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Study of Fractional Caputo Derivatives Operator by Symmetric Kernel

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ABSTRACT: The current study investigates the new type of Caputo fractional derivative operator with a symmetric kernel. We also examine some important properties related to this new type of Caputo fractional derivative operator.

Key Words: Caputo fractional integral operator, fractional calculus, hypergeometric function.

Contents

1. Introduction

Classical mechanics and mathematical physics rises to Picard-Fuchs equations and these equations are solvable in terms of hypergeometric functions [9], and the monodromy of these functions plays vital role in to elaborate properties of the solutions. Some combinatorial identities, especially which involving binomial and related coefficients, are particular cases of hypergeometric identities [7,12,13]. Hypergeometric functions are crucial in the evaluation of Watson integrals, which are used to characterize basic lattice walks. These functions are potentially valuable in solving more complex problems involving restricted lattice walks. Inspire by the importance of hypergeometric functions, here we define a new type hypergeometric function by use of symmetric kernel.

Definition 1.

$$\mathbf{F}_{(a,b)}^{(c,d)}(p,q,r;z)=\sum_{n=0}^{\infty}\frac{(p)_n(q)_n}{(q-m)_n}\frac{B_{(a,b)}^{(c,d)}(q-m+n,r-q+m)}{B(q-m,r-q+m)}\frac{z^n}{n!}. \tag{1.1}$$

Here, $\Re(r)>\Re(q)>m$, $d\geq 0$, $|z|<1$*; and* $B_{(a,b)}^{(c,d)}(x_1,x_2)$ *is generalised beta function as symmetric kernel [8].*

Fractional calculus operators extend the concept of differentiation and integration to non-integer orders and they include fractional derivatives and integrals. They are used to formulate and solve fractional differential equations. This type of equations arise in many fields like physics, biology and engineering where the system dynamics are not fully described by classical differential equations. There are several types of fractional calculus operators like, Riemann-Liouville fractional integral and derivative operators, the Caputo fractional derivative operator, the Pathway fractional integral operator. To study the properties of Caputo fractional derivative operator [1,2,3,4], here we define a new type Caputo fractional derivative operator by use of symmetric kernel.

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Definition 2.

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \frac{1}{\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right)\frac{d^m}{dt^m}f(t)dt. \tag{1.2}$$

where, $\min\{\Re(a),\Re(b)\} > 0$, $\Re(d) > 0$ $,m-1 < \Re(u) < m$, $m \in N$; *and* $E^c_{a,b}(z)$ *is the* 3*-parameter Mittag-Leffler function [5,6].*

Remark 1. *(i) If we consider* $a = b = c = 1$ *in (1.2), the new type Caputo fractional derivative operator reduced to extended Caputo fractional derivative operator.*

$$\tilde{\mathbf{D}}^{u,(1,d)}_{z,(1,1)}[f(z)] = \tilde{\mathbf{D}}^{u,d}_{z}[f(z)]. \tag{1.3}$$

(ii) If we consider $a = b = c = 1$ *and* $d = 0$ *in (1.2), the new type Caputo fractional derivative operator reduced to the classical Caputo fractional derivative operator.*

$$\tilde{\mathbf{D}}^{u,(0)}_{z,(1,1)}[f(z)] = \tilde{\mathbf{D}}^{u}_{z}[f(z)]. \tag{1.4}$$

2. Main Results

Theorem 1. *The following result hold true:*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^r] = \frac{\Gamma(r+1)}{\Gamma(r-u+1)}\frac{B^{(c,d)}_{(a,b)}(r-m+1,m-u)}{B(r-m+1,m-u)}z^{r-u}. \tag{2.1}$$

provided $m-1 < \Re(u) < m$, $\Re(u) < \Re(r)$. *Then*

Proof. By the definition of new type Caputo fractional derivative operator

$$\begin{aligned}\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^r] &= \frac{1}{\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right)\frac{d^m}{dt^m}t^r dt\\ &= \frac{1}{\Gamma(m-u)}[r(r-1)(r-2).....(r-m+1)]\int_0^z (z-t)^{m-u-1}E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right)t^{r-m}dt\\ &= \frac{\Gamma(r+1)}{\Gamma(r-m+1)\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right)t^{r-m}dt.\end{aligned} \tag{2.2}$$

