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Applications of an integral sliding mode control for secure digital data communication and synchronization of fractional order 5D non-Hamiltonian conservative hyperchaotic systems

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ABSTRACT: A conservative hyperchaotic system with fractional order in five dimensions is proposed in the study. It is found that the system is conservative but not Hamiltonian. By widening the scope of dynamical analysis, this work advances the understanding of non-Hamiltonian conservative systems. Integral sliding mode control is also used to synchronize conservative hyperchaotic systems of fractional order that are not Hamiltonian. An innovative safe digital data transfer algorithm is created using synchronized fractional order non-Hamiltonian conservative hyperchaotic systems. Numerical simulations verify the accuracy and effectiveness of theoretical conclusions.

Key Words: Fractional calculus, Integral sliding mode control, Synchronization, Digital data transmission

Contents

1. Introduction

Fractional order calculus extends classical calculus by allowing differentiation and integration of non-integer orders. Unlike traditional calculus, it provides a more flexible framework for modeling systems with memory effects and long range dependencies. Fractional calculus has become a crucial tool for accurately modeling complex real-world phenomena. Chaos theory [1,3] is a field of mathematics and science that investigates how intricate systems may exhibit erratic dynamics, even though they operate deterministically at their foundational level. A central concept in this paradigm involves the butterfly effect [14], indicating that trivial alterations in a system foundational circumstances may result in significantly divergent outcomes. In contrast to traditional sequential frameworks that presuppose an uncomplicated cause and consequences linkage, Chaos Theory focuses on nonlinear mechanisms, where systems are exceedingly responsive to their initial circumstances. This concept finds diverse applications across disciplines such as weather forecasting [28], financial modeling [33], chemistry [31,36], and even existential exploration, fundamentally transforming our understanding of how complex phenomena evolve [30]. By studying chaotic systems, scientists uncover hidden patterns in seemingly random behavior, offering profound insights into the workings of both natural and man-made systems [37,34]. Chaos can occur in three main types of systems: conservative, dissipative, and quantum [29]. In recent years, researchers have focused heavily

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on dissipative systems, uncovering many fascinating behaviors [4,5,6,16,20,24,27]. However, attention has recently shifted to conservative systems. Unlike dissipative systems, conservative systems do not produce attractors. Their dimension matches the system's order and it is always an integer. These systems are known for their richer ergodic properties and greater randomness compared to dissipative ones [10]. In 2015, Vaidyanathan introduced a novel three-dimensional non-equilibrium conservative chaotic system [7]. In 2017, Cang developed two four dimensional conservative chaotic systems [11]. In 2018, Qi designed a class of Hamiltonian conservative chaotic systems based on Euler equations [10]. In 2019, Dong categorized various types of conservative chaotic systems and introduced a class of Hamiltonian conservative chaotic systems with multiple stability [12]. The analysis of chaotic dynamics of Hamiltonian systems is mainly reflected in the integrable and non-integrable systems in the classical mechanics, such as Li and Chen [17] proposed a new periodic driven non-linear non-integrable Hamiltonian system and the chaotic phenomena are observed when the system is driven by the periodic signal with specific frequencies, Farina and Pozzoli looked the beam-plasma system as a reference of Hamiltonian system with many degrees of freedom and found that the development of self-consistent large amplitude oscillations occurs in correspondence with the onset of chaos. Gan [18] studied a quasi-integrable Hamiltonian system with two degree of freedom and this system can show chaos by changing the deterministic intensity of bounded noise. Tian introduced a five dimensional fractional order conservative chaotic system that demonstrates complex transient behavior [22].

Dynamical systems described by an ordinary differential equation are classified into these categories [8]: dissipative, conservative and expanding in phase volume depending on the sign of the divergence, $\nabla f < 0, \nabla f = 0$ respectively . where f is the vector field of the dynamical system. A chaotic system is deemed dissipative when the sum of its Lyapunov exponents is negative, signifying that the system tends to evolve toward a state of decreased complexity or increased order over time. Conversely, when the sum of the Lyapunov exponents is zero, the system is classified as conservative, indicating that it maintains its overall structure and energy over time. The properties of a system can be identified based on its equilibrium points. A chaotic system is classified as conservative when its equilibrium points consist only of a center point and a saddle point [13]. There are two types of conservative systems, they are Hamiltonian system and the non-Hamiltonian system. Conservative Hamiltonian systems strictly conserve both the total Hamiltonian energy and phase volume. Non-Hamiltonian conservative systems only maintain phase volume conservation without having a conserved Hamiltonian constant [21].

