Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44) 1** : 1–9.
ISSN-0037-8712
doi:10.5269/bspm.78457

On the Characterization of the $\overline{\gamma}$-Pseudo Left-Right Browder Essential Spectra of the Sum of Two Bounded Operators

Mahamed Beghdadi

ABSTRACT: In this paper, we study the stability of the $\overline{\gamma}$-pseudo left and right Browder essential spectra of bounded linear operators on Banach spaces. Firstly, we introduce new classes of operators called $\overline{\gamma}$-pseudo left and right Browder operators. Moreover, we examine the invariance of their corresponding spectral sets under Riesz perturbations. Finally, we study the $\overline{\gamma}$-pseudo semi-Browder spectra of the sum of two bounded linear operators, exploring its relation with the pseudo semi-Browder spectrum of each of these operators.

Key Words: Measure of noncompactness of Kuratowski, $\overline{\gamma}$-pseudo upper and lower semi-Browder operators, $\overline{\gamma}$-pseudo left and right Browder essential spectrum.

Contents

1. Introduction and Preliminaries

The study of pseudospectra and essential spectra of linear operators has been a central topic in functional analysis and operator theory, with wide-ranging applications in numerical analysis, quantum mechanics, and engineering (see [2,5,8,14,15]). The classical pseudospectrum provides crucial insights into the stability of operators under small perturbations, which is particularly important in numerical computations where rounding errors and approximations are inevitable. However, traditional pseudospectra may not fully capture the behavior of operators in infinite-dimensional Banach spaces, especially when dealing with non-compact perturbations. This limitation motivates the need for more refined tools, such as the so-called $\overline{\gamma}$-ε-pseudospectrum, which incorporates the Kuratowski measure of noncompactness $\overline{\gamma}$ to better handle operators in Banach spaces.

In this paper we extend the well-known notions of upper and lower semi-Browder essential spectra (see [1]) to the setting of $\overline{\gamma}$-ε-pseudospectra. This generalization provides a more nuanced understanding of operator behavior under non-compact perturbations. Moreover, it bridges the gap between classical spectral theory and modern applications where non-compact operators are prevalent.

First, let us recall some standard notations and definitions from Fredholm theory. Throughout this paper, let X and Y be infinite-dimensional Banach spaces, and let $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$) denote the set of all closed densely defined (resp. bounded) linear operators from X into Y. We write $\mathcal{K}(X,Y)$ for the subset of compact operators in $\mathcal{L}(X,Y)$.

For $T\in\mathcal{C}(X,Y)$, let $\alpha(T)$ denote the dimension of the kernel $\mathcal{N}(T)$, and let $\beta(T)$ denote the codimension of the range $\mathcal{R}(T)$ in Y. An operator $T\in\mathcal{L}(X,Y)$ is called:

(i) upper semi-Fredholm if $\mathcal{R}(T)$ is closed and $\alpha(T)<\infty$.

(ii) lower semi-Fredholm if $\mathcal{R}(T)$ is closed and $\beta(T)<\infty$.

We denote by $\Phi_+(X,Y)$ (resp. $\Phi_-(X,Y)$) the set of upper (resp. lower) semi-Fredholm operators. The set of Fredholm operators is defined as:

$$\Phi(X,Y):=\{T\in\mathcal{C}(X,Y):\alpha(T)<\infty,\ \beta(T)<\infty,\ \text{and } \mathcal{R}(T)\text{ is closed in } Y\}.$$

2020 *Mathematics Subject Classification:* 47A53, 47A55.
Submitted August 17, 2025. Published January 19, 2026

For $T \in \Phi(X,Y)$, the index of T is given by $i(T) := \alpha(T) - \beta(T)$. Clearly, if $T \in \Phi(X)$ then $i(T) < \infty$. If $T \in \Phi_+(X) \backslash \Phi(X)$, then $i(T) = -\infty$ and if $T \in \Phi_-(X) \backslash \Phi(X)$, then $i(T) = +\infty$.
Let $T \in \mathcal{L}(X)$. The operator T is said to be a Riesz operator if $\sigma_T(X) = \mathbb{C} \backslash \{0\}$. We denote by $\mathcal{R}(X)$ the set of Riesz operators. Recall that for $T \in \mathcal{L}(X)$, the ascent $a(T)$ and the descent $d(T)$ are defined by:

$$a(T) = \inf\{n \geq 0 : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}, \quad d(T) = \inf\{n \geq 0 : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}.$$