By substituting $t = xz$ in (2.2), we have the equation

$$\begin{aligned}&\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^r] =\\ &\frac{\Gamma(r+1)}{\Gamma(r-m+1)\Gamma(m-u)}z^{r-u}\int_0^1 x^{r-m}(1-x)^{m-u-1}E^c_{a,b}\left(\frac{-d}{x(1-x)}\right)dx.\end{aligned} \tag{2.3}$$

Then, using the definition of extended beta function as symmetric kernel

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^r] = \frac{\Gamma(r+1)}{\Gamma(r-m+1)\Gamma(m-u)}z^{r-u}B^{(c,d)}_{(a,b)}(r-m+1,m-u). \tag{2.4}$$

Again by the use of basic properties of Gamma and beta function, we have our result of Theorem (1).

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^r] = \frac{\Gamma(r+1)}{\Gamma(r-u+1)}\frac{B^{(c,d)}_{(a,b)}(r-m+1,m-u)}{B(r-m+1,m-u)}z^{r-u}. \tag{2.5}$$

□

Theorem 2. *The following result is valid when f(z) is an analytic function inside the disk* $\mid z \mid < \delta$, *and possesses the expansion* $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \sum_{n=0}^{\infty} x_n \tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^n], \tag{2.6}$$

provided that $m-1 < \Re(u) < m$.

Proof. By the definition of new type Caputo fractional derivative operator

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right) \frac{d^m}{dt^m} f(t)dt. \tag{2.7}$$

By the use, series expansion of given f,

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right) \frac{d^m}{dt^m} \left(\sum_{n=0}^{\infty} x_n t^n\right) dt. \tag{2.8}$$

Interchanging the order of integration and summation due to uniformly convergence and absolutely convergence of the power series,

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \sum_{n=0}^{\infty} x_n \left[\frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E^c_{a,b}\left(\frac{-dz^2}{t(z-t)}\right) \frac{d^m}{dt^m} t^n dt\right]. \tag{2.9}$$

Finally, by the definition of new Caputo fractional derivative operator, we get result of Theorem (2).

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[f(z)] = \sum_{n=0}^{\infty} x_n \tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^n]. \tag{2.10}$$

□

Theorem 3. *The following result holds true, when f(z) is a analytic function in the disc* $\mid z \mid < \delta$, *having the series expansion* $f(z) = \sum_{n=0}^{\infty} x_n z^n$. *Then*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e-1} f(z)] = \frac{\Gamma(e)}{\Gamma(e-u)} z^{\beta-u-1} \sum_{n=0}^{\infty} x_n \frac{(e)_n}{(e-m)_n} \frac{B^{(c,d)}_{(a,b)}(e-m+n, m-u)}{B(e-m, m-u)} z^n, \tag{2.11}$$

provided $m-1 < \Re(u) < m < \Re(\beta)$.

Proof. From the previous theorem (2), we have:

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e-1} f(z)] = \sum_{n=0}^{\infty} x_n \tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e+n-1}]. \tag{2.12}$$

Again by the use of Theorem (1), we have:

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e-1} f(z)] = \sum_{n=0}^{\infty} x_n \frac{\Gamma(e+n)}{\Gamma(e+n-u)} \frac{B^{(c,d)}_{(a,b)}(e-m+n, m-u)}{B(e-m+n, m-u)} z^{e-u-1+n}. \tag{2.13}$$

By using the basic property of Gamma function $(x)_n = \frac{\Gamma(x+n)}{\Gamma(a)}$, we get:

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e-1} f(z)] = \frac{\Gamma(e)}{\Gamma(e-u)} z^{e-u-1} \sum_{n=0}^{\infty} x_n \frac{(e)_n}{(e-u)_n} \frac{B^{(c,d)}_{(a,b)}(e-m+n, m-u)}{B(e-m+n, m-u)} z^n. \tag{2.14}$$

Again by the use of basic properties of Gamma and beta function, we get the result of Theorem (3).