Conservative Hamiltonian chaotic systems [10,11,12] are a category of chaotic systems that maintain their overall structure and energy over time, distinguishing them from dissipative systems. These systems adhere to classical Hamiltonian mechanics and exhibit chaotic behavior while conserving energy. It typically involves a finite number of degrees of freedom and can be analyzed using established mathematical techniques. Non-Hamiltonian conservative chaotic systems are chaotic systems where the sum of Lyapunov exponents equals zero, indicating that they can exhibit chaos without the strict energy conservation typical of Hamiltonian systems. In this research, a comprehensive examination is conducted to categorize the proposed five-dimensional fractional order chaotic system according to its dynamical characteristics. By calculating the divergence of the system, it is demonstrated that the divergence is consistently zero, thereby confirming the conservative nature of the system. Yet, an additional examination of the Hamiltonian structure shows that the system is not allowed to have a skew-symmetric Jacobian matrix, which is a required condition for Hamiltonian systems. Therefore, the system, though it is a conservative phase volume-preserving system, does not meet the requirements for Hamiltonian dynamics and is thus a non-Hamiltonian conservative system. This differentiation is crucial, as non-Hamiltonian conservative chaotic systems can display intricate and diverse dynamics without conforming to rigid energy conservation principles, thus expanding the possibilities for modeling and controlling chaotic systems.

Chaotic synchronization [15,19,26] is a fundamental concept of non-linear dynamics, wherein two or more chaotic systems evolve in a cooperative manner despite initial condition sensitivity. Unlike traditional synchronization in periodic systems, chaotic synchronization involves complex interactions that enable trajectories of systems to synchronize [35]. This effect has received significant interest due to its wide applications in secure communication [37], neural networks, cryptography, and synchronized physical and chemical systems [36]. A number of synchronization schemes have been developed, including complete synchronization, in which the states of interacting systems become identical over time, and generalized synchronization [32], in which a functional relation between the systems is established. Other settings, such as phase synchronization and projective synchronization [38], provide greater flexibility in the control of the degree and type of synchronization. Chaotic synchronization research not only provides us with additional understanding of the dynamics of complex systems but also enables applications in practice to fields such as information security, control systems, and biomedical signal processing. With the appropriate synchronization techniques, researchers can harness chaos for creativity while sustaining stability and coherence in non-linear dynamical systems.

Integral sliding mode control ($ISMC$) is an advanced control technique [23] widely used for systems subject to uncertainties and external disturbances. It enhances the robustness of traditional sliding mode control by introducing an integral term in the design of the sliding surface. The approach ensures that the system trajectories remain on the sliding surface from the very beginning, effectively eliminating the reaching phase and reducing chattering. ISMC [25] is particularly suitable for controlling nonlinear, uncertain, and time-varying systems, making it a popular choice in applications such as robotics, power systems [26], and chaotic system synchronization. Here, a new five-dimensional fractional order conservative hyperchaotic system is designed and carefully analyzed.

Motivated by the above discussion, in the study the Hamiltonian structure of the system, it is verified to be conservative but non-Hamiltonian. The system shows rich dynamics, with it being an ideal candidate for complex control and synchronization tasks. An integral sliding mode control method is established to ensure synchronization of two identical fractional order non-Hamiltonian conservative hyperchaotic systems. The study provides a comprehensive framework for extending integral sliding mode control based techniques to broader classes of chaotic and complex systems. A secure digital data communication algorithm is designed using this synchronized setup, which utilizes the sensitivity and randomness of chaotic dynamics to provide security against malicious attacks on insecure channels. The theoretical findings are confirmed by comprehensive numerical simulations, confirming the efficiency and stability of the suggested control strategy and its feasibility in secure data exchange.

The article is structured as follows: Section 2, outlines the fundamental preliminaries, including essential concepts and theorems of fractional order systems. Section 3, provides a detailed description of the fractional order non-Hamiltonian conservative hyperchaotic system. In Section 4, an integral sliding mode control strategy is proposed for synchronization, accompanied by numerical simulations to verify its effectiveness. In Section 5, a new secure digital data transmission algorithm is proposed by utilizing synchronized systems and verified by numerical simulations. Lastly Section 6 concludes the manuscript.

2. Preliminaries

Definition 2.1 *[19] Caputo derivative for a function $f(x)$ is defined as*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (x-\tau)^{n-\alpha+1} f^{(n)}(\tau) d\tau \tag{2.1}$$

where n is the positive integer satisfying $n-1 < \alpha < n$, Γ is gamma function. $f^{(n)}$ is n^{th} order derivative of $f(x)$ and D^{α} is called the α-order Caputo differential operator.

