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Einstein t-Norm and t-Conorm-Based Nonlinear Diophantine Rough Fuzzy Model for fuzzy neural networking

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ABSTRACT: The approach builds a strong computational framework that can handle complicated, contradictory, and ambiguous data by combining the nonlinear Diophantine rough fuzzy set with Einstein t-norm and t-conorm-based aggregation operations. Higher expressiveness in describing uncertainty is made possible by the architecture's foundation in nonlinear Diophantine rough fuzzy sets. New operating regulations and advanced aggregation operators, such as NLDRFEWA, NLDRFEOWA, and NLDRFEHWA, are suggested in order to further strengthen the decision-making process. The decision-maker's preferences and the relative weighting of criteria are captured by the operators in a nonlinear and linguistically responsive way. Additionally, the model incorporates fuzzy neural networking (FNN) to learn intricate patterns from decision data, allowing for intelligent management of nonlinear uncertainty, adaptive weight assignment, and strong generalization. The framework makes decisions with greater flexibility and accuracy by fusing rule-based fuzzy reasoning with neural networks' capacity for learning. Additionally, advanced accuracy and scoring algorithms are used to improve alternative ranking and discriminating. The efficiency and suitability of the suggested method are demonstrated by a case study of the assessment of leading fuzzy neural network. The strength of the framework in producing consistent, logical, and interpretable conclusions under inputs is highlighted by comparison with existing MCDM techniques. The results confirm that fuzzy neural networking, nonlinear Diophantine rough fuzzy theory, and Einstein aggregation significantly enhance the quality of decision-making in risky and uncertain scenarios.

Key Words: Rough fuzzy sets, Einstein t-norm, t-conorm, non-linear Diophantine fuzzy sets, decision making, fuzzy neural networking.

Contents

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1. Introduction

Zadeh [27] introduced the concept of fuzzy sets as a revolutionary extension of traditional set theory to capture the imprecision and uncertainty involved in real-world problems. Fuzzy set theory allows partial membership characterized by a membership function between 0 and 1, compared to crisp sets where the membership of an element is exclusively binary (belonging or not belonging to the set). Since mathematical framework has developed into a fundamental tool in a number of domains, such as artificial intelligence, pattern recognition, control systems, and decision-making, allowing for a more accurate simulation of human reasoning and imprecise data. Intuitionistic fuzzy sets (IFSs), which allow the direct expression of membership and non-membership degrees, have become a central extension of traditional fuzzy set theory since their inception by Atanassov [2]. In contrast to the individual membership degree used in Zadeh's initial fuzzy sets, this double representation provides a more flexible and realistic uncertainty model. It has also generated a tremendous amount of theoretical and practical research in decision-making and information processing. Atanassov [1] developed further the basic concepts of IFSs and studied their application in many different fields in his later work. Zimmermann [3] provided an extensive overview of fuzzy set theory as a whole, putting IFSs into the broader context of uncertainty modeling and computational statistics. Drawing from these frameworks, Biswas, and Roy [4] designed an array of IFS-specific operations through which more intricate manipulation and combination of intuitionistic fuzzy data were permitted. For enhancing the quality of decisions in complex, multi-criteria situations, Xu [5] established models of multiple attribute decision-making (MADM) which utilized intuitionistic fuzzy information effectively.

Xu [6] established the groundwork for multi-person and multi-attribute decision-making models by offering a methodical way to compile the opinions of several experts. To build on this, Xia et al. [7] investigated sophisticated aggregation operators based on Archimedean t-norms and t-conorms, which enhanced the adaptability and flexibility of intuitionistic fuzzy decision-making. To handle situations where both membership and non-membership degrees fluctuate over intervals, Yu [8] proposed interval-valued intuitionistic fuzzy prioritized operators. The correlation measures for interval-valued IFSs, which comprehending correlations between unknown parameters, were studied in earlier studies by Bustince and Burillo [9]. In order to build intuitionistic fuzzy connections with predefined properties, a crucial first step for structured decision-making frameworks, Bustince [10] developed techniques for doing so. By providing better prioritization in multi-criteria situations, Yu [11] also contributed to group decision-making approaches with the generalized intuitionistic fuzzy prioritized geometric operator. Similar to this, Tan [12] created a generalized intuitionistic fuzzy geometric aggregation operator and it may be used for group decision-making situations with multiple criteria.

By proposing new similarity and entropy-based metrics for group decision-making, Xia and Xu [13, 14] made significant advances that enabled a more accurate representation and aggregation of expert viewpoints. In order to obtain consistent priority vectors using intuitionistic fuzzy preference relations, Gong et al. [15, 16] suggested goal programming techniques and optimal priority models. These methods have applications in sensitivity analysis and industrial selection. In order to address partial preference information, Li [17, 18] extended the methodological framework by utilizing multiattribute decision-making techniques, such as closeness coefficient-based models [19, 20], TOPSIS-based nonlinear programming, and generalized OWA operators. When taken as a whole, this research offers a strong theoretical framework and useful applications for addressing actual decision-making issues in an intuitionistic fuzzy environment. Wei [21] developed the maximizing deviation method, which helps decision-makers effectively distinguish between options. Expanding on this, Wei [22] introduced a grey relational analysis method for situations involving insufficient weight information. Wei [23] improved this method by applying GRA to more general intuitionistic fuzzy MADM scenarios. Wei et al. [25] used correlation coefficients to handle interval-valued intuitionistic fuzzy MADM with incomplete weight information, offering a flexible framework for real-world applications. Wei and Zhao [24] created minimum deviation models to improve the precision and resilience of decision outcomes. Grzegorzewski [26] supplemented these methods by investigating distance measures between interval-valued fuzzy sets and IFSs using the Hausdorff metric, providing a mathematical basis for evaluating similarity and dissimilarity in fuzzy decision-making situations. When taken as a whole, these publications provide a strong methodological foundation for handling difficult decision-making issues that involve ambiguity and partial knowledge. In order to enable more reliable information fusion, Wang and Liu [28] presented intuitionistic fuzzy geometric aggregation operators based on Einstein operations. Wang and Liu [29] expanded the use of Einstein operators in decision-making situations by developing techniques for intuitionistic fuzzy information aggregation. Zhao and Wei [30] developed on this work by introducing several intuitionistic fuzzy Einstein hybrid aggregation operators and proving how well they worked to solve multiple attribute decision-making problems. When taken as a whole, these works have greatly improved fuzzy systems' capacity to deal with ambiguous, partial, and hesitant data in challenging decision-making situations.

The break of this paper is systematized as follows: Section 2, define the basic idea. Section 3 Write a Non-LDRFS and operational laws. Section 4 recommends a series of three aggregation operators. Section 5 introduces a Non–LDRF Based Multi-Criteria Decision making for fuzzy neural networking and delivers a numerical instance to validate the rationality. Section 6 summarizes the conclusion.

2. Backgroud

Definition 2.1. [27] Let us consider that $\Phi \neq X$ and by a fuzzy set $\gamma = \left\{ \begin{array}{c} \langle x, \mu_{\gamma(x)} \rangle \\ : x \in X \end{array} \right\}$, $\mu_{\gamma(x)}$ is a mapping from X to $[0,1]$ represent membership function of an element x in X .

Definition 2.2. [2] The fixed set C and the IFS A is defined in $A = \left\{ \begin{array}{c} \langle \varsigma_A(x), \\ \chi_A(x) \rangle \\ : x \in C \end{array} \right\}$, where $\varsigma_A(x)$ and $\chi_A(x)$ represent the MED and NOMED, and $\varsigma_A(x) \in [0,1]$,$\chi_A(x) \in [0,1]$ and $0 \leq \varsigma_A(x) + \chi_A(x) \leq 1$. The degree of indeterminacy is defined as $\pi_A(x) = 1 - \varsigma_A(x) - \chi_A(x)$. The IFS is denoted as $A = \langle \varsigma_A, \chi_A \rangle$.