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^{e-1}f(z)] = \frac{\Gamma(e)}{\Gamma(e-u)} z^{e-u-1} \sum_{n=0}^{\infty} x_n \frac{(e)_n}{(e-m)_n} \frac{B^{(c,d)}_{(a,b)}(e-m+n, m-u)}{B(e-m, m-u)} z^n. \tag{2.15}$$

□

Theorem 4. *Mellin transform of new type Caputo fractional derivative operator defined as (1.2) is given by*

$$\boldsymbol{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e]; s] = \frac{\Gamma(e+1)\Gamma^{(a,b)}_{0,c}(s)}{\Gamma(e-m+1)\Gamma(m-u)} B(m-u+s, e-m+s+1) z^{e-u}. \tag{2.16}$$

provided, $\Re(e) > m-1$ *and* $s > 0$*. Then,*

Proof. By the use of Mellin transform

$$\mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e]; s] = \int_0^{\infty} d^{s-1} \tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e] \tilde{d}d. \tag{2.17}$$

from the result of Theorem (1)

$$\mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e]; s] = \int_0^{\infty} d^{s-1} \left[\frac{\Gamma(e+1)}{\Gamma(e-u+1)} \frac{B^{(c,d)}_{(a,b)}(m-u, e-m+1)}{B(m-u, e-m+1)} z^{e-u} \right] \tilde{d}d; \tag{2.18}$$

After some manupalations,

$$\begin{aligned} &\mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e]; s] = \\ &\frac{\Gamma(e+1) z^{e-u}}{\Gamma(e-u+1) B(m-u, e-m+1)} \int_0^{\infty} d^{s-1} B^{(c,d)}_{(a,b)}(m-u, \lambda-m+1) \tilde{d}d. \end{aligned} \tag{2.19}$$

Now, using the result,

$$\int_0^{\infty} d^{(s-1)} B^{(c,d)}_{(a,b)}(x_1, x_2) \tilde{d}d = B(x_1+s, x_2+s) \Gamma^{(a,b)}_{0,c}(s), \tag{2.20}$$

provided $d \geq 0, \Re(x_1+s) > 0, \Re(x_2+s) > 0, \Re(a) > 0, \Re(b) > 0$ and $\Re(s) > 0$, then:

$$\mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^e]; s] = \frac{\Gamma(e+1)\Gamma^{(a,b)}_{0,c}(s)}{\Gamma(e-m+1)\Gamma(m-u)} B(m-u+s, e-m+s+1) z^{e-u}. \tag{2.21}$$

□

Theorem 5. *Mellin transform of new type Caputo fractional derivative operator defined as (1.2) is given by*

$$\boldsymbol{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[(1-z)^{-e}]; s] = \frac{\Gamma^{(a,b)}_{0,c}(s) z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} B(m-u+s, n+s+1)(e)_{n+m} \frac{z^n}{n!}. \tag{2.22}$$

provided $s > 0$ *and* $| z |< 1$.