Definition 2.2 *[9] Consider a commensurate fractional order system*

$$D^{\alpha} x(t) = Ax(t), x(0) = x_0, \tag{2.2}$$

where $0 < \alpha \leq 1, X \in \mathbb{R}^n$. An equilibrium point E of the system (2.2) is called a saddle point of index 1 (index 2) if the Jacobian matrix $J = \frac{\partial f(X)}{\partial X}$ at E has one eigenvalue (two eigen values) with non-negative real part.

Theorem 2.1 *[9] The commensurate fractional order system (2.2) is asymptotically stable if and only if*

$$|arg(eig(A))| > \frac{\alpha\pi}{2},$$

In this case, each component of the states decays toward 0 like $t^{-\alpha}$. Further, this system is stable if and only if $|arg(eig(A))| \geq \frac{\alpha\pi}{2}$, with those critical eigen values satisfying $|arg(eig(A))| = \frac{\alpha\pi}{2}$ having geometric multiplicity of one.

Theorem 2.2 *A necessary condition for the fractional order system (2.2) to remain chaotic is keeping at least one eigenvalue λ in the unstable region.*

$$\alpha > \frac{2}{\pi} tan^{-1}\left(\frac{|Im(\lambda)|}{Re(\lambda)}\right)$$

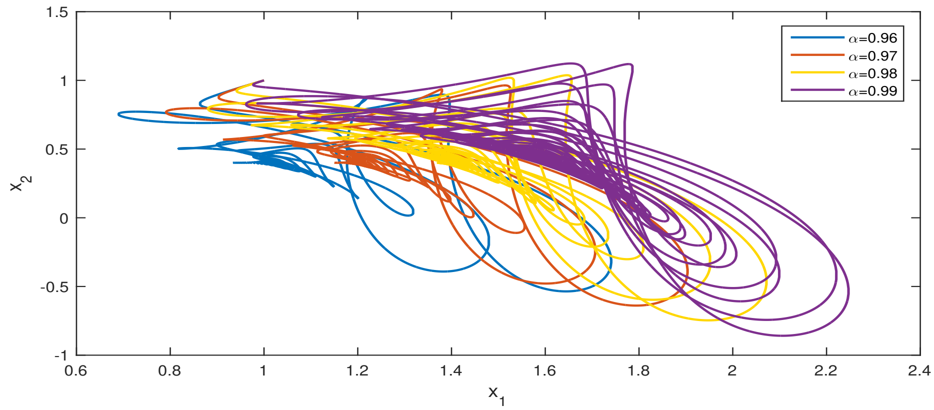

Figure 1: For $a = -3.5$, $b = 3.1$ phase trajectories of the system (3.1) corresponding to different order α (0.96 *to* 0.99) with initial condition (1,1,1,1,1).

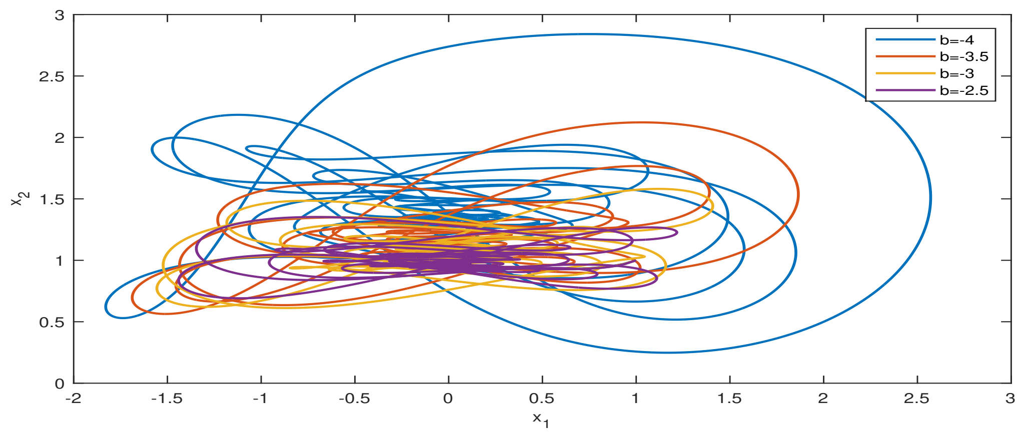

Figure 2: Phase trajectories of the system (3.1) corresponding to the parameters $b = -4, -3.5, -3, -2.5$ and $a = -3.5$ for $\alpha = 0.98$.