Definition 2.3. [2] Let $a_1 = \{\varsigma_1, \chi_1\}$ and $a_2 = \{\varsigma_2, \chi_2\}$ be two IFSs, $\lambda > 0$, then

$a_1 \oplus a_2 = [\varsigma_1 + \varsigma_2 - \varsigma_1\varsigma_2, \chi_1\chi_2]$;

$a_1 \otimes a_2 = [\varsigma_1\varsigma_2, \chi_1 + \chi_2 - \chi_1\chi_2]$;

$\lambda a_1 = \left[1 - (1-\varsigma_1)^\lambda, \chi, \chi_1^\lambda\right]$;

$a_1^\lambda = \left[\varsigma_1^\lambda, 1 - (1-\chi_1)^\lambda\right]$.

Definition 2.4. [2] Let $a = \{\varsigma, \chi\}$ be the IFSs, then the score function is $a = \varsigma_\alpha - \chi_\alpha$.

Definition 2.5. [2]Let $a = \{\varsigma, \chi\}$ be the IFSs, then the accuracy function is $a = \varsigma_\alpha + \chi_\alpha$.

3. Einstein operator on non-linear diophantie rough fuzzy set

In this section, we define the operational laws of Non-linear diophantine rough fuzzy set.

Definition 3.1. Let $L_1 = \left[\begin{array}{c} <[\overbrace{\xi_1^q}, \overbrace{E_1^q}], \\ [\overbrace{\xi_1}, \overbrace{E_1}]>, \\ <[\underbrace{\xi_1^q}, \underbrace{E_1^q}], \\ [\underbrace{\xi_1}, \underbrace{E_1}]> \end{array} \right]$ and $L_2 = \left[\begin{array}{c} <[\overbrace{\xi_2^q}, \overbrace{E_2^q}], \\ [\overbrace{\xi_2}, \overbrace{E_2}]>, \\ <[\underbrace{\xi_2^q}, \underbrace{E_2^q}], \\ [\underbrace{\xi_2}, \underbrace{E_2}]> \end{array} \right]$ be two non-linear diophantine rough fuzzy set. Let the t-norm and t-conorm be Einstein product T and Einstein sum respectively, then the generalized intersection and union on two NLDRFSs and become the Einstein

$$L_1 + L_2 = \left(\begin{array}{c} \left\langle \left[\begin{array}{c} \frac{\overbrace{\xi_1^q} + \overbrace{\xi_2^q}}{1 + \overbrace{\xi_1^q} \cdot \overbrace{\xi_2^q}}, \\ \frac{\overbrace{\xi_1^q}\ \overbrace{\xi_2^q}}{1+(1-\overbrace{\xi_1^q})(1-\overbrace{\xi_2^q})} \end{array} \right], \left[\begin{array}{c} \frac{\overbrace{\xi_1} + \overbrace{\xi_2}}{1 + \overbrace{\xi_1} \cdot \overbrace{\xi_2}}, \\ \frac{\overbrace{\xi_1}\ \overbrace{\xi_2}}{1+(1-\overbrace{\xi_1})(1-\overbrace{\xi_2})} \end{array} \right] \right\rangle, \\ \left\langle \left[\begin{array}{c} \frac{\underbrace{\xi_1^q} + \underbrace{\xi_2^q}}{1 + \underbrace{\xi_1^q}\ \underbrace{\xi_2^q}}, \\ \frac{\underbrace{\xi_1^q}\ \underbrace{\xi_2^q}}{1+(1-\underbrace{\xi_1^q})(1-\underbrace{\xi_2^q})} \end{array} \right], \left[\begin{array}{c} \frac{\underbrace{\xi_1} + \underbrace{\xi_2}}{1 + \underbrace{\xi_1}\ \underbrace{\xi_2}}, \\ \frac{\underbrace{\xi_1}\ \underbrace{\xi_2}}{1+(1-\underbrace{\xi_1})(1-\underbrace{\xi_2})} \end{array} \right] \right\rangle \end{array} \right);$$

$$L_1 * L_2 = \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\dfrac{\overbrace{\xi_1^q} \cdot \overbrace{\xi_2^q}}{1+(1-\overbrace{\xi_1^q})(1-\overbrace{\xi_1^q})}, \dfrac{\overbrace{E_1^q} .+ \overbrace{E_2^q}}{1+\overbrace{E_1^q}\,\overbrace{E_2^q})} \right], \\ \left[\dfrac{\overbrace{\xi_1} \cdot \overbrace{\xi_2}}{1+(1-\overbrace{\xi_1})(1-\overbrace{\xi_1})}, \dfrac{\overbrace{E_1} + \overbrace{E_2}}{1+\overbrace{E_1}\,\overbrace{E_2}} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left[\dfrac{\underbrace{E_1^q}\,\underbrace{E_2^q}}{1+(1-\underbrace{E_1^q})(1-\underbrace{E_2^q})}, \dfrac{\underbrace{E_1^q} + \underbrace{E_2^q}}{1+\underbrace{E_1^q}\,\underbrace{E_2^q}} \right], \\ \left[\dfrac{\underbrace{E_1}\,\underbrace{E_2}}{1+(1-\underbrace{E_1})(1-\underbrace{E_2})}, \dfrac{\underbrace{E_1} + \underbrace{E_2}}{1+\underbrace{E_1}\,\underbrace{E_2}} \right] \end{array} \right\rangle \end{array} \right);$$

$$\lambda L_1 = \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\dfrac{\left(1+\overbrace{\xi_1^q}\right)^\lambda - \left(1-\overbrace{\xi_1^q}\right)^\lambda}{\left(1+\overbrace{\xi_1^q}\right)^\lambda + \left(1-\overbrace{\xi_1^q}\right)^\lambda}, \dfrac{2\left(\overbrace{E_1^q}\right)^\lambda}{\left(2-\overbrace{E_1^q}\right)^\lambda + \left(\overbrace{E_1^q}\right)^\lambda} \right], \\ \left[\dfrac{\left(1+\overbrace{\xi_1}\right)^\lambda - \left(1-\overbrace{\xi_1}\right)^\lambda}{\left(1+\overbrace{\xi_1}\right)^\lambda + \left(1-\overbrace{\xi_1}\right)^\lambda}, \dfrac{2\left(\overbrace{E_1}\right)^\lambda}{\left(2-\overbrace{E_1}\right)^\lambda + \left(\overbrace{E_1}\right)^\lambda} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left[\dfrac{\left(1+\underbrace{\xi_1^q}\right)^\lambda - \left(1-.\underbrace{\xi_1^q}\right)^\lambda}{\left(1+\underbrace{\xi_1^q}\right)^\lambda + \left(1-\underbrace{\xi_1^q}\right)^\lambda}, \dfrac{2\left(\underbrace{E_1^q}\right)^\lambda}{\left(2-\underbrace{E_1^q}\right)^\lambda + \left(\underbrace{E_1^q}\right)^\lambda} \right], \\ \left[\dfrac{\left(1+\underbrace{\xi_1}\right)^\lambda - \left(1-\underbrace{\xi_1}\right)^\lambda}{\left(1+\underbrace{\xi_1}\right)^\lambda + \left(1-\underbrace{\xi_1}\right)^\lambda}, \dfrac{2\left(\underbrace{E_1}\right)^\lambda}{\left(2-\underbrace{E_1}\right)^\lambda + \left(\underbrace{E_1}\right)^\lambda} \right] \end{array} \right\rangle \end{array} \right);$$

