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Existence Results for Generalized Caputo Proportional-Type Fractional Langevin Equations with p-Laplacian operator via Measure of Noncompactness

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ABSTRACT: This study addresses the existence of solutions for a novel class of generalized Caputo proportional-type fractional differential Langevin equations involving the p-Laplacian operator. The analysis is conducted by integrating the theory of the p-Laplacian operator with essential concepts from fractional calculus. Using the Kuratowski measure of noncompactness in an arbitrary Banach space and applying Mönch's fixed point theorem within the framework of the measure of noncompactness approach, we establish existence results. To illustrate the applicability and effectiveness of the proposed method, a detailed example is presented.

Key Words: Generalized Caputo proportional fractional derivative, Langevin differential equation, p-Laplacian operator, measure of noncompactness, fixed point theorem.

Contents

1. Introduction

Fractional calculus has witnessed remarkable growth in recent decades, evolving into a powerful mathematical tool for modeling complex systems with memory and hereditary properties. Recent advances have enriched both its theoretical foundations and its wide range of applications [14,13,26]. On the theoretical side, significant progress has been made in developing new definitions of fractional operators, variable-order derivatives, and generalized integral transforms, which extend the classical framework and enhance analytical tractability. From a computational perspective, novel numerical schemes and approximation methods have been introduced to solve fractional differential and integro-differential equations with improved accuracy and efficiency [6,5]. In terms of applications, fractional models have found increasing use across diverse fields, including viscoelasticity, anomalous diffusion, control systems, signal processing, biology, and finance, where they provide more accurate representations than their integer-order counterparts. See [21,19,27,2,4,3,17]. These developments underscore the relevance of fractional calculus as a unifying framework for both theoretical research and practical problem-solving.

The Langevin equation, introduced by Paul Langevin in 1908, is a key component of statistical mechanics that describes Brownian motion and systems in fluctuating environments. It captures the interaction between deterministic forces and random perturbations, facilitating an understanding of particle behavior under thermal noise [16,35,12,28,10,1]. The classical Langevin equation has been extended to a fractional form, incorporating memory effects and long-range correlations. This allows for modeling fractal and anomalous diffusion processes, broadening its applications in viscoelastic materials, biological systems, and other complex, non-Markovian dynamics. See [18,7,9,33].

The p-Laplacian operator, defined as $\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$, is a nonlinear generalization of the Laplace operator. It is used in various fields, including image processing for edge-preserving smoothing,

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fluid dynamics for modeling non-Newtonian fluid diffusion, and electrical networks for nonlinear conductivity. Additionally, it models population dynamics in biology and has fractional variants that aid in understanding complex systems like blood flow and turbulent filtration. We refer readers to [24,25,30,34].

In [31] Samira and a.l. Investigated the existence of solutions for the following p-Laplacian hybrid fractional differential equation, which incorporates the generalized Caputo proportional fractional derivative:

$$\begin{cases} {}^{\mathfrak{C}}_{\delta}D^{\theta,g}_{0^+}\Phi_p\left({}^{\mathfrak{C}}_{\delta}D^{\vartheta,g}_{0^+}(\frac{w(t)}{F(t,w(t))})\right) = G(t,w(t)), \quad t\in\Sigma=[0,b], \\ \left(\frac{w(t)}{F(t,w(t))}\right)_{t=0} = w_0, \\ \left(\frac{w(t)}{F(t,w(t))}\right)'_{t=0} = 0, \qquad\qquad w_0\in\mathbb{R}, \end{cases}$$

where $0<\theta<1$, $1<\vartheta<2$, ${}^{\mathfrak{C}}_{\delta}D^{\theta,g}_{0^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order θ, $\Phi_p(x)=|x|^{p-2}x$, $p>1$ is the p-Laplacian operator, $g:\Sigma\to\mathbb{R}$, $F\in C(\Sigma\times\mathbb{R},\mathbb{R}^*)$, and $G\in C(\Sigma\times\mathbb{R},\mathbb{R})$.

They also in [32] addressed the existence of solutions for the following nonlinear boundary value Langevin fractional differential equation involving the generalized Caputo proportional fractional derivative:

$$\begin{cases} {}^{\mathfrak{C}}_{\delta}D^{\alpha,g}_{a^+}\left({}^{\mathfrak{C}}_{\delta}D^{\beta,g}_{a^+}+\chi\right)(x(t)-K(t,x(t))) = L(t,x(t)), \quad t\in\Delta=[a,b], \\ (x(t)-K(t,x(t)))\big|_{t=a} = (x(t)-K(t,x(t)))\big|_{t=b} = 0, \\ {}^{\mathfrak{C}}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t)))\big|_{t=a} = \mu, \\ {}^{\mathfrak{C}}_{\delta}D^{\beta,g}_{a^+}(x(t)-K(t,x(t)))\big|_{t=b} = \nu, \qquad \mu,\nu\in\mathbb{R}, \end{cases}$$

where $\Delta=[a,b]$ is a finite interval of $\mathbb{R}$ with $a>0$, $1<\alpha<2$, $0<\beta<1$, ${}^{\mathfrak{C}}_{\delta}D^{\alpha,g}_{a^+}(\cdot)$ is the generalized Caputo proportional fractional derivative of order α, $\chi\in\mathbb{R}^*$, and $K,L\in C(\Delta\times E,E)$ are given functions.