3. System description

Consider the fractional order non-Hamiltonian conservative hyperchaotic system [21],

$$\begin{aligned} D^\alpha x_1 &= x_1x_2 + ax_5 + bx_1 \\ D^\alpha x_2 &= -x_1^2 + x_3^2 \\ D^\alpha x_3 &= -x_2x_3 + x_4x_3 \\ D^\alpha x_4 &= -x_3^2 + x_5^2 \\ D^\alpha x_5 &= -ax_1 - x_4x_5 - bx_5 \end{aligned} \tag{3.1}$$

where D^α denotes the Caputo differential operator, $0 < \alpha \leq 1$ and a, b are parameters of the system (3.1). The divergence of the system (3.1) is calculated as

$$\begin{aligned} \nabla F &= \frac{\partial \dot{x_1}}{\partial x_1} + \frac{\partial \dot{x_2}}{\partial x_2} + \frac{\partial \dot{x_3}}{\partial x_3} + \frac{\partial \dot{x_4}}{\partial x_4} + \frac{\partial \dot{x_5}}{\partial x_5} \\ &= x_2 + b - x_2 + x_4 - b - x_4 \\ &= 0 \end{aligned} \tag{3.2}$$

$\nabla F = 0$, which demonstrates that the system is conservative. Find the Hamiltonian energy by $\dot{H} = J\,\nabla H$. Where J is a Jacobian matrix of the system (3.1).

$$\begin{pmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ D^\alpha x_3 \\ D^\alpha x_4 \\ D^\alpha x_5 \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \frac{\partial H}{\partial x_3} \\ \frac{\partial H}{\partial x_4} \\ \frac{\partial H}{\partial x_5} \end{pmatrix}$$

For the system (3.1) to be Hamiltonian, J should be skew-symmetric $(i.e)$ $J_{ij} = -J_{ji}$.

$$J = \begin{pmatrix} x_2 + b & x_1 & 0 & 0 & a \\ -2x_1 & 0 & 2x_3 & 0 & 0 \\ 0 & -x_3 & -x_2 + x_4 & x_3 & 0 \\ 0 & 0 & -2x_3 & 0 & 2x_5 \\ -a & 0 & 0 & -x_5 & -b - x_4 \end{pmatrix}$$

Here J does not satisfy the skew symmetric property. It clearly shows that (3.1) is non-Hamiltonian. Conservative systems create closed orbits close to fixed points and have a conserved phase volume. The dynamical behaviors of non-Hamiltonian conservative chaotic system (3.1) like symmetry, amplitude modulation property, transient behaviors and multi stability are investigated in [21]. Fixing parameters $a = -3.5$, $b = 3.1$, initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (1, 1, 1, 1, 1)$ and the $x_1 - x_2$ phase state trajectories of the system (3.1) with various fractional order α are depicted in Figure 1. By varying the parameter b, and plots the phase diagram in the $x_1 - x_2$ plane as depicted in Figure 2.

4. An integral sliding mode control for synchronization of non-Hamiltonian systems

In this section, an integral sliding mode control theory [23] is applied to achieve synchronization between two identical fractional order conservative hyperchaotic systems. The non-Hamiltonian conservative master system is described in (3.1) and the slave system with controller is described as follows

$$\begin{aligned} D^\alpha y_1 &= y_1y_2 + ay_5 + by_1 + u_1 \\ D^\alpha y_2 &= -y_1^2 + y_3^2 + u_2 \\ D^\alpha y_3 &= -y_2y_3 + y_4y_3 + u_3 \\ D^\alpha y_4 &= -y_3^2 + y_5^2 + u_4 \\ D^\alpha y_5 &= -ay_1 - y_4y_5 - by_5 + u_5 \end{aligned} \tag{4.1}$$

where $u_i (i = 1, 2, ..., 5)$ are controllers to be determined later. Define the synchronization error between systems (3.1) and (4.1) as

$$e_i = y_i - x_i, (i = 1, 2, ..., 5). \tag{4.2}$$

The fractional order error dynamical system is

$$\begin{aligned}
D^\alpha e_1 &= be_1 + ae_5 + y_1 y_2 - x_1 x_2 + u_1 \\
D^\alpha e_2 &= -e_1(y_1 + x_1) + e_3(y_3 + x_3) + u_2 \\
D^\alpha e_3 &= -e_3 y_2 - e_2 x_3 + e_3 y_4 + e_4 x_3 + u_3 \\
D^\alpha e_4 &= -e_3(y_3 + x_3) + e_5(y_5 + x_5) + u_4 \\
D^\alpha e_5 &= -e_4 y_5 - e_5 x_4 - ae_1 - be_5 + u_5
\end{aligned} \tag{4.3}$$

Consider the sliding surface,

$$s_i(t) = e_i(t) + \gamma_i \int_0^t e_i(\tau) d\tau, (i = 1, 2, ..., 5) \tag{4.4}$$