If no such n exists, then $a(T) = \infty$ (resp. $d(T) = \infty$). An operator T is called upper semi-Browder if $T \in \Phi_+(X)$, $i(T) \leq 0$, and $a(T) < \infty$. Similarly, T is called lower semi-Browder if $T \in \Phi_-(X)$, $i(T) \geq 0$, and $d(T) < \infty$. Let $\mathcal{B}^+(X)$ (resp. $\mathcal{B}^-(X)$) denote the set of upper (resp. lower) semi-Browder operators. An operator $T \in \mathcal{L}(X)$ is called semi-Browder if it is upper semi-Browder or lower semi-Browder, and we denote by $\mathcal{B}^{\pm}(X)$ the set of semi-Browder operators. An operator $T \in \mathcal{L}(X)$ is called Browder if it is both upper semi-Browder and lower semi-Browder. We respectively denote by $\mathcal{B}(X)$ and $\mathcal{B}^{\pm}(X)$ the sets of Browder and semi-Browder operators.
The corresponding spectra of an operator $T \in \mathcal{L}(X)$ are defined as follows:

$$\sigma_{\mathcal{B}^+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}^+(X)\} : \quad \text{upper semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}^-(X)\} : \quad \text{lower semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}^{\pm}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}^{\pm}(X)\} : \quad \text{semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}(X)\} : \quad \text{Browder essential spectrum.}$$

For more details on the set of Fredholm operators and Browder operators, see [11,13]. In 1997, V. Rakočević in [12] characterized the Browder essential spectrum for $T \in \mathcal{L}(X)$ as follows:

$$\sigma_{\mathcal{B}}(T) = \bigcap_{K \in \mathcal{K}(X), TK = KT} \sigma(T + K).$$

For $T \in \mathcal{C}(X,Y)$ and $D \in \mathcal{L}(X,Y)$ with $\|D\| < \varepsilon$, we say that:
(i) The operator T is a pseudo-upper semi-Fredholm operator if $T + D$ is upper semi-Fredholm,
(ii) T is a pseudo-lower semi-Fredholm operator if $T + D$ is lower semi-Fredholm,
(iii) T is a pseudo-Fredholm operator if $T + D$ is Fredholm.
For further details, see [2].

For every $\varepsilon > 0$, the pseudospectrum of a closed densely defined linear operator T is defined as:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}\right\}.$$

By convention, we write $\|(\lambda - T)^{-1}\| = \infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent, i.e., if $\lambda \in \sigma(T)$. In [5], Davies defined another equivalent of the pseudospectrum, one that is in terms of perturbations of the spectrum. In fact, for $T \in \mathcal{C}(X)$, we have

$$\sigma_\varepsilon(T) := \bigcup_{\|D\| < \varepsilon} \sigma(T + D).$$

We now recall some definitions and results that will be essential for the subsequent discussion. We begin by introducing the Kuratowski measure of noncompactness (abbreviated as MNC) for a bounded subset in a normed space. Let X be a normed space. We denote by $\mathcal{M}_X$ (resp. $\mathcal{Q}_X$) the family of all bounded (resp. relatively compact) subsets of X. For any $B \in \mathcal{M}_X$, the MNC of B, denoted by $\gamma(B)$, is defined as follows:

$$\gamma(B) = \inf\left\{\varepsilon > 0 : \; B \subset \bigcup_{i=1}^{n} S_i, \;\; S_i \subset X, \; \text{diam}(S_i) \leq \varepsilon, \; i = 1, 2, \ldots, n\right\}.$$

The following proposition outlines some key properties of the Kuratowski MNC.

Proposition 1.1 [9] Let X be a Banach space and let B_1 and $B_2 \in \mathcal{M}_X$. Then, we have the following properties:

(i) $\gamma(B_1) = 0$ if and only if, B_1 is relatively compact.

(ii) If $B_1 \subset B_2$, then $\gamma(B_1) \leq \gamma(B_2)$.

(iii) $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.

(iv) $\gamma(\lambda B_1) = |\lambda|\gamma(B_1)$, for every $\lambda \in \mathbb{C}$.

Definition 1.1 [10] Let X be a Banach space and let $T : \mathcal{D}(T) \subset X \longrightarrow X$ be a continuous operator. Let $k \geq 0$, then T is said to be k-set-contraction if for any bounded subset B of $\mathcal{D}(T)$, $T(B)$ is a bounded subset of X and $\gamma(T(B)) \leq k\gamma(B)$.