Inspired by the works above, this paper combines their ideas to investigate the existing results for the boundary value p-Laplacian Langevin fractional differential equation (BVPLFDE):

$$\begin{cases} {}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\theta;g}_{a^+}\left[\Phi_p\left(\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}+\lambda\right)\chi(t)\right)\right] = \mathfrak{F}(t,\chi(t)), \quad t\in\Lambda:=[a,b], \\ \left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}+\lambda\right)\chi(t)|_{t=a} = \left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}+\lambda\right)\chi(t)|_{t=b} = 0, \\ \chi(a)=\chi(b), \end{cases} \tag{1.1}$$

where $\Lambda=[a,b]$ is a finite interval of $\mathbb{R}$ with $a>0$, ${}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\theta;g}_{a^+}$, ${}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}$ are the generalized Caputo proportional fractional derivative of order $1<\theta\le 2$, and $0<\ell\le 1$, respectively, $\lambda\in\mathbb{R}^*$, $g:\Lambda\to\mathbb{R}$, $\mathfrak{F}\in C(\Lambda\times E,E)$ are given functions checking specific assumptions that will be defined later, E is a Banach space with the norm $\|\ .\ \|$, and $\Phi_p(t)$ represents the p-Laplacian operator, defined as $\Phi_p(t)=|t|^{p-2}t$ for $p>1$, its inverse operator is $\Phi_q(t)=|t|^{q-2}t$ such that $\frac{1}{P}+\frac{1}{q}=1$.

The originality of this work lies in studying a new and challenging case of fractional derivative named the generalized Caputo proportional fractional derivative introduces additional flexibility to model systems with fractional orders, further broadening the range of applications. To our knowledge, this is the first time that the p-Laplacian fractional differential equation involving the generalized Caputo proportional fractional derivative under boundary conditions (1.1) is being studied. Those conditions allow for a deeper understanding of how initial and terminal constraints influence the behavior of the modeled system.

This paper is organized as follows: Section 2 reviews key notations and results related to fractional calculus and the p-Laplacian operator. In Section 3, we examine the existence of solutions for the problem in equation (1.1) using the Kuratowski measure of noncompactness and Mönch's fixed-point theorem. Finally, Section 4 presents an example to illustrate the main result.

2. Preliminaries

In this section, we will provide definitions and properties related to the Kuratowski measure of noncompactness and introduce the generalized Caputo proportional fractional derivative, along with the properties of the p-Laplacian operator. These will be applied in the following sections.

Let $C(\Lambda, E)$ be the Banach space of continuous functions with the norm $\|\chi\|_\infty = \sup\{\| \chi(t) \| : t \in \Lambda\}$. The function $\mathfrak{f} : \Lambda \times E \to E$ is said to satisfy the Caratheodory conditions if the following criteria are met:

(i) $\mathfrak{f}(t, \chi)$ is measurable with respect to t for every $\chi \in E$;

(ii) $\mathfrak{f}(t, \chi)$ is continuous with respect to $\chi \in E$ for every $t \in \Lambda$.

Additionally, we consider a function $g : \Lambda \to \mathbb{R}$ that is strictly positive, increasing, and differentiable.

Definition 2.1 [8] Let E be a Banach space and F be a bounded subset of E. The Kuratowski measure of noncompactness is given by the mapping $\kappa : F \to [0, \infty)$ is defined as follows

$$\kappa(U) = \inf\left\{\alpha > 0 / U \subseteq \cup_{i=1}^{n} U_i \quad and \quad diam(U_i) \leq \alpha\right\}.$$

Lemma 2.1 *[8] Let V and W be two bounded subset of the Banach space E, then we have the following proprieties:*

1. $V \subseteq W \Rightarrow \kappa(V) \leq \kappa(W)$.
2. $\kappa(V + W) \leq \kappa(V) + \kappa(W)$.
3. $\kappa(\beta V) = |\beta|\kappa(V)$, $\beta \in \mathbb{R}$.
4. $\kappa(V) = 0 \Leftrightarrow V$ *is relatively compact in* E.
5. $\kappa(V \cup W) = max\{\kappa(V), \kappa(W)\}$.
6. $\kappa(V) = \kappa(\overline{V}) = \kappa(\overline{conv} V)$, *where* $\overline{V}$ *and* $convV$ *denote the closure and the convex hull of* V *respectively.*

Definition 2.2 [15] Let $\Upsilon : F \to E$ be a continuous, bounded operator. Then Υ is κ-Lipschitz if there exists a constant $\Theta > 0$, such that for all $V \subset F$ we have

$$\kappa(\Upsilon(V)) \leq \Theta\kappa(V).$$

Lemma 2.2 *[20] Consider that $V \in C(\Lambda, E)$ is a bounded and equicontinuous subset. Then, the function $t \to \kappa(V(t))$ is continuous on Λ and*

$$\kappa\left(\int_a^b \chi(s)ds\right) \leq \int_a^b \kappa(V(s))ds,$$

where $V(s) = \{\chi(s) : \chi \in V\}$, $s \in \Lambda$.