For every $\gamma_i > 0$, the design of the sliding surface and the control law ensure finite time stability, with synchronization errors converging to zero within a finite time frame. The integral sliding surface is defined in the space of the synchronization error. In the case of non-Hamiltonian conservative chaotic systems, the sliding surface must be carefully designed to stabilize or synchronize the chaotic behavior while preserving the system$'s$ conservative property. According to the sliding mode control method, when the error system reaches the sliding surface, it must satisfy the condition $s_i(t) = 0$ and $\dot{s}_i(t) = 0$, $i = 1, 2, ..., 5$. The ultimate goal is to find the suitable controller u_i such that $\lim_{t \to \infty} \|e_i(t)\| = 0$. Differentiating the surface equations given in (4.4),

$$D^\alpha s_i(t) = D^\alpha e_i(t) + \gamma_i e_i(t), (i = 1, 2, ..., 5) \tag{4.5}$$

Since $\dot{s}_i(t) = 0$, we obtain the following sliding mode dynamics

$$D^\alpha e_i(t) = -\gamma_i e_i(t) \tag{4.6}$$

Theorem 4.1 *Synchronization between non-Hamiltonian conservative hyperchaotic systems (3.1) and (4.1) is achieved for the following integral sliding mode control law*

$$u_i = u_i^{eq} + u_i^d, \quad i = 1, 2, ..., 5. \tag{4.7}$$

where u_i is the total control input, u_i^{eq} is the equivalent control laws, and u_i^d is a discontinuous reaching laws $(i = 1, 2, ..., 5)$.

Proof: The total control input u_i is selected as the summation of the equivalent control laws and the discontinuous reaching laws. According to the integral sliding mode control theory, u_i^{eq} and u_i^d are selected as follows

$$\begin{aligned}
u_1^{eq} &= -be_1 - ae_5 - y_1 y_2 + x_1 x_2 - \gamma_1 e_1 \\
u_2^{eq} &= e_1(y_1 + x_1) - e_3(y_3 + x_3) - \gamma_2 e_2 \\
u_3^{eq} &= e_3 y_2 + e_2 x_3 - e_3 y_4 - e_4 x_3 - \gamma_3 e_3 \\
u_4^{eq} &= e_3(y_3 + x_3) - e_5(y_5 + x_5) - \gamma_4 e_4 \\
u_5^{eq} &= e_4 y_5 + e_5 x_4 + ae_1 + be_5 - \gamma_5 e_5
\end{aligned} \tag{4.8}$$

and

$$u_i^d = -\eta_i sign(s_i) - k_i s_i, \quad i = 1, 2, ..., 5. \tag{4.9}$$

Where $\eta_i, k_i (i = 1, 2, ..., 5)$ are positive gains and

$$sign(s_i) = \begin{cases} +1 & if \quad s_i > 0 \\ 0 & if \quad s_i = 0 \\ -1 & if \quad s_i < 0 \end{cases}$$

Substituting (4.7) into (4.3), one can obtain that

$$\begin{aligned} D^\alpha e_1(t) &= -\eta_1 sign(s_1) - k_1 s_1 - \gamma_1 e_1(t) \\ D^\alpha e_2(t) &= -\eta_2 sign(s_2) - k_2 s_2 - \gamma_2 e_2(t) \\ D^\alpha e_3(t) &= -\eta_3 sign(s_3) - k_3 s_3 - \gamma_3 e_3(t) \\ D^\alpha e_4(t) &= -\eta_4 sign(s_4) - k_4 s_4 - \gamma_4 e_4(t) \\ D^\alpha e_5(t) &= -\eta_5 sign(s_5) - k_5 s_5 - \gamma_5 e_5(t) \end{aligned} \tag{4.10}$$

Consider the Lyapunov function as $V(t) = \frac{1}{2} \sum_{i=1}^{5} s_i^2$. Then the time derivative of $V(t)$ is

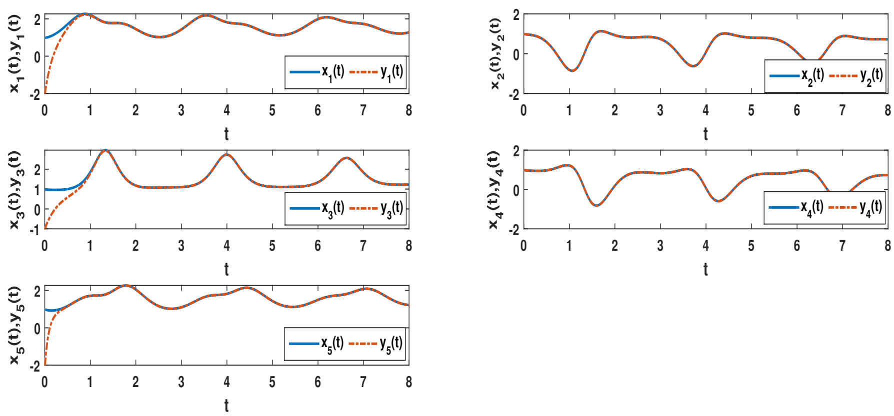

Figure 3: Time response of synchronized states of the master system (3.1) and the slave system (4.1)