We now give the definition of the Kuratowski MNC of $T \in \mathcal{L}(X)$:

$$\overline{\gamma}(T) := \inf\left\{k \colon T \text{ is } \ k\text{-set-contraction}\right\}.$$

The Kuratowski MNC of a bounded operator T can also be defined by the following formula:

Definition 1.2 Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then,

$$\overline{\gamma}(T) = \sup_{B \in \mathcal{M}_X \backslash \mathcal{Q}_X} \frac{\gamma(T(B))}{\gamma(B)}.$$

Proposition 1.2 [6] Let X be a Banach space and let $T_1, T_2 \in \mathcal{L}(X)$. Then the following properties are satisfied.

(i) $\bar{\gamma}(T_1) = 0$ if and only if, T_1 is compact.

(ii) $\bar{\gamma}(T_1 T_2) \leq \bar{\gamma}(T_1)\bar{\gamma}(T_2)$.

(iii) If $K \in \mathcal{K}(X)$, then $\bar{\gamma}(T_1 + K) = \bar{\gamma}(T_1)$.

(iv) If $B \in \mathcal{M}_X$, then $\gamma(T_1(B)) \leq \bar{\gamma}(T_1)\gamma(B)$.

The relative-$\overline{\gamma}$-pseudospectrum of a closed densely defined operator T parameterized by $\varepsilon > 0$ on a Banach space X is defined in [3] by:

$$\sigma_{\varepsilon}^{\overline{\gamma}}(T) := \sigma(T) \cup \left\{\lambda \in \mathbb{C} : \overline{\gamma}((\lambda - T)^{-1}) > \frac{1}{\varepsilon}\right\}. \tag{1.1}$$

By convention, we write $\overline{\gamma}((\lambda - T)^{-1}) = \infty$ if $(\lambda - T)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$. We can refer the reader to [3], for more information.

In this paper, motivated by the work done in [1], we define new essential pseudospectra for bounded linear operators in Banach spaces called the $\overline{\gamma}$-pseudo upper and lower semi-Browder essential spectra. Furthermore, we study the stability of the $\overline{\gamma}$-pseudo upper and lower semi-Browder essential spectra under Riesz operator perturbations. Finally, we give the relationship between the $\overline{\gamma}$-pseudo semi-Browder spectra of the sum of two bounded linear operators and the pseudo semi-Browder spectrum of each of these operators.

The paper is organized as follows: In Section 2, we introduce the definition the $\overline{\gamma}$-pseudo upper and lower semi-Browder operators and their corresponding essential spectra of bounded linear operators in Banach spaces. We further study the stability of these spectra under Riesz operator perturbations. In Section 3, we show the relationship between the $\overline{\gamma}$-pseudo upper (resp. lower) semi-Browder spectra of the sum of two bounded linear operators and the pseudo upper (resp. lower) semi-Browder spectrum of each of these operators.

2. Stability of the $\overline{\gamma}$-Pseudo Browder Essential Spectrum

In this section, we study the stability of $\overline{\gamma}$-pseudo semi-Browder spectra under Riesz operator perturbations. We start by defining the $\overline{\gamma}$-pseudo semi-Browder operators and their corresponding spectra.

Definition 2.1 Let X be a Banach space, $\varepsilon > 0$ and $T \in \mathcal{L}(X)$. Then,

(i) T is called a $\overline{\gamma}$-pseudo upper (resp. lower) semi-Browder operator if $T + D$ is an upper (resp. lower) semi-Browder operator for all $D \in \mathcal{L}(X)$ such that $\overline{\gamma}(D) < \varepsilon$ and $TD = DT$.

(ii) T is called a $\overline{\gamma}$-pseudo semi-Browder operator if $T + D$ is a semi-Browder operator for all $D \in \mathcal{L}(X)$ such that $\overline{\gamma}(D) < \varepsilon$ and $TD = DT$.

(iii) T is called a $\overline{\gamma}$-pseudo Browder operator if $T + D$ is a Browder operator for all $D \in \mathcal{L}(X)$ such that $\overline{\gamma}(D) < \varepsilon$ and $TD = DT$.

We denote by $\mathcal{B}_{\varepsilon}^{\overline{\gamma}}(X)$, $\mathcal{B}_{\varepsilon}^{\pm,\overline{\gamma}}(X)$, $\mathcal{B}_{\varepsilon}^{+,\overline{\gamma}}(X)$ (resp. $\mathcal{B}_{\varepsilon}^{-,\overline{\gamma}}(X)$) the set of $\overline{\gamma}$-pseudo-Browder operators, $\overline{\gamma}$-pseudo semi-Browder operators, $\overline{\gamma}$-pseudo-upper semi-Browder (resp. lower semi-Browder) operators, respectively.