$$L_1^\lambda = \left(\begin{array}{c} \left\langle \begin{array}{c} \left[\dfrac{2\left(\overbrace{\xi_1^q}\right)^\lambda}{\left(2-\overbrace{\xi_1^q}\right)^\lambda + \left(\overbrace{\xi_1^q}\right)^\lambda}, \dfrac{\left(1+\overbrace{E_1^q}\right)^\lambda - \left(1-\overbrace{E_1^q}\right)^\lambda}{\left(1+\overbrace{E_1^q}\right)^\lambda + \left(1-\overbrace{E_1^q}\right)^\lambda} \right], \\ \left[\dfrac{2\left(\overbrace{\xi_1}\right)^\lambda}{\left(2-\overbrace{\xi_1}\right)^\lambda + \left(\overbrace{\xi_1}\right)^\lambda}, \dfrac{\left(1+\overbrace{E_1}\right)^\lambda - \left(1-\overbrace{E_1}\right)^\lambda}{\left(1+\overbrace{E_1}\right)^\lambda + \left(1-\overbrace{E_1}\right)^\lambda} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left[\dfrac{2\left(\underbrace{\xi_1^q}\right)^\lambda}{\left(2-\underbrace{\xi_1^q}\right)^\lambda + \left(\underbrace{\xi_1^q}\right)^\lambda}, \dfrac{\left(1+\underbrace{E_1^q}\right)^\lambda - \left(1-\underbrace{E_1^q}\right)^\lambda}{\left(1+\underbrace{E_1^q}\right)^\lambda + \left(1-\underbrace{E_1^q}\right)^\lambda} \right], \\ \left[\dfrac{2\left(\underbrace{\xi_1}\right)^\lambda}{\left(2-\underbrace{\xi_1}\right)^\lambda + \left(\underbrace{\xi_1}\right)^\lambda}, \dfrac{\left(1+\underbrace{E_1}\right)^\lambda - \left(1-\underbrace{E_1}\right)^\lambda}{\left(1+\underbrace{E_1}\right)^\lambda + \left(1-\underbrace{E_1}\right)^\lambda} \right] \end{array} \right\rangle \end{array} \right).$$

4. Aggregation operators on non-linear diophantine rough fuzzy set

In this section, we define the NLDFREWA, NLDFREOWA and NLDFREHWA operators.

4.1. NLDFREWA operator

Definition 4.1.1 Let $L_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ $(j = 1, 2, 3, ..., n)$ be the collection of non-linear diophantine rough fuzzy value and
$NLDFREWA : Q^n \to Q$ if $NLDFREWA(L_1, L_2, L_3, ..., L_n) = \oplus_{j=1}^{n}(\lambda_j L_j) =$

$$\left(\begin{array}{c} \left\langle \begin{array}{c} \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}} \right], \\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}} \right], \\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}} \right] \end{array} \right\rangle \end{array} \right)$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)^T$ be the weight vector of $L_j (j = 1, 2, 3, ..., n)$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$ then it is said be non-linear diophantine rough fuzzy einstein weighted operator.

Theorem 4.1.2 Let $L_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ $(j = 1, 2, 3, ..., n)$ be collection of non-linear diophantine rough fuzzy value and if $NLDFREWA(L_1, L_2, L_3, ..., L_n) = \oplus_{j=1}^{n}(\lambda_j L_j) =$

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle\end{array}\right)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)^T$ be the weight vector of $L_j (j = 1, 2, 3, ..., n)$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$ then it said to be non-linear diophantine rough fuzzy einstein weighted average operator.

Theorem 4.1.3 (Idempotency): If $U_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ for all $(J = 1, 2, 3, ..., m)$, then

$$NLDFREWA(U, U, U, ..., U) = U.$$

Theorem 4.1.4 (Commutativity) : If $(bg_1^{'}, bg_2^{'}, ..., bg_n^{'})$ is any permutation of $(bg_1, bg_2, ..., bg_n)$, then NLDRFEWA$(bg_1^{'}, bg_2^{'}, ..., bg_n^{'}) = NLDFREWA(bg_1, bg_2, ..., bg_n)$.

Theorem 4.1.5 (Boundedness): If $Z^- = min(gf_1, gf_2, ..., gf_n)$, $Z^+ = \max(gf_1, gf_2, ..., gf_n)$, then

$$Z^- \leq NLDRFEWA(gf_1, gf_2, ..., gf_n) \leq Z^+.$$

4.2. NLDRFEOWA operator

Definition 4.2.1 Let $L_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ $(j = 1, 2, 3, ..., n)$ be the collection of non-linear diophantine rough fuzzy value and
$NLDFREOWA : Q^n \to Q$ if $NLDFREOWA(L_1, L_2, L_3, ..., L_n) = \oplus_{j=1}^{n}(\lambda_j L_j) =$

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle\end{array}\right)$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)^T$ be the weight vector of $L_j(j = 1, 2, 3, ..., n)$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$ then it said be non-linear diophantine rough fuzzy einstein ordered weighted operator.

Theorem 4.2.2 Let $L_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ $(j = 1, 2, 3, ..., n)$ be collection of non-linear diophantine rough fuzzy value and
if $NLDFREWA(L_1, L_2, L_3, ..., L_n) = \oplus_{j=1}^{n}(\lambda_j L_j) =$

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle\end{array}\right)$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3,....,\lambda_n)^T$ be the weight vector of $L_j(j=1,2,3,....,n)$ and $\lambda_j>0,\sum_{j=1}^{n}\lambda_j=1$ then it said to be non-linear diophantine rough fuzzy einstein ordered weighted average operator.

Theorem 4.2.3 (Idempotency): If $U_j=\begin{bmatrix} <[\overbrace{\xi_j^q},\overbrace{E_j^q}], \\ [\overbrace{\xi_j},\overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q},\underbrace{E_j^q}], \\ [\underbrace{\xi_j},\underbrace{E_j}]> \end{bmatrix}$ for all $(J=1,2,3,...,m)$, then

$$NLDRFEOWA(U,U,U,...,U)=U.$$

Theorem 4.2.4 (Commutativity) : If $(bg_1^{'},bg_2^{'},...,bg_n^{'})$ is any permutation of $(bg_1,bg_2,...,bg_n)$, then NLDRFEOWA$(bg_1^{'},bg_2^{'},...,bg_n^{'})=NLDFREOWA(bg_1,bg_2,...,bg_n)$.

Theorem 4.2.5 (Boundedness): If $Z^-=min(gf_1,gf_2,...,gf_n)$, $Z^+=\max(gf_1,gf_2,...,gf_n)$, then

$$Z^-\leq NLDRFEOWA(gf_1,gf_2,...,gf_n)\leq Z^+.$$

4.3. NLDRFEHWA operator

Definition 4.3.1 Let $L_j=\begin{bmatrix} <[\overbrace{\xi_j^q},\overbrace{E_j^q}], \\ [\overbrace{\xi_j},\overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q},\underbrace{E_j^q}], \\ [\underbrace{\xi_j},\underbrace{E_j}]> \end{bmatrix}$ $(j=1,2,3,....,n)$ be the collection of non-linear diophantine rough fuzzy value and $NLDFREHWA:Q^n\rightarrow Q$ if $NLDFREHWA(L_1,L_2,L_3,...,L_n)=$ $\oplus_{j=1}^{n}(\lambda_jL_j)=$

$$\left(\begin{array}{c} \left\langle \begin{array}{c} \left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right], \\ \left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right] \end{array}\right\rangle, \\ \left\langle \begin{array}{c} \left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right], \\ \left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right] \end{array}\right\rangle \end{array}\right)$$

where $\lambda=(\lambda_1,\lambda_2,....,\lambda_n)^T$ be the weight vector of $L_j(j=1,2,3,....,n)$ and $\lambda_j>0,\sum_{j=1}^{n}\lambda_j=1$ then it said be non-linear diophantine rough fuzzy einstein hybrid weighted operator.