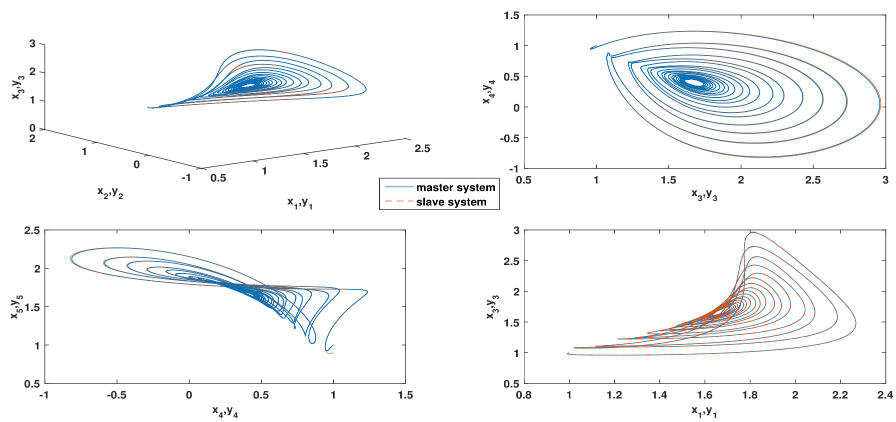

Figure 4: Different phase portraits of synchronized chaotic attractors of the master system (3.1) and the slave system (4.1)

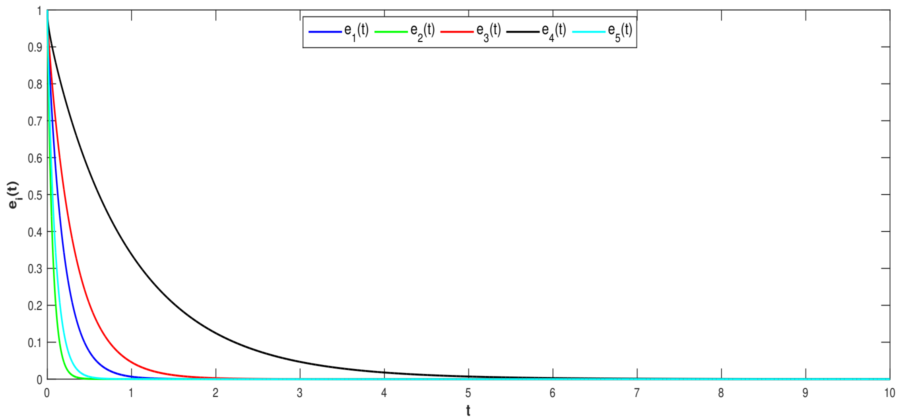

Figure 5: Time response of synchronized error (4.10) between the master system (3.1) and the slave system (4.1)

$$\begin{aligned}
D^\alpha V(t) &= s_1 D^\alpha s_1 + s_2 D^\alpha s_2 + s_3 D^\alpha s_3 + s_4 D^\alpha s_4 + s_5 D^\alpha s_5 \\
&= s_1(-\eta_1 sign(s_1) - k_1 s_1) + s_2(-\eta_2 sign(s_2) - k_2 s_2) + s_3(-\eta_3 sign(s_3) - k_3 s_3) + \\
&\quad s_4(-\eta_4 sign(s_4) - k_4 s_4) + s_5(-\eta_5 sign(s_5) - k_5 s_5) \\
&= \sum_{i=1}^{5} (s_i(-\eta_i sign(s_i) - k_i s_i)) \\
&= -\sum_{i=1}^{5} (\eta_i |s_i| + k_i s_i^2)
\end{aligned}$$

Clearly $D^\alpha V(t)$ is negative definite function since $\eta_i, k_i > 0$ for $i = 1, 2, ..., 5$. Using Lyapunov stability theory, $s_i \to 0, (i = 1, 2, ..., 5)$ as $t \to \infty$. It follows that $\eta_i \to 0, (i = 1, 2, ..., 5)$ as $t \to \infty$. The state trajectories of the error system $\lim_{t\to 0} \|e_i(t)\| = 0$ as $t \to \infty$ for $i = 1, 2, 3, 4, 5$ is asymptotically stable. Hence the two identical non-Hamiltonian fractional order conservative hyperchaotic systems are synchronized successfully. □