In general, we have:

$$\mathcal{B}_{\varepsilon}^{+}(X) \subset \mathcal{B}_{\varepsilon}^{+,\overline{\gamma}}(X),\ \mathcal{B}_{\varepsilon}^{-}(X) \subset \mathcal{B}_{\varepsilon}^{-,\overline{\gamma}}(X) \text{ and } \mathcal{B}_{\varepsilon}(X) \subset \mathcal{B}_{\varepsilon}^{\overline{\gamma}}(X).$$

Note that if $\varepsilon \to 0$, we recover the usual definitions of the upper semi-Browder, lower semi-Browder, semi-Browder and Browder essential spectra of a bounded linear operator T, respectively.

Their corresponding spectra are given in the following definition.

Definition 2.2 Let $\varepsilon > 0$ and $T \in \mathcal{L}(X)$, we define the following $\overline{\gamma}$-pseudo semi-Browder essential spectra:

$$\sigma_{\mathcal{B}1,\varepsilon}^{\overline{\gamma}}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{\varepsilon}^{+,\overline{\gamma}}(X)\} : \quad \overline{\gamma}\text{-pseudo upper semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}2,\varepsilon}^{\overline{\gamma}}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{\varepsilon}^{-,\overline{\gamma}}(X)\} : \quad \overline{\gamma}\text{-pseudo lower semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}3,\varepsilon}^{\overline{\gamma}}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{\varepsilon}^{\pm,\overline{\gamma}}(X)\} : \quad \overline{\gamma}\text{-pseudo semi-Browder essential spectrum,}$$

$$\sigma_{\mathcal{B}4,\varepsilon}^{\overline{\gamma}}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}_{\varepsilon}^{\overline{\gamma}}(X)\} : \quad \overline{\gamma}\text{-pseudo Browder essential spectrum.}$$

As an extension of the concept of pseudo Browder essential spectrum given in [12], we define the $\overline{\gamma}$-pseudo Browder essential spectrum of $T \in \mathcal{L}(X)$ as:

$$\sigma_{\mathcal{B}4,\varepsilon}^{\overline{\gamma}}(T) := \bigcap_{K \in \mathcal{K}(X), TK = KT} \sigma_{\varepsilon}^{\overline{\gamma}}(T + K).$$

Next, we present the following useful lemma.

Lemma 2.1 Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\overline{\gamma}(T) < 1$. Then, $I - T$ is invertible.

Proof: First, let us prove the surjectivity of $I - T$. We aim to show that for every $y \in X$, there exists $x \in X$ such that:

$$(I - T)x = y.$$

The equation $(I - T)x = y$ can be rewritten as: $x = Tx + y$. This suggests that x is a fixed point of the mapping $\Psi(x) = Tx + y$. Obviously, for any bounded set $M \subset X$, we have:

$$\Psi(M) = T(M) + \{y\}.$$

Using the subadditivity and monotonicity of γ (see Proposition 1.1), we get

$$\gamma(\Psi(M)) = \gamma(T(M) + \{y\}) \le \gamma(T(M)) + \gamma(\{y\}).$$

Since $\{y\}$ is a singleton, $\gamma(\{y\}) = 0$. Thus, $\gamma(\Psi(M)) \leq \gamma(T(M))$. By Proposition 1.2, we have $\gamma(T(M)) \leq \overline{\gamma}(T)\gamma(M)$. Since $\overline{\gamma}(T) < 1$, it follows that Ψ is $\overline{\gamma}(T)$-set-contractive.

Now, by Darbo's Fixed point theorem, Ψ has a fixed point $x \in X$, satisfying:

$$x = Tx + y.$$

This implies that $I - T$ is surjective.
Moreover, since $\overline{\gamma}(T) < 1$, then by applying Lemma 2.11 and Theorem 3.2 in [4], we infer that $I - T$ is a Fredholm operator of index zero. This result combined with the fact that $I - T$ is surjective enables us to deduce that $I - T$ is injective. Consequently, $I - T$ is invertible. □

Theorem 2.1 Let X be a Banach space, $T \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,
(i) If $S \in \mathcal{R}(X)$ and $TS = ST$, then for all $D \in \mathcal{L}(X)$ such that $\overline{\gamma}(D) < \varepsilon$ and $DS = SD$, we have:

$$\sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T + S), \quad i \in \{1, \dots, 4\}.$$

(ii) If $K \in \mathcal{K}(X)$ and $TK = KT$, then for all $D \in \mathcal{L}(X)$ such that $\overline{\gamma}(D) < \varepsilon$ and $DK = KD$, we have:

$$\sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T + K), \quad i \in \{1, \dots, 4\}.$$

