilon, \end{cases} \tag{4.5}$$

where

$$\omega_1 = \frac{1}{\Gamma(\delta+1)}\left(\aleph_\epsilon + \frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\aleph_\kappa + \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\aleph_{\eta_1}\right),$$

and

$$\omega_2 = \frac{1}{\Gamma(\delta+1)}\left(\aleph_\epsilon + \frac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_\kappa + \frac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_{\eta_2}\right). \tag{4.6}$$

Proof 4.2 *From* (4.4) *and using the operators defined in* (3.5)*, we have*

$$\begin{cases} \bar{v}(\epsilon) = \Upsilon_1(\bar{v},\bar{u}) + \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta \\ \qquad - \frac{\varrho_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta \\ \qquad + \frac{\sigma_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_1(\vartheta)\,d\vartheta, \\ \bar{u}(\epsilon) = \Upsilon_2(\bar{v},\bar{u}) + \frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta \\ \qquad - \frac{\varrho_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta \\ \qquad + \frac{\sigma_1}{\beta_1+\varrho_1-\sigma_1}\frac{1}{\Gamma(\delta)}\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}\psi_2(\vartheta)\,d\vartheta. \end{cases} \tag{4.7}$$

This implies that

$$\begin{aligned} |\bar{v}(\epsilon) - \Upsilon_1(\bar{v},\bar{u})| \quad &\leq \tfrac{1}{\Gamma(\delta)}\textstyle\int_{\kappa_1}^{\epsilon} \vartheta^{\rho-1}\left(\frac{\epsilon^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\psi_1(\vartheta)|\,d\vartheta \\ &+ \tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{1}{\Gamma(\delta)}\textstyle\int_{\kappa_1}^{\kappa_2} \vartheta^{\rho-1}\left(\frac{\kappa_2^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\psi_1(\vartheta)|\,d\vartheta \\ &+ \tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\tfrac{1}{\Gamma(\delta)}\textstyle\int_{\kappa_1}^{\eta_1} \vartheta^{\rho-1}\left(\frac{\eta_1^\rho-\vartheta^\rho}{\rho}\right)^{\delta-1}|\psi_1(\vartheta)|\,d\vartheta \\ &\leq \tfrac{1}{\Gamma(\delta+1)}\left(\aleph_\epsilon + \tfrac{|\varrho_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_\kappa + \tfrac{|\sigma_1|}{|\beta_1+\varrho_1-\sigma_1|}\aleph_{\eta_1}\right)\varepsilon = \omega_1\varepsilon. \end{aligned}$$

Similarly

$$|\bar{u}(\epsilon) - \Upsilon_2(\bar{v},\bar{u})| \quad \leq \omega_2\varepsilon.$$

Theorem 4.1 *The solution of system* (4.3) *is H-U and generalized H-U stable, if* $\xi < 1$ *holds.*

Proof 4.3 *If* $(v(t),u(t)) \in \Omega$ *be the unique solution of* (1.1) *and* $(\bar{v},\bar{u}) \in \Omega$ *is any solution, then*

$$\begin{aligned} |\bar{v}(\epsilon) - v(\epsilon)| &\leq |\bar{v}(\epsilon) - \Upsilon_1(\bar{v},\bar{u})(\epsilon) + \Upsilon_1(\bar{v},\bar{u})(\epsilon) - v(\epsilon)| \\ &\leq |\bar{v}(\epsilon) - \Upsilon_1(\bar{v},\bar{u})(\epsilon)| + |\Upsilon_1(\bar{v},\bar{u})(\epsilon) - v(\epsilon)|. \end{aligned}$$

By applying inequalities (3.8) *and* (4.5) *and taking the maximum, we obtain*

$$\|\bar{v} - v\| \leq \omega_1\varepsilon + \xi_1\left[\|v-\bar{v}\| + \|u-\bar{u}\|\right]. \tag{4.8}$$

In a similar manner, by applying inequalities (3.9) *and* (4.5) *and considering the maximum, we derive*

$$\|\bar{u} - u\| \leq \omega_2\varepsilon + \xi_2\left[\|v-\bar{v}\| + \|u-\bar{u}\|\right]. \tag{4.9}$$