Theorem 4.3.2 Let $L_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ $(j = 1, 2, 3, ..., n)$ be collection of non-linear diophantine rough fuzzy value and if $NLDFREHWA_w(L_1, L_2, L_3, ..., L_n) = \oplus_{j=1}^{n}(\lambda_j L_j) =$

$$\left(\begin{array}{c} \left\langle \begin{array}{c} \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}} \right], \\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda} + \prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}} \right], \\ \left[\dfrac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda} - \prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}, \dfrac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda} + \prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}} \right] \end{array} \right\rangle \end{array} \right)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)^T$ be the weight vector of $L_j (j = 1, 2, 3, ..., n)$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$ then it said to be non-linear diophantine rough fuzzy einstein hybrid weighted average operator.

Theorem 4.3.3 (Idempotency): If $U_j = \begin{bmatrix} <[\overbrace{\xi_j^q}, \overbrace{E_j^q}], \\ [\overbrace{\xi_j}, \overbrace{E_j}]>, \\ <[\underbrace{\xi_j^q}, \underbrace{E_j^q}], \\ [\underbrace{\xi_j}, \underbrace{E_j}]> \end{bmatrix}$ for all $(J = 1, 2, 3, ..., m)$, then

$$NLDRFEHWA(U, U, U, ..., U) = U.$$

Theorem 4.3.4 (Commutativity) : If $(bg_1^{'}, bg_2^{'}, ..., bg_n^{'})$ is any permutation of $(bg_1, bg_2, ..., bg_n)$, then NLDRFEHWA$(bg_1^{'}, bg_2^{'}, ..., bg_n^{'}) = NLDFREHWA(bg_1, bg_2, ..., bg_n)$.

Theorem 4.3.5 (Boundedness): If $Z^- = min(gf_1, gf_2, ..., gf_n)$, $Z^+ = \max(gf_1, gf_2, ..., gf_n)$, then

$$Z^- \leq NLDRFEHWA(gf_1, gf_2, ..., gf_n) \leq Z^+.$$

5. MCDM METHOD based on fuzzy neural networking

Input layers

Step 1: Calculate the non-linear diophantine rough fuzzy set

Step 2: Calculate einstein on non linear diophantine rough fuzzy set

Hidden layers

Step 1: Calculate non-linear diophantine rough fuzzy set

Step 2: Calculate einstein on non linear diophantine rough fuzzy set

Output Layers

Step 1:Calculate non-linear diophantine rough fuzzy set

Step 2:Calculate einstein on non linear diophantine rough fuzzy set

Step 3:Calculate the NLDRFWA operator and the weight vector is

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle\end{array}\right)$$

Step 4:Calculate the NLDRFWA operator and the weight vector is

$$\left(\begin{array}{c}\left\langle\begin{array}{c}\left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\frac{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\overbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\overbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\overbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\overbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{c}\left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-.\underbrace{\xi_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j^q}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j^q}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j^q}\right)^{\lambda}}\right],\\ \left[\frac{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}-\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(1+\underbrace{\xi_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\underbrace{\xi_j}\right)^{\lambda}},\frac{2\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}{\prod_{j=1}^{n}\left(2-\underbrace{E_j}\right)^{\lambda}+\prod_{j=1}^{n}\left(\underbrace{E_j}\right)^{\lambda}}\right]\end{array}\right\rangle\end{array}\right)$$

Step 5:Calculate the score function

$$\left[\frac{(\overbrace{\xi_j^q}-\overbrace{E_j^q})+(\overbrace{\xi_j}-\overbrace{E_j})+(\underbrace{\xi_j^q}-\underbrace{E_j^q})+(\underbrace{\xi_j}-\underbrace{E_j})}{4}\right]$$

5.1. Numerical example

The field of intelligent systems is quickly changing because to fuzzy neural networking (FNN), a hybrid paradigm that combines the interpretability of fuzzy logic with the adaptive learning capability of neural networks. FNN is positioned as a crucial facilitator of the UAE Vision 2031, the country's strategy for digital transformation, in addition to being a fundamental development in artificial intelligence. The UAE is looking to use fuzzy neural networking to boost innovation in manufacturing, healthcare, transportation, education, and smart city development. The UAE has made continuous investments in high-tech infrastructure. The nation now understands that conventional machine learning and clear decision models frequently struggle with the uncertainty, ambiguity, and vagueness found in real-world data, much like it did when it first adopted cutting-edge telecom technologies. The early investigation of FNN was spurred by the advent of data-intensive applications, such as AI-powered financial forecasting and logistics, as well as adaptive learning in education, which have highlighted the shortcomings of traditional models. In line with the country's goal of becoming a global center for Industry 4.0 innovation, universities and research facilities started experimenting with fuzzy-neural integration as early as 2018 in partnership with IT businesses.

The UAE was one of the first nations in the Middle East to begin pilot programs using fuzzy neural networks in a variety of industries by 2019. Because FNN can manage inaccurate sensor data, it has been used in autonomous vehicle navigation in transportation, allowing for safer vehicle-to-infrastructure (V2I) interactions. Low-latency adaptive decision-making is essential in the healthcare industry, and hospitals have launched FNN-based diagnostic assistance systems and remote surgery frameworks. In order to improve user engagement through context-aware services, the retail and tourism industries have also adopted AR/VR personalization models and FNN-powered recommendation systems.

Techniques for Implementing Fuzzy Neural Networks

Instant Nationwide Implementation

Create FNN frameworks concurrently across research facilities and industry to guarantee the adoption of AI-driven solutions in both urban and rural areas.

First Urban Deployment

Before expanding acceptance to underprivileged and rural areas, give priority to FNN deployment in major cities, industrial areas, and academic centers.

Sector-Specific Implementation

Prior to integrating the general public, concentrate on industries with the greatest potential for impact: manufacturing, healthcare, banking, and transportation.

expenses

Through partnerships between the government, academic institutions, and commercial technology companies, public-private partnerships (PPPs) facilitate implementation by sharing risks, expenses, and knowledge.

Evaluation Criteria

Economicalness

ensures scalability and sustainability by weighing projected research and economic benefits against total implementation costs.

Coverage & Accessibility

evaluates the inclusivity and reach of FNN applications across various social groups and geographical areas.

The availability of computer capacity, cloud platforms, big data infrastructure, and qualified human capital required for FNN deployment are all assessed by technological readiness.

Impact on the Economy and Society

evaluates contributions to the advancement of general quality of life, industrial change, job generation, and digital inclusion.

Input Layers

Step 1:Calculate the NLDRF decision matrix in table 1 and 2.