Proof. By using the binomial identity $(1-z)^{-e} = \sum_{n=0}^{\infty} (e)_n \frac{z^n}{n!}$ and use Theorem (4), we get:

$$\begin{aligned} \mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[(1-z)^{-e}]; s] &= \mathbf{M}\left[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)} \left(\sum_{n=0}^{\infty} (e)_n \frac{z^n}{n!} \right); s \right] \\ &= \sum_{n=0}^{\infty} \frac{(e)_n}{n!} \mathbf{M}[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[z^n]; s] \\ = \sum_{n=m}^{\infty} \frac{(e)_n}{n!} &\left[\frac{\Gamma(n+1)\Gamma^{(a,b)}_{0,c}(s)}{\Gamma(n-m+1)\Gamma(m-u)} B(m-u+s, n-m+s+1) z^{n-u} \right]. \end{aligned} \tag{2.23}$$

Rearranging the terms

$$\mathbf{M}\big[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[(1-z)^{-e}];s\big]=\frac{\Gamma^{(a,b)}_{0,c}(s)z^{-u}}{\Gamma(m-u)}\sum_{n=m}^{\infty}\frac{B(m-u+s,n-m+s+1)}{\Gamma(n-m+1)}(\alpha)_n z^n. \tag{2.24}$$

By putting $n=n+m$, we get our result.

$$\mathbf{M}\big[\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[(1-z)^{-e}];s\big]=\frac{\Gamma^{(a,b)}_{0,c}(s)z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B(m-u+s,n+s+1)(e)_{n+m}\frac{z^n}{n!}. \tag{2.25}$$

□

3. Applications

As we know fractional calculus is mathematical tool that has many applications in science and engineering. In this section, we have found some applications of new type Caputo fractional derivative operator by means of symmetric kernel.

Theorem 6. *Generating function for new type hypergeometric function is given by*

$$\mathbf{F}^{(c,d)}_{(a,b)}(l,k,u;z)=\frac{\Gamma(u)}{\Gamma(k)}z^{1-u}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}[z^{k-1}(1-z)^{-l}]. \tag{3.1}$$

provided $m-1<\Re(k-u)<m<\Re(k)$ *and* $|z|<1$.

Proof. From the binomial identity $(1-z)^{-l}=\sum_{n=0}^{\infty}(l)_n\frac{z^n}{n!}$ and (1), we have:

$$\begin{aligned}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}[z^{k-1}(1-z)^{-l}]&=\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}\left[z^{k-1}\left(\sum_{n=0}^{\infty}(l)_n\frac{z^n}{n!}\right)\right]\\&=\sum_{n=0}^{\infty}\frac{(l)_n}{n!}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}[z^{k-1+n}]\\&=\sum_{n=0}^{\infty}\frac{(l)_n}{n!}\frac{\Gamma(k+n)}{\Gamma(u+n)}\frac{B^{(c,d)}_{(a,b)}(k-m+n,m-k+u)}{B(k-m+n,m-k+u)}z^{u-1+n}.\end{aligned} \tag{3.2}$$

By using basic properties of Gamma and beta function we have:

$$\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}[z^{k-1}(1-z)^{-l}]=\frac{\Gamma(k)}{\Gamma(u)}z^{u-1}\sum_{n=0}^{\infty}\frac{(k)_n(l)_n}{(k-m)_n}\frac{B^{(c,d)}_{(a,b)}(k-m+n,m-k+u)}{B(k-m,m-k+u)}\frac{z^n}{n!}. \tag{3.3}$$

By means of new hypergeometric function and rearrangement of terms, we have our result of Theorem (6).

$$\mathbf{F}^{(c,d)}_{(a,b)}(l,k,u;z)=\frac{\Gamma(u)}{\Gamma(k)}z^{1-u}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}[z^{k-1}(1-z)^{-l}]. \tag{3.4}$$

□

Theorem 7. *Generating relation for the new type hypergeometric function holds true:*

$$\sum_{n=0}^{\infty}\frac{(e)_n}{n!}\mathbf{F}^{(c,d)}_{(a,b)}(e+n,k,u;z)t^n=(1-t)^{-e}\mathbf{F}^{(c,d)}_{(a,b)}\left(e,k,u;\frac{z}{(1-t)}\right), \tag{3.5}$$

provided $m-1<\Re(k-u)<m<\Re(k)$ *and* $|z|<\min(1,|1-t|)$.