Theorem 2.1 *(Mönch's fixed point theorem [29]) Let X be a closed, bounded and convex subset of a Banach space E such that $0 \in E$ and let $I : X \to X$ be a continuous mapping satisfying:*

$$Q = \overline{conv} I(Q), \quad or \quad Q = I(Q) \cup \{0\} \Rightarrow \kappa(Q) = 0, \forall Q \subset X. \tag{2.1}$$

Then, the mapping I has a fixed point.

Definition 2.3 [22], [23] Let $0 < \delta < 1$, $\ell > 0$, $\chi \in L^1(\Lambda, E)$. The left-sided generalized proportional fractional integral concerning g of order ℓ of the function χ is given by

$$_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t) = \frac{1}{\delta^{\ell}\Gamma(\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{\ell-1}\chi(s)ds,$$

where $\Gamma(\ell) = \displaystyle\int_a^{+\infty} e^{-\sigma}\sigma^{\ell-1}d\sigma$ is the Euler gamma function.

Definition 2.4 [22], [23] Let $0 < \delta < 1$, z, $\hbar : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous functions such that $\lim\limits_{\delta\to0^+} z(\delta,t) = 0$, $\lim\limits_{\delta\to1^-} z(\delta,t) = 1$, $\lim\limits_{\delta\to0^+} \hbar(\delta,t) = 1$, $\lim\limits_{\delta\to1^-} \hbar(\delta,t) = 0$, and $z(\delta,t) \neq 0$, $\hbar(\delta,t) \neq 0$ for each $\delta \in [0,1]$, $t \in \mathbb{R}$. The proportional derivative of order δ with respect to g for the function χ is defined as follows:

$$_{\delta}\mathfrak{D}^{g}\chi(t) = \hbar(\delta,t)\chi(t) + z(\delta,t)\frac{\chi^{'}(t)}{g^{'}(t)}.$$

Specifically, if $z(\delta,t) = \delta$ and $\hbar(\delta,t) = 1-\delta$, we establish the following:

$$_{\delta}\mathfrak{D}^{g}\chi(t) = (1-\delta)\chi(t) + \delta\frac{\chi^{'}(t)}{g^{'}(t)}.$$

Definition 2.5 [22], [23] Let $\delta \in (0,1]$. The left-sided generalized Caputo proportional fractional derivative of order $n-1 < \ell < n$ is defined as follows:

$$\begin{aligned}{}^{C}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}\chi(t) &= {}_{\delta} I^{n-\ell;g}_{a^+}({}_{\delta}\mathfrak{D}^{n,g}\chi(t))\\ &= \frac{1}{\delta^{n-\ell}\Gamma(n-\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{n-\ell-1}({}_{\delta}\mathfrak{D}^{n,g}\chi)(s)ds,\end{aligned}$$

where $n = [\ell]+1$ and ${}_{\delta}\mathfrak{D}^{n,g} = \underbrace{{}_{\delta}\mathfrak{D}^{g}.{}_{\delta}\mathfrak{D}^{g}...{}_{\delta}\mathfrak{D}^{g}}_{n-times}$.

Lemma 2.3 *[22], [23] Let $t \in \Lambda$, $\delta \in (0,1]$, $(\ell,\theta > 0)$, and $\chi \in L^1(\Lambda,E)$. Then, we have*

$${}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}({}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\chi(t)) = {}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}({}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t)) = {}_{\delta}\mathfrak{I}^{\ell+\theta;g}_{a^+}\chi(t).$$

For simplicity, we assume throughout this paper.

$$\Omega^{\ell-1}_{g}(t,a) = e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))^{\ell-1}.$$

Lemma 2.4 *[22], [23] Let $\ell > 0$, $\theta > 0$, and $\delta \in (0,1]$. Then, we have*

(i) $({}_{\delta}\mathfrak{I}^{\ell,g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\sigma)-g(a))}(g(\sigma)-g(a))^{\theta-1})(t) = \frac{\Gamma(\theta)}{\delta^{\ell}\Gamma(\ell+\theta)}\Omega^{\ell+\theta-1}_{g}(t,a)$.

(ii) $({}^{C}_{\delta}\mathfrak{D}^{\ell,g}_{a^+}e^{\frac{\delta-1}{\delta}(g(\sigma)-g(a))}(g(\sigma)-g(a))^{\theta-1})(t) = \frac{\delta^{\ell}\Gamma(\theta)}{\Gamma(\theta-\ell)}\Omega^{\theta-\ell-1}_{g}(t,a)$.