NLDRF decision matrix table 1

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.4, 0.3], [0.8, 0.6], [0.02, 0.09], [0.06, 0.08] \right\rangle$	$\left\langle [0.5, 0.2], [0.8, 0.5], [0.09, 0.01], [0.07, 0.05] \right\rangle$	$\left\langle [0.7, 0.2], [0.3, 0.5], [0.04, 0.03], [0.08, 0.05] \right\rangle$	$\left\langle [0.3, 0.2], [0.1, 0.5], [0.09, 0.03], [0.07, 0.05] \right\rangle$
L_2	$\left\langle [0.7, 0.8], [0.7, 0.6], [0.09, 0.07], [0.01, 0.02] \right\rangle$	$\left\langle [0.2, 0.3], [0.5, 0.6], [0.05, 0.06], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.8], [0.9, 0.6], [0.05, 0.06], [0.03, 0.05] \right\rangle$	$\left\langle [0.2, 0.5], [0.7, 0.8], [0.05, 0.06], [0.02, 0.08] \right\rangle$
L_3	$\left\langle [0.4, 0.2], [0.1, 0.5], [0.03, 0.01], [0.04, 0.02] \right\rangle$	$\left\langle [0.4, 0.1], [0.6, 0.9], [0.07, 0.05], [0.03, 0.01] \right\rangle$	$\left\langle [0.5, 0.9], [0.9, 0.3], [0.06, 0.03], [0.04, 0.02] \right\rangle$	$\left\langle [0.7, 0.5], [0.1, 0.7], [0.07, 0.09], [0.03, 0.07] \right\rangle$
L_4	$\left\langle [0.5, 0.3], [0.7, 0.9], [0.04, 0.01], [0.02, 0.05] \right\rangle$	$\left\langle [0.7, 0.9], [0.8, 0.6], [0.08, 0.02], [0.02, 0.01] \right\rangle$	$\left\langle [0.6, 0.5], [0.1, 0.6], [0.01, 0.05], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.5], [0.3, 0.1], [0.03, 0.01], [0.02, 0.04] \right\rangle$

NLDRF decision matrix table 2

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\langle [0.5, 0.2], [0.6, 0.4], [0.04, 0.01], [0.07, 0.09] \rangle$	$\langle [0.2, 0.5], [0.3, 0.7], [0.01, 0.02], [0.03, 0.04] \rangle$	$\langle [0.4, 0.5], [0.6, 0.7], [0.01, 0.03], [0.02, 0.04] \rangle$	$\langle [0.7, 0.9], [0.2, 0.6], [0.04, 0.07], [0.01, 0.02] \rangle$
L_2	$\langle [0.8, 0.9], [0.1, 0.2], [0.02, 0.07], [0.03, 0.01] \rangle$	$\langle [0.4, 0.3], [0.3, 0.2], [0.01, 0.03], [0.04, 0.02] \rangle$	$\langle [0.2, 0.5], [0.3, 0.6], [0.01, 0.07], [0.05, 0.06] \rangle$	$\langle [0.4, 0.6], [0.7, 0.1], [0.01, 0.07], [0.05, 0.01] \rangle$
L_3	$\langle [0.9, 0.7], [0.5, 0.3], [0.06, 0.03], [0.07, 0.01] \rangle$	$\langle [0.3, 0.5], [0.8, 0.2], [0.03, 0.05], [0.04, 0.01] \rangle$	$\langle [0.7, 0.4], [0.3, 0.9], [0.07, 0.09], [0.01, 0.09] \rangle$	$\langle [0.5, 0.9], [0.2, 0.4], [0.01, 0.05], [0.08, 0.07] \rangle$
L_4	$\langle [0.3, 0.4], [0.3, 0.6], [0.02, 0.04], [0.05, 0.07] \rangle$	$\langle [0.7, 0.4], [0.5, 0.7], [0.02, 0.01], [0.03, 0.04] \rangle$	$\langle [0.3, 0.1], [0.2, 0.4], [0.03, 0.06], [0.02, 0.01] \rangle$	$\langle [0.2, 0.1], [0.4, 0.5], [0.02, 0.03], [0.03, 0.01] \rangle$

Step 2:Calculate the NLDREFEWA operator in table 3.

NLDREFEWA operator in table 3

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.75, 0.03], [0.94, 0.19], [0.05, 0.0004], [0.12, 0.0039] \right\rangle$	$\left\langle [0.63, 0.09], [0.88, 0.30], [0.099, 0.0001], [0.099, 0.001] \right\rangle$	$\left\langle [0.85, 0.07], [0.76, 0.30], [0.04, 0.0004], [0.099, 0.001] \right\rangle$	$\left\langle [0.82, 0.16], [0.29, 0.25], [0.12, 0.001], [0.07, 0.0005] \right\rangle$
L_2	$\left\langle [0.96, 0.70] , [0.74, 0.09], [0.10, 0.002] , [0.03, 0.001] \right\rangle$	$\left\langle [0.5, 0.006] , [0.69, 0.09], [0.05, 0.0009] , [0.06, 0.004] \right\rangle$	$\left\langle [0.78, 0.36] , [0.94, 0.31], [0.05, 0.002] , [0.07, 0.001] \right\rangle$	$\left\langle [0.5, 0.25] , [0.93, 0.06], [0.05, 0.002] , [0.06, 0.0004] \right\rangle$
L_3	$\left\langle [0.95, 0.11], [0.57, 0.1], [0.08, 0.0001], [0.1, 0.0001] \right\rangle$	$\left\langle [0.62, 0.03], [0.94, 0.16], [0.09, 0.001], [0.06, 0.00005] \right\rangle$	$\left\langle [0.8, 0.33], [0.94, 0.25], [0.12, 0.001], [0.04, 0.009] \right\rangle$	$\left\langle [0.8, 0.42], [0.29, 0.23], [0.07, 0.002], [0.10, 0.002] \right\rangle$
L_4	$\left\langle [0.69, 0.08] , [0.82, 0.51], [0.005, 0.0002] , [0.06, 0.001] \right\rangle$	$\left\langle [0.93, 0.33] , [0.92, 0.37], [0.09, 0.0001] , [0.04, 0.0002] \right\rangle$	$\left\langle [0.76, 0.03] , [0.29, 0.19], [0.03, 0.001] , [0.04, 0.0002] \right\rangle$	$\left\langle [0.78, 0.003] , [0.62, 0.03], [0.04, 0.001] , [0.05, 0.0002] \right\rangle$

Hidden Layers

Step 1:Calculate the NLDRF decision matrix in table 1 and 2.

NLDRF decision matrix table 1

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.4, 0.3], [0.8, 0.6], [0.02, 0.09], [0.06, 0.08] \right\rangle$	$\left\langle [0.5, 0.2], [0.8, 0.5], [0.09, 0.01], [0.07, 0.05] \right\rangle$	$\left\langle [0.7, 0.2], [0.3, 0.5], [0.04, 0.03], [0.08, 0.05] \right\rangle$	$\left\langle [0.3, 0.2], [0.1, 0.5], [0.09, 0.03], [0.07, 0.05] \right\rangle$
L_2	$\left\langle [0.7, 0.8], [0.7, 0.6], [0.09, 0.07], [0.01, 0.02] \right\rangle$	$\left\langle [0.2, 0.3], [0.5, 0.6], [0.05, 0.06], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.8], [0.9, 0.6], [0.05, 0.06], [0.03, 0.05] \right\rangle$	$\left\langle [0.2, 0.5], [0.7, 0.8], [0.05, 0.06], [0.02, 0.08] \right\rangle$
L_3	$\left\langle [0.4, 0.2], [0.1, 0.5], [0.03, 0.01], [0.04, 0.02] \right\rangle$	$\left\langle [0.4, 0.1], [0.6, 0.9], [0.07, 0.05], [0.03, 0.01] \right\rangle$	$\left\langle [0.5, 0.9], [0.9, 0.3], [0.06, 0.03], [0.04, 0.02] \right\rangle$	$\left\langle [0.7, 0.5], [0.1, 0.7], [0.07, 0.09], [0.03, 0.07] \right\rangle$
L_4	$\left\langle [0.5, 0.3], [0.7, 0.9], [0.04, 0.01], [0.02, 0.05] \right\rangle$	$\left\langle [0.7, 0.9], [0.8, 0.6], [0.08, 0.02], [0.02, 0.01] \right\rangle$	$\left\langle [0.6, 0.5], [0.1, 0.6], [0.01, 0.05], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.5], [0.3, 0.1], [0.03, 0.01], [0.02, 0.04] \right\rangle$