4.1. Numerical Simulations

For numerical simulations, assume that $a = -3.5$,$b = 3.1$, the initial conditions are $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (1, 1, 1, 1, 1)$ and $(y_1(0), y_2(0), y_3(0), y_4(0), y_5(0)) = (-2, 1, -1, 1, -2)$ .The control parameters are chosen as follows: $\eta_1 = 0.3, \eta_2 = 0.5, \eta_3 = 0.2, \eta_4 = 0.4, \eta_5 = 0.7$, $k_1 = 2.5, k_2 = 3, k_3 = 3.9, k_4 = 7.8, k_5 = 5$ and $\gamma_1 = 5, \gamma_2 = 15, \gamma_3 = 3, \gamma_4 = 4, \gamma_5 = 10$. The time response of each states of (3.1) and (4.1) is depicted in Figure 3. Different phase portraits of synchronized chaotic attractors of master and slave system is depicted in Figure 4. The time response of the error system (4.10) is depicted in Figure 5.

5. Application of the proposed synchronization scheme

This section explores the application of synchronized identical fractional order chaotic systems in the development of a digital data transmission via insecure channel. Taking advantage of the unpredictable and sensitive nature of chaotic behavior, this method ensures enhanced security. The synchronization of these systems allows for precise digital data recovery during decryption. This approach provides robustness against external attacks and noise interference. In general, it presents a secure and efficient technique for modern digital communication systems.

5.1. Secure digital data transmission algorithm:

The proposed algorithm is structured into three distinct phases to ensure secure digital communication : First is the key generation phase, where cryptographic keys are derived using synchronized fractional order chaotic systems. Next is the encryption phase, where the original digital data is transformed into an unintelligible form using the generated keys. This is followed by the decryption phase, where the encrypted digital data is accurately reconstructed using synchronization and the proposed keys. To construct the encryption and decryption algorithm, the following assumptions are made:

Consider the fractional order drive system (3.1) as the sender's (S) system and the response system (4.1) as receiver's (R) system. Both S and R agree on the parameters (t_0, t_1, g, p, α) where the synchronization error between (3.1) and (4.1) tends to zero from time t_0 onward.

1. Key generation phase

i. Choose a random number $t_1 \geq t_0$ and solve the system (3.1) at time t_1 then use the results to compute the private key K_1 and generate the corresponding public key K_2 by

$$K_1 \equiv \lfloor \frac{1}{\alpha} * (x_1(t_1) + x_2(t_1) + x_3(t_1) + x_4(t_1) + x_5(t_1)) \ * 10^4 \rfloor \ (mod\ p)$$
$$K_2 \equiv g^{K_1} \ (mod\ p)$$

The sender kept their private key K_1 secret.

ii. R solves the system (4.1) at time t_1. Then computes the private keys K_3 and public key K_4 by

$$K_3 \equiv \lfloor \frac{1}{\alpha} * (y_1(t_1) + y_2(t_1) + y_3(t_1) + y_4(t_1) + y_5(t_1)) \ * 10^4 \rfloor \ (mod\ p)$$
$$K_4 \equiv g^{K_3} \ (mod\ p)$$

The receiver kept their private key K_3 secret.

iii. Sender and receiver are published their public keys K_2 and K_4.

2. Encryption phase

iv. S wants to send a digital message $\mathbb{M}$ secretly via insecure channel.

v. S computes an encrypted message $\mathbb{E}$ of $\mathbb{M}$ by

$$\mathbb{E} \equiv \mathbb{M} * (K_1) * (K_4)^{-1} * g \ (mod\ p)$$

vi. Further, S sends $\mathbb{E}$ to receiver.

3. Decryption phase

vii. R receives $\mathbb{E}$ from the sender and recovers an original digital message $\mathbb{M}$ by decrypting $\mathbb{D}$ encrypted message $\mathbb{E}$

$$\mathbb{D} \equiv \mathbb{E} * (K_2) * (K_3)^{-1} * g^{-1} \ (mod\ p) \tag{5.1}$$

For,

$$\begin{aligned}\mathbb{D} \equiv& \mathbb{E} * (K_2) * (K_3)^{-1} * g^{-1} \ (mod\ p)\\ \equiv& \mathbb{M} * (K_1) * (K_4)^{-1} * (K_2) * (K_3)^{-1} \ (mod\ p)\\ \equiv& \mathbb{M}\end{aligned}$$

Remark 5.1 *In the proposed algorithm, $\mathbb{M}$ is the any type of digital message. Therefore, the proposed algorithm is suitable for all types of digital data transmission like text, image, audio and video.*