Proof: (i) Firstly, we prove that $\sigma^{\overline{\gamma}}_{\mathcal{B}_1,\varepsilon}(T + S) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_1,\varepsilon}(T)$. Let $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}_1,\varepsilon}(T)$. Then $\lambda - T \in \mathcal{B}^{+,\overline{\gamma}}_{\varepsilon}(X)$. It follows that

$$\lambda - (T + D) \in \mathcal{B}_+(X), \quad \forall \overline{\gamma}(D) < \varepsilon \text{ and } TD = DT.$$

Since $(T + D)S = S(T + D)$, then by using [[12], Corollary 2], we deduce that

$$\lambda - (T + S + D) \in \mathcal{B}_+(X), \quad \forall \overline{\gamma}(D) < \varepsilon \text{ and } (T + S)D = D(T + S).$$

Therefore, $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}_1,\varepsilon}(T + S)$. The converse inclusion follows by symmetry.

Next, we show the same inclusion for $i = 2$. Let $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T)$. Then $\lambda - T \in \mathcal{B}^{-,\overline{\gamma}}_{\varepsilon}(X)$. This implies that

$$\lambda - (T + D) \in \mathcal{B}^-(X), \quad \forall \overline{\gamma}(D) < \varepsilon \text{ and } TD = DT.$$

Since $(T + D)S = S(T + D)$, it yields from [[12], Corollary 2] that

$$\lambda - (T + S + D) \in \mathcal{B}^-(X), \quad \forall \overline{\gamma}(D) < \varepsilon \text{ and } (T + S)D = D(T + S).$$

Consequently, $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T + S)$. The inverse inclusion can be checked by using a similar reasoning.
For $i = 3, 4$, we use the fact that

$$\sigma^{\overline{\gamma}}_{\mathcal{B}3,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T) \cap \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T) \subseteq \sigma^{\overline{\gamma}}_{\mathcal{B}4,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T),$$

to deduce that

$$\sigma^{\overline{\gamma}}_{\mathcal{B}3,\varepsilon}(T + S) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T + S) \cap \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T + S) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T) \cap \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}3,\varepsilon}(T), \quad (2.1)$$

$$\sigma^{\overline{\gamma}}_{\mathcal{B}4,\varepsilon}(T + S) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T + S) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T + S) = \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}4,\varepsilon}(T). \quad (2.2)$$

(ii) As $\mathcal{K}(X) \subset \mathcal{R}(X)$ and by using assertion (i), we obtain that

$$\sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T) = \sigma^{\overline{\gamma}}_{\mathcal{B}_i,\varepsilon}(T + K), \quad i \in \{1, \dots, 4\}.$$ □

Theorem 2.2 Let $\varepsilon > 0$, $T \in \mathcal{L}(X)$, and $S \in \mathcal{R}(X)$ such that S is commuting with $T + K$ for all $K \in \mathcal{K}(X)$. If $\overline{\gamma}(S) < \varepsilon$, then there exist ε_0 and ε_1 such that $0 < \varepsilon_0 < \varepsilon < \varepsilon_1$ satisfying:

$$\sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_0}(T + S) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_1}(T + S).$$

Proof: Firstly, we will show that there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon$ and $\sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_0}(T+S) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T)$. To do so, we assume that $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T)$ and $\lambda \notin \sigma(T+K)$. By using Equation (1.1), we infer that $\lambda \notin \{\lambda \in \mathbb{C} : \overline{\gamma}((T+K-\lambda I)^{-1}) > \frac{1}{\varepsilon}, \forall K \in \mathcal{K}(X)\}$. This implies that $\overline{\gamma}((T+K-\lambda I)^{-1}) \leq \frac{1}{\varepsilon}$.
Now, we can write the operator $T+K+S-\lambda$ in the following form:

$$T+K+S-\lambda = (T+K-\lambda)\left(I+(T+K-\lambda)^{-1}S\right). \tag{2.3}$$

By using the fact that

$$\overline{\gamma}((T+K-\lambda I)^{-1}S) \leq \overline{\gamma}((T+K-\lambda I)^{-1})\overline{\gamma}(S) < 1,$$

it follows from Lemma 2.1 that $I+(T+K-\lambda)^{-1}S$ is invertible, and its inverse can be written as:

$$(T+K+S-\lambda)^{-1} = \left(I+(T+K-\lambda)^{-1}S\right)^{-1}(T+K-\lambda)^{-1}.$$