NLDRF decision matrix table 2

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.5, 0.2], [0.6, 0.4], [0.04, 0.01], [0.07, 0.09] \right\rangle$	$\left\langle [0.2, 0.5], [0.3, 0.7], [0.01, 0.02], [0.03, 0.04] \right\rangle$	$\left\langle [0.4, 0.5], [0.6, 0.7], [0.01, 0.03], [0.02, 0.04] \right\rangle$	$\left\langle [0.7, 0.9], [0.2, 0.6], [0.04, 0.07], [0.01, 0.02] \right\rangle$
L_2	$\left\langle [0.8, 0.9], [0.1, 0.2], [0.02, 0.07], [0.03, 0.01] \right\rangle$	$\left\langle [0.4, 0.3], [0.3, 0.2], [0.01, 0.03], [0.04, 0.02] \right\rangle$	$\left\langle [0.2, 0.5], [0.3, 0.6], [0.01, 0.07], [0.05, 0.06] \right\rangle$	$\left\langle [0.4, 0.6], [0.7, 0.1], [0.01, 0.07], [0.05, 0.01] \right\rangle$
L_3	$\left\langle [0.9, 0.7], [0.5, 0.3], [0.06, 0.03], [0.07, 0.01] \right\rangle$	$\left\langle [0.3, 0.5], [0.8, 0.2], [0.03, 0.05], [0.04, 0.01] \right\rangle$	$\left\langle [0.7, 0.4], [0.3, 0.9], [0.07, 0.09], [0.01, 0.09] \right\rangle$	$\left\langle [0.5, 0.9], [0.2, 0.4], [0.01, 0.05], [0.08, 0.07] \right\rangle$
L_4	$\left\langle [0.3, 0.4], [0.3, 0.6], [0.02, 0.04], [0.05, 0.07] \right\rangle$	$\left\langle [0.7, 0.4], [0.5, 0.7], [0.02, 0.01], [0.03, 0.04] \right\rangle$	$\left\langle [0.3, 0.1], [0.2, 0.4], [0.03, 0.06], [0.02, 0.01] \right\rangle$	$\left\langle [0.2, 0.1], [0.4, 0.5], [0.02, 0.03], [0.03, 0.01] \right\rangle$

Step 2:Calculate the NLDREFEWA operator in table 3.

NLDREFEWA operator in table 3

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\langle [0.75, 0.03], [0.94, 0.19], [0.05, 0.0004], [0.12, 0.0039] \rangle$	$\langle [0.63, 0.09], [0.88, 0.30], [0.099, 0.0001], [0.099, 0.001] \rangle$	$\langle [0.85, 0.07], [0.76, 0.30], [0.04, 0.0004], [0.099, 0.001] \rangle$	$\langle [0.82, 0.16], [0.29, 0.25], [0.12, 0.001], [0.07, 0.0005] \rangle$
L_2	$\langle [0.96, 0.70], [0.74, 0.09], [0.10, 0.002], [0.03, 0.001] \rangle$	$\langle [0.5, 0.006], [0.69, 0.09], [0.05, 0.0009], [0.06, 0.004] \rangle$	$\langle [0.78, 0.36], [0.94, 0.31], [0.05, 0.002], [0.07, 0.001] \rangle$	$\langle [0.5, 0.25], [0.93, 0.06], [0.05, 0.002], [0.06, 0.0004] \rangle$
L_3	$\langle [0.95, 0.11], [0.57, 0.1], [0.08, 0.0001], [0.1, 0.0001] \rangle$	$\langle [0.62, 0.03], [0.94, 0.16], [0.09, 0.001], [0.06, 0.00005] \rangle$	$\langle [0.8, 0.33], [0.94, 0.25], [0.12, 0.001], [0.04, 0.009] \rangle$	$\langle [0.8, 0.42], [0.29, 0.23], [0.07, 0.002], [0.10, 0.002] \rangle$
L_4	$\langle [0.69, 0.08], [0.82, 0.51], [0.005, 0.0002], [0.06, 0.001] \rangle$	$\langle [0.93, 0.33], [0.92, 0.37], [0.09, 0.0001], [0.04, 0.0002] \rangle$	$\langle [0.76, 0.03], [0.29, 0.19], [0.03, 0.001], [0.04, 0.0002] \rangle$	$\langle [0.78, 0.003], [0.62, 0.03], [0.04, 0.001], [0.05, 0.0002] \rangle$

Output Layers

Step 1:Calculate the NLDRF decision matrix in table 1 and 2.

NLDRF decision matrix table 1

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.4, 0.3], [0.8, 0.6], [0.02, 0.09], [0.06, 0.08] \right\rangle$	$\left\langle [0.5, 0.2], [0.8, 0.5], [0.09, 0.01], [0.07, 0.05] \right\rangle$	$\left\langle [0.7, 0.2], [0.3, 0.5], [0.04, 0.03], [0.08, 0.05] \right\rangle$	$\left\langle [0.3, 0.2], [0.1, 0.5], [0.09, 0.03], [0.07, 0.05] \right\rangle$
L_2	$\left\langle [0.7, 0.8], [0.7, 0.6], [0.09, 0.07], [0.01, 0.02] \right\rangle$	$\left\langle [0.2, 0.3], [0.5, 0.6], [0.05, 0.06], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.8], [0.9, 0.6], [0.05, 0.06], [0.03, 0.05] \right\rangle$	$\left\langle [0.2, 0.5], [0.7, 0.8], [0.05, 0.06], [0.02, 0.08] \right\rangle$
L_3	$\left\langle [0.4, 0.2], [0.1, 0.5], [0.03, 0.01], [0.04, 0.02] \right\rangle$	$\left\langle [0.4, 0.1], [0.6, 0.9], [0.07, 0.05], [0.03, 0.01] \right\rangle$	$\left\langle [0.5, 0.9], [0.9, 0.3], [0.06, 0.03], [0.04, 0.02] \right\rangle$	$\left\langle [0.7, 0.5], [0.1, 0.7], [0.07, 0.09], [0.03, 0.07] \right\rangle$
L_4	$\left\langle [0.5, 0.3], [0.7, 0.9], [0.04, 0.01], [0.02, 0.05] \right\rangle$	$\left\langle [0.7, 0.9], [0.8, 0.6], [0.08, 0.02], [0.02, 0.01] \right\rangle$	$\left\langle [0.6, 0.5], [0.1, 0.6], [0.01, 0.05], [0.03, 0.04] \right\rangle$	$\left\langle [0.7, 0.5], [0.3, 0.1], [0.03, 0.01], [0.02, 0.04] \right\rangle$

NLDRF decision matrix table 2

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.5, 0.2], [0.6, 0.4], [0.04, 0.01], [0.07, 0.09] \right\rangle$	$\left\langle [0.2, 0.5], [0.3, 0.7], [0.01, 0.02], [0.03, 0.04] \right\rangle$	$\left\langle [0.4, 0.5], [0.6, 0.7], [0.01, 0.03], [0.02, 0.04] \right\rangle$	$\left\langle [0.7, 0.9], [0.2, 0.6], [0.04, 0.07], [0.01, 0.02] \right\rangle$
L_2	$\left\langle [0.8, 0.9], [0.1, 0.2], [0.02, 0.07], [0.03, 0.01] \right\rangle$	$\left\langle [0.4, 0.3], [0.3, 0.2], [0.01, 0.03], [0.04, 0.02] \right\rangle$	$\left\langle [0.2, 0.5], [0.3, 0.6], [0.01, 0.07], [0.05, 0.06] \right\rangle$	$\left\langle [0.4, 0.6], [0.7, 0.1], [0.01, 0.07], [0.05, 0.01] \right\rangle$
L_3	$\left\langle [0.9, 0.7], [0.5, 0.3], [0.06, 0.03], [0.07, 0.01] \right\rangle$	$\left\langle [0.3, 0.5], [0.8, 0.2], [0.03, 0.05], [0.04, 0.01] \right\rangle$	$\left\langle [0.7, 0.4], [0.3, 0.9], [0.07, 0.09], [0.01, 0.09] \right\rangle$	$\left\langle [0.5, 0.9], [0.2, 0.4], [0.01, 0.05], [0.08, 0.07] \right\rangle$
L_4	$\left\langle [0.3, 0.4], [0.3, 0.6], [0.02, 0.04], [0.05, 0.07] \right\rangle$	$\left\langle [0.7, 0.4], [0.5, 0.7], [0.02, 0.01], [0.03, 0.04] \right\rangle$	$\left\langle [0.3, 0.1], [0.2, 0.4], [0.03, 0.06], [0.02, 0.01] \right\rangle$	$\left\langle [0.2, 0.1], [0.4, 0.5], [0.02, 0.03], [0.03, 0.01] \right\rangle$

Step 2:Calculate the NLDREFEWA operator in table 3.