Proof. By rearranging binomial identity:
$[(1-z)-t]^{-e} = (1-t)^{-e}\left(1-\frac{z}{(1-t)}\right)^{-e}$.
After re-arranging its terms, we recover:
$(1-z)^{-e}\left(1-\frac{t}{1-z}\right)^{-e} = (1-t)^{-e}[\left(1-\frac{z}{(1-t)}\right)^{-e}$.
Using binomial identity $\left(1-\frac{t}{1-z}\right)^{-e}$, we have:

$$(1-z)^{-e}\sum_{n=0}^{\infty}\frac{(e)_n}{n!}\left(\frac{t}{1-z}\right)^n = (1-t)^{-e}\left(1-\frac{z}{(1-t)}\right)^{-e}. \tag{3.6}$$

Now, by multiplication of B.H.S by z^{k-1}, we have

$$z^{k-1}(1-z)^{-e}\sum_{n=0}^{\infty}\frac{(e)_n}{n!}\left(\frac{t}{1-z}\right)^n = z^{k-1}(1-t)^{-e}\left(1-\frac{z}{(1-t)}\right)^{-e}. \tag{3.7}$$

By applying of new type Caputo fractional derivative operator $\tilde{\mathbf{D}}z,(a,b)^{k-u,(c,d)}$ to B.H.S of the above equation

$$\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}\left[\sum_{n=0}^{\infty}\frac{(e)_n}{n!}(1-z)^{-e}\left(\frac{t}{1-z}\right)^n z^{k-1}\right] = (1-t)^{-e}\tilde{\mathbf{D}}^{k-u,(r)}_{z,(a,b)}\left[z^{k-1}\left(1-\frac{z}{(1-t)}\right)^{-e}\right]. \tag{3.8}$$

Interchanging order of summation and operator:

$$\sum_{n=0}^{\infty}\frac{(e)_n}{n!}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}\left[z^{k-1}(1-z)^{-e-n}\right]t^n = (1-t)^{-e}\tilde{\mathbf{D}}^{k-u,(c,d)}_{z,(a,b)}\left[z^{k-1}\left(1-\frac{z}{(1-t)}\right)^{-e}\right]. \tag{3.9}$$

By using Theorem (6), we have result of Theorem (7).

$$\sum_{n=0}^{\infty}\frac{(e)_n}{n!}\mathbf{F}^{(c,d)}_{(a,b)}(e+n,k,u;z)t^n = (1-t)^{-e}\mathbf{F}^{(c,d)}_{(a,b)}\left(e,k,u;\frac{z}{(1-t)}\right). \tag{3.10}$$

□

4. Examples

In this section, we have found several new type Caputo derivatives of some known special functions.

Example 1. *New type Caputo derivatives of exponential function is given as:*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[e^z] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B^{(c,d)}_{(a,b)}(m-u,n+1)\frac{z^n}{n!}, \tag{4.1}$$

provided that $z \in C$.

Proof. By power series representation of $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ and using results (1) and (2),we have:

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[e^z] = \sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\mathbf{D}}z,(a,b)^{u,(c,d)}[z^n] = \sum_{n=m}^{\infty}\frac{1}{n!}\frac{\Gamma(n+1)}{\Gamma(n-u+1)}\frac{B^{(c,d)}_{(a,b)}(n-m+1,m-u)}{B(n-m+1,m-u)}z^{n-u}. \tag{4.2}$$

On putting $n = n+m$, we get:

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[e^z] = \sum_{n=0}^{\infty}\frac{1}{(n+m)!}\frac{\Gamma(n+m+1)}{\Gamma(n+m-u+1)}\frac{B^{(c,d)}_{(a,b)}(n+1,m-u)}{B(n+1,m-u)}z^{n+m-u}. \tag{4.3}$$

By using known identities, we get result of Theorem (1).