Since

$$\overline{\gamma}(\left(I+(T+K-\lambda)^{-1}S\right)^{-1}) \leq \frac{\varepsilon}{\varepsilon-\overline{\gamma}(S)},$$

we conclude that

$$\overline{\gamma}((T+K+S-\lambda)^{-1}) \leq \overline{\gamma}((T+K-\lambda)^{-1}) \cdot \frac{\varepsilon}{\varepsilon-\overline{\gamma}(S)}.$$

Consequently

$$\overline{\gamma}((T+K+S-\lambda)^{-1}) \leq \frac{1}{\varepsilon-\overline{\gamma}(S)}.$$

By choosing $\varepsilon_0 = \varepsilon - \overline{\gamma}(S)$, where $0 < \varepsilon_0 < \varepsilon$, we conclude that by using Equation (1.1) that $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_0}(T+S)$. Hence,

$$\sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_0}(T+S) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T).$$

Next, we will show that there exists ε_1 such that $0 < \varepsilon < \varepsilon_1$ and $\sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T) \subset \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon_1}(T+S)$. Let $\varepsilon_1 = \varepsilon + \overline{\gamma}(S)$. For

$$\lambda \notin \{\lambda \in \mathbb{C} : \overline{\gamma}((T+K+S-\lambda)^{-1}) > \frac{1}{\varepsilon_1}\},$$

we have

$$\overline{\gamma}((T+K+S-\lambda)^{-1}) \leq \frac{1}{\varepsilon_1}.$$

The operator $T+K-\lambda$ can be written as follows:

$$T+K-\lambda = (T+K+S-\lambda)\left(I-(T+K+S-\lambda)^{-1}S\right).$$

As

$$\overline{\gamma}((T+K+S-\lambda)^{-1}S) < 1,$$

it yields from Lemma 2.1 that $I-(T+K+S-\lambda)^{-1}S$ is an invertible operator. Using Equation (2.3), we have

$$(T+K-\lambda)^{-1} = \left(I-(T+K+S-\lambda)^{-1}S\right)^{-1}(T+K+S-\lambda)^{-1}.$$

Since $(S+K)T = T(S+K)$, we obtain that

$$\overline{\gamma}((I+(T+K+S-\lambda)^{-1}S)^{-1}) < \frac{\varepsilon_1}{\varepsilon_1-\overline{\gamma}(S)}.$$

Thus,

$$\overline{\gamma}((T+K-\lambda)^{-1}) \leq \frac{1}{\varepsilon}.$$

Consequently, $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}_4,\varepsilon}(T)$. $\square$

3. $\overline{\gamma}$-Pseudo Semi-Browder Essential Spectra of the Sum of Two Operators

This section is devoted to the study of the $\overline{\gamma}$-pseudo semi-Browder essential spectra of the sum of two bounded linear operators under commuting Riesz operator perturbations. Our first result is the following.

Theorem 3.1 Let T, S, and D be three operators in $\mathcal{L}(X)$ such that $TS = ST$, $TD = DT$, and $T(S+D) \in \mathcal{R}(X)$, and let $\varepsilon > 0$. Then:

$$\sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T+S)\backslash\{0\} = \sigma_{\mathcal{B}1}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)\backslash\{0\}.$$

Proof: For $\lambda \in \mathbb{C}$, we can write:

$$(\lambda - T)(\lambda - S - D) = T(S+D) + \lambda(\lambda - T - S - D), \tag{3.1}$$

$$(\lambda - S - D)(\lambda - T) = (S+D)T + \lambda(\lambda - T - S - D). \tag{3.2}$$

Suppose that $\lambda \neq 0$ such that $\lambda \notin \sigma_{\mathcal{B}1}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)$. Then $(\lambda - T) \in \mathcal{B}_1(X)$ and $(\lambda - S) \in \mathcal{B}^{+,\overline{\gamma}}_{\varepsilon}(X)$. It follows that

$$(\lambda - T) \in \mathcal{B}^+(X) \quad \text{and} \quad (\lambda - S - D) \in \mathcal{B}^+(X),\ \forall \overline{\gamma}(D) < \varepsilon,\ SD = DS. \tag{3.3}$$

Since $T(S+D) = (S+D)T$, we have

$$(\lambda - T)(\lambda - S - D) = (\lambda - S - D)(\lambda - T). \tag{3.4}$$

By applying [[7], Theorem 7.9.2] to Equations (3.3), (3.4), we obtain that

$$(\lambda - T)(\lambda - S - D) \in \mathcal{B}^+(X),\ \forall \overline{\gamma}(D) < \varepsilon \text{ and } SD = DS.$$