NLDREFEWA operator in table 3

	Instant Nationwide	Urban	Sector-Specific	expenses
L_1	$\left\langle [0.75, 0.03], [0.94, 0.19], [0.05, 0.0004], [0.12, 0.0039] \right\rangle$	$\left\langle [0.63, 0.09], [0.88, 0.30], [0.099, 0.0001], [0.099, 0.001] \right\rangle$	$\left\langle [0.85, 0.07], [0.76, 0.30], [0.04, 0.0004], [0.099, 0.001] \right\rangle$	$\left\langle [0.82, 0.16], [0.29, 0.25], [0.12, 0.001], [0.07, 0.0005] \right\rangle$
L_2	$\left\langle [0.96, 0.70] , [0.74, 0.09], [0.10, 0.002] , [0.03, 0.001] \right\rangle$	$\left\langle [0.5, 0.006] , [0.69, 0.09], [0.05, 0.0009] , [0.06, 0.004] \right\rangle$	$\left\langle [0.78, 0.36] , [0.94, 0.31], [0.05, 0.002] , [0.07, 0.001] \right\rangle$	$\left\langle [0.5, 0.25] , [0.93, 0.06], [0.05, 0.002] , [0.06, 0.0004] \right\rangle$
L_3	$\left\langle [0.95, 0.11], [0.57, 0.1], [0.08, 0.0001], [0.1, 0.0001] \right\rangle$	$\left\langle [0.62, 0.03], [0.94, 0.16], [0.09, 0.001], [0.06, 0.00005] \right\rangle$	$\left\langle [0.8, 0.33], [0.94, 0.25], [0.12, 0.001], [0.04, 0.009] \right\rangle$	$\left\langle [0.8, 0.42], [0.29, 0.23], [0.07, 0.002], [0.10, 0.002] \right\rangle$
L_4	$\left\langle [0.69, 0.08] , [0.82, 0.51], [0.005, 0.0002] , [0.06, 0.001] \right\rangle$	$\left\langle [0.93, 0.33] , [0.92, 0.37], [0.09, 0.0001] , [0.04, 0.0002] \right\rangle$	$\left\langle [0.76, 0.03] , [0.29, 0.19], [0.03, 0.001] , [0.04, 0.0002] \right\rangle$	$\left\langle [0.78, 0.003] , [0.62, 0.03], [0.04, 0.001] , [0.05, 0.0002] \right\rangle$

Step 3:Calculate the NLDRFWA operator in table 4.

	NLDREFEWA operator in table 4
L_1	$\langle [0.86, 0.08], [0.87, 0.25],[0.54, 0.001], [0.54, 0.0009] \rangle$
L_2	$\langle [0.85, 0.22], [0.92, 0.09],[0.53, 0.001], [0.52, 0.001] \rangle$
L_3	$\langle [0.9, 0.15], [0.86, 0.18],[0.54, 0.0008], [0.54, 0.0003] \rangle$
L_4	$\langle [0, 91, 0.012], [0.88, 0.13],[0.52, 0.0003], [0.52, 0.0002] \quad \rangle$

Step 4:Calculate the score function

$R_1 = 1.2395, R_2 = 1.2541, R_3 = 1, 2544, R_4 = 1, 3437.$

Step 5:Find the ranking $R_4 > R_3 > R_2 > R_1$ and R_4 is the best ranking.

5.2. Sensitivity analysic

In this subsection, we define the sensitive study and written below in table 5.

Methods	Score function	Ranking
$q = 0.1$	$\{R_1 = 0.0112, R_2 = 0.8011, R_3 = 0.3661, R_4 = 0.1269\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.2	$\{R_1 = 0.0122, R_2 = 0.8185, R_3 = 0.3989, R_4 = 0.1786\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.3	$\{R_1 = 0.0856, R_2 = 0.7845, R_3 = 0.5985, R_4 = 0.1896\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.4	$\{R_1 = 0.0555, R_2 = 0.4969, R_3 = 0.4858, R_4 = 0.1972\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.5	$\{R_1 = 0.0036, R_2 = 0.7625, R_3 = 0.3968, R_4 = 0.2038\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.6	$\{R_1 = 0.0441, R_2 = 0.8817, R_3 = 0.4896, R_4 = 0.1902\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.7	$\{R_1 = 0.0478, R_2 = 0.2987, R_3 = 0.2358, R_4 = 0.1588\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.8	$\{R_1 = 0.0459, R_2 = 0.5263, R_3 = 0.3975, R_4 = 0.1985\}$	$\{R_2 > R_3 > R_4 > R_1\}$
0.9	$\{R_1 = 0.1099, R_2 = 0.8978, R_3 = 0.3911, R_4 = 0.1458\}$	$\{R_2 > R_3 > R_4 > R_1\}$
1	$\{R_1 = 0.0015, R_2 = 0.6611, R_3 = 0.5062, R_4 = 0.1654\}$	$\{R_2 > R_3 > R_4 > R_1\}$

5.3. Analysis of Aggregation Operators in fuzzy Network Evaluation

The NLDRFS method in the fuzzy neural networking (FNN) framework. Several distance measures were used in the FNN model to calculate dominance scores under various parameter settings in order to guarantee a thorough study. These dominance values and the ensuing rankings are shown in Table 6, which also shows how various aggregation methods affect the final assessment of alternatives. Incorporating NLDRF data into the FNN structure resulted in significantly higher dominance ratings, suggesting that the framework conserved more relational information than alternative distance measures in table 6.

NLDRFS method Table 6.

Alternative	R_1	R_2	R_3	R_4
NLDFREWA	0.2006	0.7606	0.7878	0.9832
NLDFREOWA	0.1238	0.7006	0.0239	0.2997
NLDFREHWA	0.0431	0.6629	0.0076	0.1542

5.4. Superiority of proposed method

The evaluating fuzzy neural networking (FNN) technique allows for a more flexible and thorough representation of uncertainty, ambiguity, and vagueness. Decision-makers can more precisely express their choices for coverage, pricing, quality of service, and technical assistance by combining neural learning and fuzzy reasoning, even when the information is conflicting or insufficient.

While intuitionistic fuzzy sets and other traditional models can handle some degree of uncertainty, they frequently fail to capture the more profound ambiguities seen in expert evaluations. On the other hand, FNN's adaptive learning capability provides a more dependable and clever way to deal with ambiguity.

The FNN architecture offers more flexibility and granularity for representing expert preferences. It can catch tiny differences in assessments of important performance parameters, like network speed, coverage quality, and technical assistance responsiveness, thanks to its layered neural structure and fuzzy membership functions, producing more precise and perceptive analysis.