Lemma 2.5 *[22], [23] Let $\ell > 0$, $\delta \in (0,1]$, and $\chi \in L^1(\Lambda,E)$. Then, we have*

$$\lim_{t\to a}({}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t)) = 0.$$

Lemma 2.6 *[22], [23] Let $\delta \in (0,1]$, $n-1 < \ell < n$, $n = [\ell]+1$. Then, we have*

$${}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}({}^{C}_{\delta}\mathfrak{D}^{\ell,g}_{a^+}\chi(t)) = \chi(t) - \sum_{k=0}^{n-1}\frac{({}_{\delta}\mathfrak{D}^{k,g}\chi)(a)}{\delta^{k}\Gamma(k+1)}\Omega^{k}_{g}(t,a).$$

Lemma 2.7 *[11] The p-Laplacian operator $\Phi_p : \mathbb{R} \to \mathbb{R}$ defined by $\Phi_p(\chi) = |\chi|^{p-2}\chi$ satisfies the following proprieties:*

1. *The p-Laplacian operator Φ_p is invertible, moreover we have $\Phi_p^{-1} = \Phi_q$ such that $\frac{1}{p} + \frac{1}{q} = 1$.*
2. *If $p \in (1, 2)$, $|\chi|, |\zeta| \geq C_1 > 0$, and $\chi\zeta > 0$. Then, we have*

$$|\Phi_p(\chi) - \Phi_p(\zeta)| \leq (p-1)C_1^{p-2}|\chi - \zeta|.$$

3. *If $p > 2$, $|\chi|, |\zeta| \leq C_2$. Then, we have*

$$|\Phi_p(\chi) - \Phi_p(\zeta)| \leq (p-1)C_2^{p-2}|\chi - \zeta|.$$

3. Main Result

This section establishes the equivalence between an integral equation and the problem delineated in (1.1). Furthermore, we present an existence theorem for the solution of the boundary value p-Laplacian Langevin fractional differential equation (BVPLFDE) outlined in (1.1).

Definition 3.1 A function $\chi \in \mathcal{C}(\Lambda, E)$ is recognized as a solution to problem (1.1) when it successfully fulfills the given equation ${}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\theta;g}_{a^+}\left[\Phi_p\left(\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)\right)\right]$, a.e on Λ and the conditions $\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)|_{t=a} = \left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)|_{t=b} = 0,\ x(a) = x(b)$.

Lemma 3.1 *Let $a \geq 0$, $1 < \theta \leq 2$, $0 < \ell \leq 1$, $\mathfrak{F} \in \mathcal{C}(\Lambda, E)$, $\lambda \in \mathbb{R}^*$. Then the function χ is a solution of the boundary value problem (1.1) if and only if*

$$\begin{aligned}\chi(t) = {}& {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\left(\Phi_q\left[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t, \chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a))\right]\right)\\ & - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t) + \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))},\end{aligned} \tag{3.1}$$

where

$$\Sigma = \frac{-{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(b, \chi(b))}{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b) - g(a))}, \tag{3.2}$$

$$\Pi = \frac{{}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\left(\Phi_q\left[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(b, \chi(b)) + \Sigma e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b) - g(a))\right]\right) - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(b)}{1 - e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}. \tag{3.3}$$

Proof: Applying the operator ${}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}$ on both sides of the fractional differential Equation (1.1) we obtain by using Lemma 3.1.

$$\Phi_p\left(\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)\right) = {}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t, \chi(t)) + d_0 e^{\frac{\delta-1}{\delta}(g(t)-g(a))} + d_1\frac{e^{\frac{\delta-1}{\delta}(g(t)-g(a))}}{\delta}(g(t) - g(a)), \tag{3.4}$$

where d_0 and d_1 are constants. Next, by using the fact that
$\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)|_{t=a} = \left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t)|_{t=b} = 0$ in (3.4) we obtain $d_0 = 0$ and

$$d_1 = \delta\left(\frac{-{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(b, \chi(b))}{e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b) - g(a))}\right).$$

Substituting the value of d_0 and d_1 in (3.4). Then

$$\Phi_p\left({}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+} + \lambda\right)\chi(t) = {}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t, \chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t) - g(a)). \tag{3.5}$$

Using the inverse operator of Φ_p, we find

$$ {}^{\mathfrak{C}}_{\delta}\mathfrak{D}^{\ell;g}_{a^+}\chi(t) + \lambda\chi(t) = \Phi_q\Big[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\Big]. \tag{3.6} $$

Applying the operator ${}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}$ on both sides of the fractional differential Equation (3.6) and using Lemma. Then, we have

$$ \begin{aligned} \chi(\iota) =& {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\bigg(\Phi_q\Big[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\Big]\bigg) \\ & - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t) + d_2 e^{\frac{\delta-1}{\delta}(g(t)-g(a))}, \end{aligned} \tag{3.7} $$

by using the boundary condition $\chi(a) = \chi(b)$ in equation (3.7), we arrive at the following result:

$$ \begin{aligned} d_2 =& {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\bigg(\Phi_q\Big[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(b,\chi(b)) + \Sigma e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))\Big]\bigg) \\ & - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(b) + d_2 e^{\frac{\delta-1}{\delta}(g(b)-g(a))}, \end{aligned} $$

which implies that

$$ d_2 = \frac{{}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\bigg(\Phi_q\bigg[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(b,\chi(b)) + \Sigma e^{\frac{\delta-1}{\delta}(g(b)-g(a))}(g(b)-g(a))\bigg]\bigg) - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(b)}{1 - e^{\frac{\delta-1}{\delta}(g(b)-g(a))}}. $$