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}[e^z] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}. \tag{4.4}$$

□

Example 2. *New type Caputo derivatives of Prabhakar-type function is given by*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[E^{\gamma}_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+m}}{\Gamma(n\alpha + m\alpha + \beta)} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}, \tag{4.5}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the Prabhakar-type function given in [10].

Proof. By using Prabhakar-type function defined in [10]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\beta + n\alpha)} \frac{z^n}{n!}, \tag{4.6}$$

and by applications of proof of Theorem (1), we get result of Theorem (2):

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[E^{\gamma}_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+m}}{\Gamma(n\alpha + m\alpha + \beta)} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}. \tag{4.7}$$

□

Example 3. *The following result holds true:*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[E_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(n\alpha + m\alpha + \beta)} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}, \tag{4.8}$$

where $E_{\alpha,\beta}(z)$ is the Mittag Leffler function given in [5].

Proof. Considering $\gamma = 1$ in Theorem (2), require result is obtained. □

Example 4. *The following result holds true:*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[E_{\alpha}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(n\alpha + m\alpha + 1)} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}, \tag{4.9}$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function given in [6,11].

Proof. On substituting $\gamma = 1$ and $\beta = 1$ in Theorem (2), require result is obtained. □

Example 5. *New type Caputo derivatives of generalized hypergeometric function is given by*

$$\begin{aligned} &\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[{}_pF_q(p_1, p_2..p_p; q_1, q_2..q_q; z)\right] = \\ &\frac{z^{m-u}}{\Gamma(m-u)} \frac{\prod_{j=1}^{q}\Gamma(q_j)}{\prod_{i=1}^{p}\Gamma(p_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(p_i+n+m)}{\prod_{j=1}^{q}\Gamma(q_j+n+m)} B^{(c,d)}_{(a,b)}(m-u, n+1) \frac{z^n}{n!}. \end{aligned} \tag{4.10}$$

Here, ${}_pF_q(p_1, p_2..p_p; q_1, q_2..q_q; z)$ is the generalized hypergeometric function given in [9].

Proof. By applying the generalized hypergeometric function and utilizing a similar method as in Theorem (1), we obtain the result outlined in Theorem (5) □

Example 6. *New type Caputo derivatives of Gauss hypergeometric function is given by*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[{}_2F_1(p_1,p_2;q_1;z)\right] = \frac{z^{m-u}}{\Gamma(m-u)}\frac{\Gamma(q_1)}{\Gamma(p_1)\Gamma(p_2)}\sum_{n=0}^{\infty}\frac{\Gamma(p_1+n+m)\Gamma(p_2+n+m)}{\Gamma(q_1+n+m)}B^{(c,d)}_{(a,b)}(m-u,n+1)\frac{z^n}{n!}, \tag{4.11}$$

provided that $\mid z \mid < 1$.

Proof. Substituting $p=2$ and $q=1$ into Theorem (5), the desired result is obtained. □

Example 7. *New type Caputo derivatives of confluent hypergeometric function is given by*

$$\tilde{\mathbf{D}}^{u,(c,d)}_{z,(a,b)}\left[{}_1F_1(p_1,q_1;z)\right] = \frac{z^{m-u}}{\Gamma(m-u)}\frac{\Gamma(q_1)}{\Gamma(p_1)}\sum_{n=0}^{\infty}\frac{\Gamma(p_1+n+m)}{\Gamma(q_1+n+m)}B^{(c,d)}_{(a,b)}(m-u,n+1)\frac{z^n}{n!}, \tag{4.12}$$

given that $z \in C$.

Proof. Put $p=1$ and $q=1$ in Theorem (5), our result is obtained. □

5. Conclusion

Here, firstly we define new type of Caputo fractional derivative operator by means of symmetric kernel then found some basic properties of that operator. Then we have found some important applications and examples of this new type of Caputo fractional derivative operator. We can conclude here that result are new and important in the field of fractional calculus.

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