Combining inequalities (4.8) *and* (4.9) *and utilizing the conditions* $\omega_1+\omega_2 = \omega$ *and* $\xi_1+\xi_2 = \xi$*, we obtain*

$$\|\bar{v} - v\| + \|\bar{u} - u\| \leq \omega\varepsilon + \xi\left[\|v-\bar{v}\| + \|u-\bar{u}\|\right].$$

Then,

$$\|\bar{v}-v\|+\|\bar{u}-u\| \leq \frac{\omega\varepsilon}{1-\xi}.$$

Or

$$\|(\bar{v},\bar{u})-(v,u)\|\leq \frac{\omega\varepsilon}{1-\xi}.$$

From which we can deduce that the system (1.1) is H-U stable.

Corollary 4.1 *If we have a non-decreasing function $\mathcal{F}:\Lambda\to\mathbb{R}$, such that*

$$\begin{aligned}\mathcal{F}(\varepsilon) &= \varepsilon,\\ \mathcal{F}(0) &= 0.\end{aligned}$$

Then,

$$\|\bar{v}-v\|+\|\bar{u}-u\| \leq \frac{\omega}{1-\xi}\mathcal{F}(\varepsilon).$$

Let $\mathbb{C}_{\omega,\xi}=\frac{\omega}{1-\xi}$, we have

$$\|\bar{v}-v\|+\|\bar{u}-u\| \leq \mathbb{C}_{\omega,\xi}\mathcal{F}(\varepsilon).$$

Or

$$\|(\bar{v},\bar{u})-(v,u)\|\leq \mathbb{C}_{\omega,\xi}\mathcal{F}(\varepsilon).$$

Thus, the proposed system is generalized H-U stable.

5. Pertinent example

In this section, we provide illustrative example to support our theoretical findings. For $\epsilon\in[0,1]$, considering the given problem

$$\begin{cases} {}^{\frac{2}{3}}_{c}D^{\frac{1}{2}}_{0^+}v(\epsilon)=\frac{|v(\epsilon)|}{100+|v(\epsilon)|}+\frac{\sin|u(\epsilon)|}{100}+\frac{\sin\epsilon}{100}, \quad \epsilon\in\Lambda=[0,1],\\ {}^{\frac{2}{3}}_{c}D^{\frac{1}{2}}_{0^+}u(\epsilon)=\frac{|u(\epsilon)|}{100+|u(\epsilon)|}+\frac{\sin|v(\epsilon)|}{100}+\frac{\cos\epsilon}{100},\\ \frac{1}{2}v(0)+\frac{1}{3}v(1)=\frac{1}{100}v(\frac{1}{2}),\\ \frac{1}{4}u(0)+\frac{1}{5}u(1)=\frac{1}{100}u(\frac{1}{2}). \end{cases} \tag{5.1}$$

Here $\mathcal{G}_1(\epsilon,v(\epsilon),u(\epsilon))=\frac{|v(\epsilon)|}{100+|v(\epsilon)|}+\frac{\sin|u(\epsilon)|}{100}+\frac{\sin t}{100}$ and $\mathcal{G}_2(\epsilon,v(\epsilon),u(\epsilon))=\frac{|u(\epsilon)|}{100+|u(\epsilon)|}+\frac{\sin|v(\epsilon)|}{100}+\frac{\cos t}{100}$. Let (v,u) ,$(\bar{v},\bar{u})\in\Omega$, then

$$\begin{aligned}|\mathcal{G}_1(\epsilon,v(\epsilon),u(\epsilon))-\mathcal{G}_1(\epsilon,\bar{v}(\epsilon),\bar{u}(\epsilon))| &= |\frac{|v(\epsilon)|}{100+|v(\epsilon)|}-\frac{|\bar{v}(\epsilon)|}{100+|\bar{v}(\epsilon)|}+\frac{\sin|u(\epsilon)|}{100}-\frac{\sin|\bar{u}(\epsilon)|}{100}|\\ &\leq \frac{1}{100}\left[|v-\bar{v}|+|u-\bar{u}|\right],\end{aligned}$$

and

$$|\mathcal{G}_1(\epsilon,v(\epsilon),u(\epsilon))-\mathcal{G}_1(\epsilon,\bar{v}(\epsilon),\bar{u}(\epsilon))|\leq \frac{1}{100}\left[|v-\bar{v}|+|u-\bar{u}|\right].$$