Substituting the value of d_2 in (3.7) we obtain

$$ \begin{aligned} \chi(t) =& \;{}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\bigg(\Phi_q\bigg[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\bigg]\bigg) \\ & - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t) + \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}. \end{aligned} $$

□

Before we discuss our results, let's first outline the following hypotheses:

$(\mathcal{H}_1)$: The function $\mathfrak{F} \in C(\Lambda \times E, E)$ satisfies the Carathéodory conditions.

$(\mathcal{H}_2)$: The inverse of the p-Laplacian operator $\Phi_q : \mathbb{R} \to \mathbb{R}$ is κ-Lipschitz, i.e., such that for all $V \subset E$ we have

$$ \kappa(\Phi_q(V)) \leq \kappa(V), $$

$(\mathcal{H}_3)$: There exist $\varpi \in C(\Lambda, \mathbb{R}^+)$ and $\rho \in C(\mathbb{R}^+, \mathbb{R}^+)$, with ρ being nondecreasing. Then, for all $t \in \Lambda$ and $\chi \in E$, we have

$$ \|\mathfrak{F}(t,\chi)\| \leq \varpi(t)\rho(\|\chi\|), \quad \text{with } \varpi^* = \max_{t\in\Lambda}\{\varpi(t)\}. $$

$(\mathcal{H}_4)$: For every bounded subset $V \subset E$ and for each $t \in \Lambda$, we have

$$ \kappa(\mathfrak{F}(t,V)) \leq \varpi(t)\kappa(V). $$

Let $\chi \in \mathcal{C}(\Lambda, E)$ and $t \in \Lambda$, then we define the operator $\mathfrak{X}$ as follows:

$$ \begin{aligned} (\mathfrak{X}\chi)(t) =& \;{}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\bigg(\Phi_q\bigg[\,{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\bigg]\bigg) \\ & - \lambda {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t) + \Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))}, \end{aligned} \tag{3.8} $$

where Σ and Π are given by (3.2) and (3.3), respectively.
It is important to note that if the operator $\mathfrak{X}$ has a fixed point, then the boundary value differential equation (BVPLFDE) (1.1) has a solution in $\mathcal{C}(\Lambda, E)$. We will now present the main result of this paper, which is the existence theorem for problem (1.1).

Theorem 3.1 *Assuming that all hypotheses* $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ *are valid, the BVPLFDE* (1.1) *has at least one solution on* Λ *if the following inequality is satisfied:*

$$\Phi = \left\{ \frac{\varpi^*(g(b)-g(a))^{\ell+\theta}}{\delta^{\ell+\theta}\Gamma(\ell+1)\Gamma(\theta+1)} + \frac{|\lambda|(g(b)-g(a))^{\ell}}{\delta^{\ell}\Gamma(\ell+1)} \right\} < 1. \tag{3.9}$$

Proof: Let $\mathfrak{X} : \mathcal{C}(\Lambda, E) \to \mathcal{C}(\Lambda, E)$. Then we consider the set $\mathcal{B}_\varrho$ defined as:

$$\mathcal{B}_\varrho = \{\chi \in \mathcal{C}(\Lambda, E) : \|\chi\| \le \varrho\},$$

with

$$\varrho \le \frac{\mathfrak{B}}{1-\mathfrak{C}}, \ \textit{such that } 1-\mathfrak{C} \neq 0, \tag{3.10}$$

$$\mathfrak{C} = \frac{|\lambda|(g(b)-g(a))^{\ell}}{\delta^{\ell}\,\Gamma(\ell+1)}, \tag{3.11}$$

$$\mathfrak{B} = \frac{(g(b)-g(a))^{\ell}}{\delta^{\ell}\,\Gamma(\ell+1)}\mathfrak{A}^{q-1} + |\Pi|, \tag{3.12}$$

where

$$\mathfrak{A} = (g(b)-g(a))\left(\frac{\varpi^*\rho(\varrho)(g(b)-g(a))^{\theta-1}}{\delta^{\theta}\Gamma(\theta+1)} + |\Sigma| \right).$$