So, by using Equation (3.1), we conclude that

$$T(S+D) + \lambda(\lambda - T - S - D) \in \mathcal{B}^+(X). \tag{3.5}$$

Since $TS = ST$, $TD = DT$, and $SD = DS$, we have

$$T(S+D)(\lambda - T - S - D) = (\lambda - T - S - D)T(S+D). \tag{3.6}$$

Taking into account that $T(S+D) \in \mathcal{R}(X)$, then by using [[12], Corollary 2] to Equations (3.5) and (3.6), we get

$$\lambda - T - S - D \in \mathcal{B}^+(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon, \text{ and } (T+S)D = D(S+T).$$

This implies $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T+S)\backslash\{0\}$. Consequently,

$$\sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T+S)\backslash\{0\} \subset \sigma_{\mathcal{B}1}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)\backslash\{0\}.$$

For the converse inclusion, let $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T+S)\backslash\{0\}$. Then $(\lambda - T - S) \in \mathcal{B}^+_{\varepsilon}(X)$. This shows that

$$(\lambda - T - S - D) \in \mathcal{B}^+(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon, \text{ and } (\lambda - T - S)D = D(\lambda - T - S).$$

Since $T(S+D) \in \mathcal{R}(X)$ and $(S+D)T \in \mathcal{R}(X)$, then the use of [[12], Corollary 2] to Equation (3.6) leads to

$$T(S+D) + \lambda(\lambda - T - S - D) \in \mathcal{B}^+(X),$$

and

$$(S+D)T + \lambda(\lambda - T - S - D) \in \mathcal{B}^+(X).$$

So, by using Equations (3.1) and (3.2), we infer that

$$(\lambda - T)(\lambda - S - D) \in \mathcal{B}^+(X) \quad \text{and} \quad (\lambda - S - D)(\lambda - T) \in \mathcal{B}^+(X).$$

Again, by Equation (3.4) and Theorem 7.9.2 in [[7], page 276], we deduce that

$$\lambda - T \in \mathcal{B}^+(X) \quad \text{and} \quad (\lambda - S - D) \in \mathcal{B}^+(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon,\ (\lambda - S)D = D(\lambda - S).$$

Hence, $\lambda \notin \sigma_{\mathcal{B}1}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)\backslash\{0\}$. Therefore,

$$\sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(T+S)\backslash\{0\} = \sigma_{\mathcal{B}1}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)\backslash\{0\}.$$

□

Theorem 3.2 Let T, S, and D be three operators in $\mathcal{L}(X)$ and let $\varepsilon > 0$ such that $\overline{\gamma}(D) < \varepsilon$, $TS = ST$, $TD = DT$, and $T(S+D) \in \mathcal{R}(X)$. Then

$$\sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T+S)\backslash\{0\} = \sigma_{\mathcal{B}2}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(S)\backslash\{0\}.$$

Proof: Suppose that $\lambda \notin [\sigma_{\mathcal{B}2}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(S)]\backslash\{0\}$. Then

$$(\lambda - T) \in \mathcal{B}^-(X) \quad \text{and} \quad (\lambda - S - D) \in \mathcal{B}^-(X),\ \forall \overline{\gamma}(D) < \varepsilon \text{ and } SD = DS.$$

Thanks to the use of [[7], Theorem 7.9.2] to Equation (3.4), then we obtain that

$$(\lambda - T)(\lambda - S - D) \in \mathcal{B}^-(X).$$

This combined with the use of Equation (3.1) allows us to infer that

$$T(S+D) + \lambda(\lambda - T - S - D) \in \mathcal{B}^-(X).$$

Again, since $T(S+D) \in \mathcal{R}(X)$, then by applying [[12], Corollary 2] to Equation (3.6) we conclude that

$$\lambda - T - S - D \in \mathcal{B}^-(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon,\ \text{and } (T+S)D = D(T+S).$$

Therefore $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T+S)\backslash\{0\}$. Conversely, let $\lambda \notin \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(T+S)\backslash\{0\}$. Then

$$(\lambda - T - S - D) \in \mathcal{B}^-(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon,\ \text{and } (T+S)D = D(T+S).$$

Since $T(S+D) \in \mathcal{R}(X)$ and $(S+D)T \in \mathcal{R}(X)$, it follows from [[12], Corollary 2] and Equation (3.6) that

$$T(S+D) + \lambda(\lambda - T - S - D) \in \mathcal{B}^-(X),$$
$$(S+D)T + \lambda(\lambda - T - S - D) \in \mathcal{B}^-(X).$$

Due to Equations (3.1) and (3.2), we get

$$(\lambda - T)(\lambda - S - D) \in \mathcal{B}^-(X) \quad \text{and} \quad (\lambda - S - D)(\lambda - T) \in \mathcal{B}^-(X).$$

Combining Equation (3.4) and [[7], Theorem 7.9.2], we deduce that

$$\lambda - T \in \mathcal{B}^-(X) \quad \text{and} \quad (\lambda - S - D) \in \mathcal{B}^-(X),\ \forall D \in \mathcal{L}(X),\ \overline{\gamma}(D) < \varepsilon,\ \text{and } (\lambda - S)D = D(\lambda - S).$$

Consequently, $\lambda \notin [\sigma_{\mathcal{B}2}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(S)]\backslash\{0\}$. □

At the end of this section, we state the following corollary.