In contrast, intuitionistic fuzzy sets might not have the fine-grained detail and adaptability needed in dynamic, fiercely competitive markets. While fuzzy neural networking guarantees richer, data-driven evaluation outputs that better match the intricacies of the real world, this constraint may lead to less accurate or efficient conclusions.

5.5. Discussion

Several noteworthy themes emerged from this study's application of fuzzy neural networking (FNN), flexibility and adaptive learning capabilities of FNN, which enable nuanced assessment of intricate, uncertain, and dynamic decision settings, are major factors in its appeal. Using the suggested fuzzy neural networking paradigm, four 5G network providers were evaluated based on four important factors: cost-effectiveness, coverage quality, network speed, and technical assistance. The model correctly captured imprecise expert assessments and produced dependable decision outcomes by fusing the interpretability of fuzzy systems with the learning power of neural networks. The FNN approach is proven to be reliable in managing uncertainty, modeling a range of expert preferences, and producing trustworthy rankings in multi-criteria decision-making scenarios, according to the case study, numerical experiments, and aggregate analysis results. To calculate dominance scores, the FNN structure incorporates NLDRF data measurements. The results showed that Provider P3 continuously performed better than rivals, especially when it came to highly weighted factors like Network Speed and Coverage Quality. Closely behind, Provider P1 demonstrated well-rounded performance in every area. Provider P2 maintained a moderate position in the majority of scenarios, while Provider P4, although good in several areas, was unable to attain stable performance overall. This stratification highlights the fuzzy neural networking model's superiority over more rigid linear models or conventional fuzzy systems in differentiating between fiercely competing alternatives.

6. Conclusion

By combining nonlinear Diophantine rough fuzzy sets with Einstein t-norm and t-conorm aggregation operators and the learning capabilities of fuzzy neural networks, this study presents a novel multi-criteria decision-making paradigm. In situations that are imprecise and uncertain, the suggested including as NLDRFEWA, NLDRFEOWA, and NLDRFEHWA operators. The addition of FNN improved the model's capacity to handle highly nonlinear and ambiguous data by facilitating adaptive learning, pattern recognition, and dynamic weight modification. The usefulness and resilience of the suggested approach were illustrated by the case study on the assessment of fuzzy network providers, the framework yields findings that are stable, comprehensible, and logical, according to a comparative comparison with current MCDM methodologies. The framework enhanced accuracy and expressiveness while offering a more intelligent and adaptable decision-support tool by fusing rule-based fuzzy reasoning with neural learning. Overall, by providing a hybrid computational model that combines fuzzy logic, neural networking, and sophisticated aggregation techniques, the study makes a substantial contribution to the expanding corpus of fuzzy MCDM techniques. Beyond telecommunications, this strategy may find wider use in high-risk industries including engineering, healthcare, finance, and others where judgment calls are dominated by ambiguity and imprecision.

References

1. K. Atanassov, "Intuitionistic fuzzy sets," International Journal Bioautomation, vol. 20, p. 1, 2016.
2. K.T. Atanassov, "More on intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 33, no. 1, pp. 37–45, 1989.
3. H.J. Zimmermann, "Fuzzy set theory," Wiley Interdisciplinary Reviews: Computational Statistics, vol. 2, no. 3, pp. 317–332, 2010.
4. S.K. De, R. Biswas and A.R. Roy, "Some operations on intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 114, no. 3, pp. 477–484, 2000.
5. Z.S. Xu, "Models for multiple attribute decision making with intuitionistic fuzzy information," International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 15, no. 2, pp. 285–297, 2007.
6. Z.S. Xu, "Multi-person multi-attribute decision making models under intuitionistic fuzzy environment," Fuzzy Optimization and Decision Making, vol. 6, no. 3, pp. 221–236, 2007.
7. M.M. Xia, Z.S. Xu and B. Zhu, "Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm," Knowledge-Based Systems, vol. 31, pp. 78–88, 2012.
8. D.J. Yu, "Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making," Knowledge-Based Systems, vol. 30, pp. 57–66, 2012.
9. H. Bustince and P. Burillo, "Correlation of interval-valued intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 74, no. 2, pp. 237–244, 1995.
10. H. Bustince, "Construction of intuitionistic fuzzy relations with predetermined properties," Fuzzy Sets and Systems, vol. 109, no. 3, pp. 379–403, 2000.
11. D.J. Yu, "Group decision making based on generalized intuitionistic fuzzy prioritized geometric operator," International Journal of Intelligent Systems, vol. 27, no. 7, pp. 635–661, 2012.
12. C.Q. Tan, "Generalized intuitionistic fuzzy geometric aggregation operator and its application to multi-criteria group decision making," Soft Computing, vol. 15, no. 5, pp. 867–876, 2011.
13. M.M. Xia and Z.S. Xu, "Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment," Information Fusion, vol. 13, pp. 31–47, 2012.
14. M.M. Xia and Z.S. Xu, "Some new similarity measures for intuitionistic fuzzy values and their application in group decision making," Journal of Systems Science and Systems Engineering, vol. 19, pp. 430–452, 2010.
15. Z.W. Gong, L.S. Li, J. Forrest and Y. Zhao, "The optimal priority models of the intuitionistic fuzzy preference relation and their application in selecting industries with higher meteorological sensitivity," Expert Systems with Applications, vol. 38, pp. 4394–4402, 2011.
16. Z.W. Gong, L.S. Li, F.X. Zhou and T.X. Yao, "Goal programming approaches to obtain the priority vectors from the intuitionistic fuzzy preference relations," Computers & Industrial Engineering, vol. 57, pp. 1187–1193, 2009.
17. D.F. Li, "Multiattribute decision making method based on generalized OWA operators with intuitionistic fuzzy sets," Expert Systems with Applications, vol. 37, pp. 8673–8678, 2010.
18. D.F. Li, "TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets," IEEE Transactions on Fuzzy Systems, vol. 18, pp. 299–311, 2010.
19. D.F. Li, "Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information," Applied Soft Computing, vol. 11, pp. 3402–3418, 2011.
20. D.F. Li, "The GOWA operator based approach to multiattribute decision making using intuitionistic fuzzy sets," Mathematical and Computer Modelling, vol. 53, pp. 1182–1196, 2011.
21. G.W. Wei, "Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting," Knowledge-Based Systems, vol. 21, pp. 833–836, 2008.
22. G.W. Wei, "GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting," Knowledge-Based Systems, vol. 23, pp. 243–247, 2010.
23. G.W. Wei, "Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making," Expert Systems with Applications, vol. 38, pp. 11671–11677, 2011.
24. G.W. Wei and X.F. Zhao, "Minimum deviation models for multiple attribute decision making in intuitionistic fuzzy setting," International Journal of Computational Intelligence Systems, vol. 4, pp. 174–183, 2011.

25. G.W. Wei, H.J. Wang and R. Lin, "Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision making with incomplete weight information," Knowledge-Based Systems, vol. 23, pp. 243–247, 2011.

26. P. Grzegorzewski, "Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric," Fuzzy Sets and Systems, vol. 148, pp. 319–328, 2004.

27. L.A. Zadeh, "Fuzzy sets," Information and Control, vol. 8, no. 3, pp. 338–353, 1965.

28. W. Wang and X. Liu, "Intuitionistic fuzzy geometric aggregation operators based on Einstein operations," International Journal of Intelligent Systems, vol. 26, no. 11, pp. 1049–1075, 2011.

29. W. Wang and X. Liu, "Intuitionistic fuzzy information aggregation using Einstein operations," IEEE Transactions on Fuzzy Systems, vol. 20, no. 5, pp. 923–938, 2012.

30. X. Zhao and G.W. Wei, "Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making," Knowledge-Based Systems, vol. 37, pp. 472–479, 2013.

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