It is clear that $\mathcal{B}_\varrho$ is a convex, closed, and bounded subset of the Banach space $\mathcal{C}(\Lambda, E)$.
The proof is presented in several steps:
Step 1. $\mathfrak{X}(\mathcal{B}_\varrho) \subset \mathcal{B}_\varrho$,
Let $\chi \in \mathcal{B}_\varrho$ and $t \in \Lambda$, then by using hypothesis $\mathcal{H}_3$ and the fact that $e^{\frac{\delta-1}{\delta}(g(\cdot)-g(\cdot))} < 1$, we get

$$\begin{aligned}
&\left\| {}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a)) \right\| \\
&\le \frac{1}{\delta^{\theta}\Gamma(\theta)}\int_a^t \Omega_g^{\theta-1}(t,a)g'(s)\|\mathfrak{F}(s,\chi(s))\|ds + |\Sigma|(g(b)-g(a)) \\
&\le (g(b)-g(a))\left(\frac{\varpi^*\rho(\varrho)(g(b)-g(a))^{\theta-1}}{\delta^{\theta}\Gamma(\theta+1)} + |\Sigma| \right) \\
&=: \mathfrak{A}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|(\mathfrak{X}\chi)(t)\| \le\ & {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\left\| \Phi_q\left[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t)) + \Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a)) \right] \right\| + |\lambda|\ {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\|\chi(t)\| + |\Pi| \\
\le\ & \frac{1}{\delta^{\ell}\,\Gamma(\ell)}\int_a^t g'(s)\big(g(t)-g(s)\big)^{\ell-1} \left\| {}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(s,\chi(s)) + \Sigma e^{\frac{\delta-1}{\delta}(g(s)-g(a))}(g(s)-g(a)) \right\|^{q-1} ds \\
&+ \frac{|\lambda|}{\delta^{\ell}\,\Gamma(\ell)}\int_a^t g'(s)\big(g(t)-g(s)\big)^{\ell-1}\|\chi(s)\|\,ds + |\Pi| \\
\le\ & \frac{(g(b)-g(a))^{\ell}}{\delta^{\ell}\,\Gamma(\ell+1)}\mathfrak{A}^{q-1} + |\Pi| + \frac{|\lambda|(g(b)-g(a))^{\ell}}{\delta^{\ell}\,\Gamma(\ell+1)}\varrho \\
=:\ & \mathfrak{B} + \mathfrak{C}\varrho \le \varrho.
\end{aligned}$$

Therefore, $\mathfrak{X}$ maps $\mathcal{B}_\varrho$ into itself.
Step 2. The operator $\mathfrak{X}$ is continuous. Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_\varrho$ such that χ_n converges to χ in $\mathcal{B}_\varrho$ as $n \to +\infty$. Then, we have

$$\begin{aligned}\|(\mathfrak{X}\chi_n)(t)-(\mathfrak{X}\chi)(t)\| \leq & \Bigg\| {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\Bigg(\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi_n(t))+\Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\Bigg]\Bigg)\\ & - {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\Bigg(\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t,\chi(t))+\Sigma e^{\frac{\delta-1}{\delta}(g(t)-g(a))}(g(t)-g(a))\Bigg]\Bigg)\Bigg\|\\ & + |\lambda|\, {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\|\chi_n(t)-\chi(t)\|.\end{aligned}$$

Utilizing the fact that Φ_q is a continuous operator and the function $\mathfrak{F}$ satisfies the Carathéodory conditions, along with the Lebesgue-dominated convergence theorem, we can derive the following result from the previous inequality:

$$\|(\mathfrak{X}\chi_n)(t)-(\mathfrak{X}\chi)(t)\| \to 0 \text{ as } n \to +\infty.$$

This shows that the operator $\mathfrak{X}$ is continuous.

Step 3. The operator $\mathfrak{X}$ is equicontinuous.

Let $t_1, t_2 \in \Lambda$, with $t_1 < t_2$, and $\chi \in \mathcal{B}_\varrho$. Then, we have

$$\begin{aligned}&\|(\mathfrak{X}\chi)(t_2)-(\mathfrak{X}\chi)(t_1)\|\\ &\leq \Bigg\| {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\Bigg(\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t_2,\chi(t_2))+\Sigma e^{\frac{\delta-1}{\delta}(g(t_2)-g(a))}(g(t_2)-g(a))\Bigg]\Bigg)\\ &- {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\Bigg(\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(t_1,\chi(t_1))+\Sigma e^{\frac{\delta-1}{\delta}(g(t_1)-g(a))}(g(t_1)-g(a))\Bigg]\Bigg)\Bigg\|\\ &+|\lambda|\Big\| {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t_2) - {}_{\delta}\mathfrak{I}^{\ell;g}_{a^+}\chi(t_1)\Big\|\\ &\leq \frac{1}{\delta^\ell\Gamma(\ell)}\int_a^{t_1}\left(\Omega_g^{\ell-1}(t_2,a)-\Omega_g^{\ell-1}(t_1,a)\right)g'(s)\\ &\times\Bigg\|\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(s,\chi(s))+\Sigma e^{\frac{\delta-1}{\delta}(g(s)-g(a))}(g(s)-g(a))\Bigg]\Bigg\|ds+\frac{1}{\delta^\ell\Gamma(\ell)}\int_{t_1}^{t_2}\Omega_g^{\ell-1}(t_2,a)g'(s)\\ &\times\Bigg\|\Phi_q\Bigg[{}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(s,\chi(s))+\Sigma e^{\frac{\delta-1}{\delta}(g(s)-g(a))}(g(s)-g(a))\Bigg]\Bigg\|ds\\ &+|\lambda|\Bigg[\frac{1}{\delta^\ell\Gamma(\ell)}\int_a^{t_1}\left(\Omega_g^{\ell-1}(t_2,a)-\Omega_g^{\ell-1}(t_1,a)\right)g'(s)\|\chi(s)\|\,ds\\ &+\frac{1}{\delta^\ell\Gamma(\ell)}\int_{t_1}^{t_2}\Omega_g^{\ell-1}(t_2,a)g'(s)\|\chi(s)\|\,ds\Bigg]\\ &\leq \frac{\mathfrak{A}^{q-1}}{\delta^\ell\Gamma(\ell)}\int_a^{t_1}\left(\Omega_g^{\ell-1}(t_2,a)-\Omega_g^{\ell-1}(t_1,a)\right)g'(s)ds+\frac{\mathfrak{A}^{q-1}}{\delta^\ell\Gamma(\ell)}\int_{t_1}^{t_2}\Omega_g^{\ell-1}(t_2,a)g'(s)ds\\ &+|\lambda|\Bigg[\frac{\varrho}{\delta^\ell\Gamma(\ell)}\int_a^{t_1}\left(\Omega_g^{\ell-1}(t_2,a)-\Omega_g^{\ell-1}(t_1,a)\right)g'(s)ds\\ &+\frac{\varrho}{\delta^\ell\Gamma(\ell)}\int_{t_1}^{t_2}\Omega_g^{\ell-1}(t_2,a)g'(s)ds\Bigg].\end{aligned}$$