Corollary 3.1 Let T, S, and D be three operators in $\mathcal{L}(X)$ such that $TS = ST$, $TD = DT$, and $T(S+D) \in \mathcal{R}(X)$, and let $\varepsilon > 0$. Then,

(i) $\sigma^{\overline{\gamma}}_{\mathcal{B}3,\varepsilon}(T+S)\backslash\{0\} = \Big(\sigma_{\mathcal{B}3}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}3,\varepsilon}(S)\Big) \cup \Big(\sigma_{\mathcal{B}1}(T) \cap \sigma^{\overline{\gamma}}_{\mathcal{B}2,\varepsilon}(S)\Big) \cup \Big(\sigma_{\mathcal{B}2}(T) \cap \sigma^{\overline{\gamma}}_{\mathcal{B}1,\varepsilon}(S)\Big)\backslash\{0\}$.

(ii) $\sigma^{\overline{\gamma}}_{\mathcal{B}4,\varepsilon}(T+S)\backslash\{0\} = \sigma_{\mathcal{B}4}(T) \cup \sigma^{\overline{\gamma}}_{\mathcal{B}4,\varepsilon}(S)\backslash\{0\}$.

Proof: The two results follow immediately form the use of Theorem 3.1 and Theorem 3.2 to Equations (2.1) and (2.2). □

References

1. F. Abdmouleh, B. Elgabeur.: *On the pseudo semi-Browder essential spectra and application to* 2×2 *block operator matrices*, Filomat 37:19 (2023), 6373–6386.

2. A. Ammar and A. Jeribi.: *A characterization of the essential pseudo-spectra and application to a transport equation*, Extracta Math. 28 (1) (2013) 95–112.

3. M. Beghdadi and B. Krichen.: *Some Results on* γ-ε-*Condition Pseudospectrum*. To appear in Facta Universitatis Series Mathematics and Informatics (2025).

4. W. Chaker, A. Jeribi, B. Krichen.: *Demicompact linear operators, essential spectrum and some perturbation results*. Math. Nachr. 288: 1476–1486 (2015).

5. E. B. Davies.: *Spectral Theory and Differential Operators*. Cambridge University Press, Cambridge (1995).

6. D. E. Edmunds, W. D. Evans.: *Spectral theory and differential operators*. Oxford University Press, 2018.

7. R. Harte.: *Invertibility and singularity for bounded linear operators*, Marcel Dekker, New York, 1988:

8. D. Hinrichsen, A. J. Pritchard.: *Robust stability of linear evolution operators on Banach spaces*. SIAM J. Control Optim. 32, 1503–1541 (1994).

9. K. Kuratowski.: *Sur les espaces complets*. Fundamenta mathematicae, 1(15) 301-309, 1930.

10. K. Kuratowski.: *Topology*: Volume I, volume 1. Elsevier, 2014.

11. V. Müller.: *Spectral Theory of Linear Operator and Spectral Systems in Banach Algebras, Operator Theory: Advances and Applications*. Birkhäuser, Verlag, Basel-Boston-Berlin (2007).

12. V. Rakočević.: *Semi-Browder operators and perturbations*. Stud. Math. 112(2) (1997) 131-137.

13. M. Schechter.: *Principles of functional analysis*. Graduate studies in Mathematics, vol 36, American Mathematical Society, (2002).

14. L. N. Trefethen.: *Pseudospectra of Matrices, Numerical Analysis*, Pitman Res. Notes Math. Ser., 260 (1992) 234–266.

15. J. Varah.: *The computation of bounds for the invariant subspaces of a general matrix operator*, cs 66. Computer Science Dept., Stanford University (Ph.D thesis), 1967.

Mahamed Beghdadi,

Faculty of Natural and Life Sciences,

University of Saida-Dr. Moulay Tahar. B.P. 138. Nasr, Saida 20000,

Algeria.

E-mail address: mahamedbeghdadi32@gmail.com