Therefore,

$$\mathbb{K}_{\mathcal{G}_1}=\mathbb{K}_{\mathcal{G}_2}=\frac{1}{100}.$$

We have $\delta = \frac{1}{2}$, $\rho = \frac{2}{3}$, $\aleph_\epsilon = \sqrt{\frac{3}{2}}\epsilon^{\frac{1}{3}}$, $\mathbb{K}_{\mathcal{G}_1} = \mathbb{K}_{\mathcal{G}_2} = \frac{1}{100}$, $\aleph_\kappa = (\frac{3}{2})^{\frac{1}{2}}$, $\aleph_{\eta_1} = \aleph_{\eta_2} = 0,9721$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{1}{4}$, $\varrho_1 = \frac{1}{3}$, $\varrho_2 = \frac{1}{5}$, $\sigma_1 = \sigma_2 = \frac{1}{100}$, $\eta_1 = \eta_2 = \frac{1}{2}$. Then,

$$\begin{aligned}\xi &= \frac{\aleph_\epsilon}{\Gamma(\delta+1)}(\mathbb{K}_{\mathcal{G}_1} + \mathbb{K}_{\mathcal{G}_2}) + \Big(\frac{|\varrho_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\mathbb{K}_{\mathcal{G}_1} + \frac{|\varrho_2|}{|\beta_2 + \varrho_2 - \sigma_2|}\mathbb{K}_{\mathcal{G}_2}\Big)\frac{\aleph_\kappa}{\Gamma(\delta+1)} \\ &+ \frac{|\sigma_1|}{|\beta_1 + \varrho_1 - \sigma_1|}\frac{\mathbb{K}_{\mathcal{G}_1}\aleph_{\eta_1}}{\Gamma(\delta+1)} + \frac{|\sigma_2|}{|\beta_2 + \varrho_2 - \sigma_2|}\frac{\mathbb{K}_{\mathcal{G}_2}\aleph_{\eta_2}}{\Gamma(\delta+1)} \le 0,1370075257 < 1.\end{aligned}$$

Since Theorem 3.1 hypotheses hold, (5.1) admits a unique solution in Λ.

We have

$$|\mathcal{G}_1(\epsilon, v, u)| \le 1 + \frac{1}{100} + \frac{1}{100} = 1.02,$$

and

$$|\mathcal{G}_2(\epsilon, v, u)| \le 1 + \frac{1}{100} + \frac{1}{100} = 1.02.$$

Therefore, hypothesis $\mathcal{H}(2)$ is verified with the constants $\ell_1 = \ell_2 = 1.02$.

We have

$$|\mathcal{G}_1(\epsilon, v, u)| \le 1.02 + 0|v| + 0|u|,$$

and

$$|\mathcal{G}_2(\epsilon, v, u)| \le 1.02 + 0|v| + 0|u|.$$

Then, hypothesis $\mathcal{H}(2)$ is verified with the constants $\varphi_0 = \varpi_0 = 1.02 > 0$ and $\varphi_i = \varpi_i = 0 \le 0$.

As all requirements of Theorem 3.2 are satisfied, it follows that (5.1) possesses at least one solution in Λ.

6. Conclusion

In this work, we studied a coupled system of fractional differential equations with Caputo-type modified Erdélyi-Kober fractional derivatives. Using the Banach contraction principle, we established the uniqueness of the solution, while the Leray-Schauder alternative fixed point theorem was employed to prove the existence of at least one solution under appropriate conditions. Additionally, we derived Hyers-Ulam (H-U) stability results for the system, ensuring the robustness of solutions under small perturbations. To reinforce our theoretical findings, we presented two illustrative examples, demonstrating the applicability of the obtained results.

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