By utilizing the continuity of the function $\Omega_g^{(\cdot)}(t,a)$, along with the Lebesgue-dominated convergence theorem, we derive the following inequality:

$$\|(\mathfrak{X}\chi)(t_2)-(\mathfrak{X}\chi)(t_1)\| \to 0 \quad \text{as } t_1 \to t_2.$$

Hence, the operator $\mathfrak{X}$ is equicontinuous.

Step 4. Let's check Condition (2.1) of Theorem 2.1.
Let $Q \subseteq \overline{conv}(\mathfrak{X}(Q) \cup \{0\})$ be a bounded and equicontinuous subset. Then, the function $I(t) = \kappa(Q(t))$ is continuous on Λ. Thanks to Lemma 2.2, Hypotheses $(\mathcal{H}_2)$, and $(\mathcal{H}_4)$, we get:

$$\begin{aligned}
I(t) &= \kappa(Q(t)) \leq \kappa\left(conv((\mathfrak{X}Q)(t) \cup \{0\})\right) \leq \kappa((\mathfrak{X}Q)(t)) \\
&\leq \kappa\left\{\frac{1}{\delta^{\ell}\Gamma(\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t)-g(s))^{\ell-1}\right. \\
&\times \left.\Phi_q\left({}_{\delta}\mathfrak{I}^{\theta;g}_{a^+}\mathfrak{F}(s,\chi(s)) + \Sigma e^{\frac{\delta-1}{\delta}(g(s)-g(a))}(g(s)-g(a))\right)ds : \chi \in Q\right\} \\
&+ \kappa\left\{\frac{\lambda}{\delta^{\ell}\Gamma(\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t)-g(s))^{\ell-1}\chi(s)ds : \chi \in Q\right\} \\
&+ \kappa\left\{\Pi e^{\frac{\delta-1}{\delta}(g(t)-g(a))} : t \in \Lambda\right\} \\
&\leq \frac{1}{\delta^{\ell}\Gamma(\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t)-g(s))^{\ell-1} \\
&\times \frac{1}{\delta^{\theta}\Gamma(\theta)}\int_a^s e^{\frac{\delta-1}{\delta}(g(s)-g(\tau))} g'(\tau)(g(s)-g(\tau))^{\theta-1}\kappa\left(\mathfrak{F}(\tau, Q(\tau))\right)d\tau ds \\
&+ \frac{|\lambda|}{\delta^{\ell}\Gamma(\ell)}\int_a^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t)-g(s))^{\ell-1}\kappa(Q(s))ds \\
&\leq \frac{\varpi^* \parallel I \parallel_\infty}{\delta^{\ell+\theta}\Gamma(\ell)\Gamma(\theta)}\int_a^t g'(s)(g(t)-g(s))^{\ell-1}\int_a^s g'(\tau)(g(s)-g(\tau))^{\theta-1}d\tau ds \\
&+ \frac{|\lambda| \parallel I \parallel_\infty}{\delta^{\ell}\Gamma(\ell)}\int_a^t g'(s)(g(t)-g(s))^{\ell-1}ds \\
&\leq \left\{\frac{\varpi^*(g(b)-g(a))^{\ell+\theta}}{\delta^{\ell+\theta}\Gamma(\ell+1)\Gamma(\theta+1)} + \frac{|\lambda|(g(b)-g(a))^{\ell}}{\delta^{\ell}\Gamma(\ell+1)}\right\} \parallel I \parallel_\infty .
\end{aligned}$$

Therefore,

$$\parallel I \parallel_\infty \leq \Phi \parallel I \parallel_\infty,$$

where Φ is given by (3.9).
From Condition (3.9), it follows that $\parallel I \parallel_\infty = 0$. Consequently, for every $t \in \Lambda$, we have $I(t) = 0$, which implies $\kappa(Q(t)) = 0$. Thus, Q is relatively compact in E. By applying the Arzelá–Ascoli theorem, we deduce that Q is relatively compact in $\mathcal{B}_\varrho$.

Furthermore, we note that all the conditions of Theorem 3.1 are fulfilled. Therefore, the operator $\mathfrak{X}$ has a fixed point in $\mathcal{B}_\varrho$, which corresponds to the solution of the BVPLFDE (1.1). This concludes the proof. □

4. Example

In this section, we present an example to demonstrate our results.
Let $t \in [1, 2]$, $E = \mathbb{R}$, $\delta = \ell = \frac{1}{2}$, $\theta = \frac{3}{2}$, $\lambda = 0.02$, $g(t) = t$, $p = 3$, $q = \frac{3}{2}$, and $\mathfrak{F} : \Lambda \times \mathbb{R} \to \mathbb{R}$, such that

$$\mathfrak{F}(t, \chi(t)) := \frac{1}{49 + e^t}\left(\frac{e}{7} + \sin(\chi(t))\right).$$

Then, consider the boundary value problem for the p-Laplacian Langevin fractional differential equation (BVPLFDE) on the interval $\Lambda = [1, 2]$ defined by

$$\begin{cases}
{}^{\mathfrak{C}}_{\frac{1}{2}}\mathfrak{D}^{\frac{3}{2};t}_{1^+}\left[\Phi_3\left({}^{\mathfrak{C}}_{\frac{1}{2}}\mathfrak{D}^{\frac{1}{2};t}_{1^+}\chi(t) + \lambda\chi(t)\right)\right] = \frac{1}{49+e^t}\left(\frac{e}{7} + \sin(\chi(t))\right), \quad t \in [1, 2], \\
\left({}^{\mathfrak{C}}_{\frac{1}{2}}\mathfrak{D}^{\frac{1}{2};t}_{1^+} + \lambda\right)\chi(t)\Big|_{t=1} = \left({}^{\mathfrak{C}}_{\frac{1}{2}}\mathfrak{D}^{\frac{1}{2};t}_{1^+} + \lambda\right)\chi(t)\Big|_{t=2} = 0, \\
\chi(1) = \chi(2).
\end{cases} \tag{4.1}$$

Now, we check for hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_4)$:
The function $\mathfrak{F}$ that satisfies the Carathéodory conditions is straightforward. We have

$$\|\mathfrak{F}(t,\chi)\| \leq \frac{1}{49+e^t}\left(\frac{e}{7} + \| \chi \|\right).$$

Then, the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold with $\varpi(t) = \frac{1}{49+e^t}$, $\varpi^* = \frac{1}{49+e} \simeq 0,0193$, and $\rho(\| \chi \|) = \frac{e}{7} + \| \chi \|$.
For each bounded set $V \subset \mathbb{R}$ and $t \in [1,2]$, we have

$$\kappa(\mathfrak{F}(t,V)) \leq \varpi(t)\kappa(V),$$

$$\kappa(\Phi_q(V)) \leq \kappa(V).$$

Thus, the hypotheses $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$ are satisfied.
Next, we check the condition (3.9). Then, we have

$$\begin{aligned}\frac{\varpi^*(g(b)-g(a))^{\ell+\theta}}{\delta^{\ell+\theta}\Gamma(\ell+1)\Gamma(\theta+1)} + \frac{|\lambda|(g(b)-g(a))^{\ell}}{\delta^{\ell}\Gamma(\ell+1)} &\leq \frac{0,0193(2-1)^{\frac{1}{2}+\frac{3}{2}}}{\left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{3}{2}}\Gamma(\frac{1}{2}+1)\Gamma(\frac{3}{2}+1)} + \frac{0.02(2-1)^{\frac{1}{2}}}{\frac{1}{2}^{\frac{1}{2}}\Gamma(\frac{1}{2}+1)} \\ &\approx 0.6553 + 0.0902 \\ &= 0.7455 < 1.\end{aligned}$$

It follows that all the conditions of Theorem 3.1 are satisfied. Consequently, the BVPLFDE (4.1) admits at least one solution on the interval $[1,2]$.

5. Conclusion

In this paper, we have studied and investigated the result of the existence of a new class of the boundary value p-Laplacian Langevin fractional differential equations. The novelty of the considered problem is that it has been investigated under the generalized Caputo proportional fractional derivative in an arbitrary Banach space. By employing the measure of the noncompactness method and Mönch's fixed point theorem, we established the existence of solutions. Lastly, a relevant example is provided to demonstrate our findings effectively.
As a potential direction for future research, we aim to extend these findings to investigate the ψ-Hilfer generalized proportional derivative and explore the Ulam–Hyers stability. In addition, we intend to extend it to the hybrid p-Laplacian Langevin equations and their inclusion.

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