5.2. Security analysis

Assume that an adversary needs to locate the private keys K_1 and K_3 in order to recover the original message $\mathbb{M}$. In this case, K_1 and K_3 deal with the fractional order systems (3.1) and (4.1) at time t_1 respectively, with the fractional order α. Furthermore, the fractional order systems (3.1) and (4.1) are chaotic systems which are sensitivity depends on initial conditions. Also the public keys K_2 and K_4 are discrete logarithm problem, which involves the private keys K_1 and K_3 respectively. Apart from any other existing algorithms, the public keys are also secure in the proposed algorithm due to the complexity of private keys and solving discrete logarithm problem. No one cannot be decrypted the original message $\mathbb{M}$ except by the receiver. Comprehensively, the proposed degital data transmission algorithm is more secure for sharing digital messages via insecure channel. Due to the complexity of solving fractional order systems, it is extremely difficult to hack the private keys. Therefore, the suggested algorithm's private keys are more safe and secure.

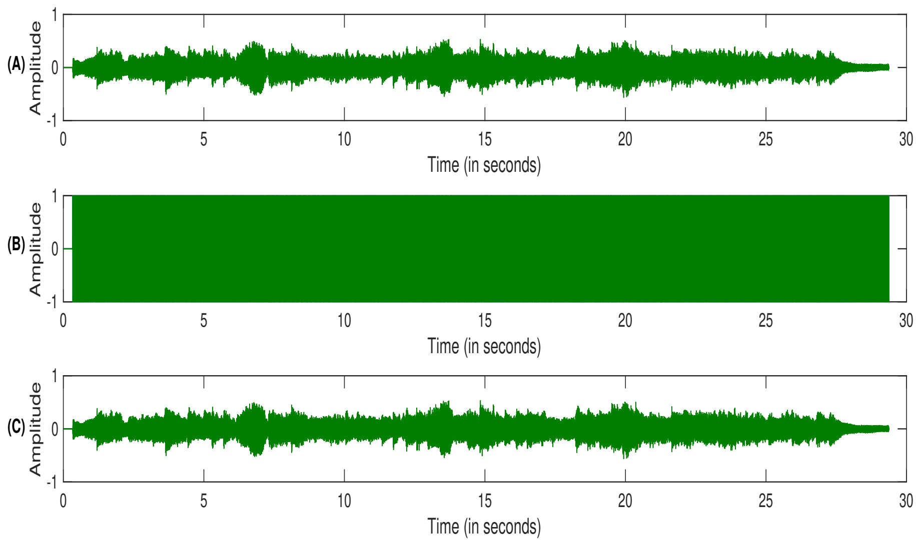

Figure 6: Digital audio signals: (A) original $\mathbb{M}$ (B) encrypted $\mathbb{E}$ and (C) decrypted $\mathbb{D}$

5.3. Demonstration of the proposed algorithm

Consider the digital audio signal as $\mathbb{M}$. Assume that S and R agree on $t_0 = 6, p = 353, g = 11, \alpha = 0.98$. When $\alpha = 0.98$, the sender computed the solutions $x_1(t_1) = 1, x_2(t_1) = 0.8, x_3(t_1) = 1.2, x_4(t_1) = 0.7, x_5(t_1) = 0.8$ of the fractional order system (3.1) at time $t_1 = 8$. When $\alpha = 0.98$ the receiver computed the solutions $y_1(t_1) = 1, y_2(t_1) = 0.8, y_3(t_1) = 1.2, y_4(t_1) = 0.7, y_5(t_1) = 0.8$ of the fractional order system (4.1) at time $t_1 = 8$. Further, S and R computed the corresponding keys $K_1 = 28 = K_3$ and $K_2 = 38 = K_4$. Then

$$\begin{aligned}\mathbb{E} \equiv& \mathbb{M} * (K_1) * (K_4)^{-1} * g \ (mod\ p)\\ \equiv& \mathbb{M} * 28 * (38)^{-1} * 11 \ (mod\ 353)\\ \equiv& \mathbb{M} * 101 \ (mod\ 353)\end{aligned}$$

S sends the encrypted message E to R. Then R recovers the original message by computing

$$\begin{aligned} \mathbb{D} \equiv & \mathbb{E} * (K_2) * (K_3)^{-1} * g^{-1} \ (mod\ p) \\ \equiv & \mathbb{M} * 101 * 38 * (28)^{-1} * (11)^{-1} \ (mod\ 353) \\ \mathbb{D} \equiv & \mathbb{M} \end{aligned} \tag{5.2}$$

The original, encrypted and decrypted audio signals are depicted in Figure 6 and their corresponding spectrogram is depicted in Figure 7 respectively. It shows that there is no similarity between original and encrypted digital audio signals.

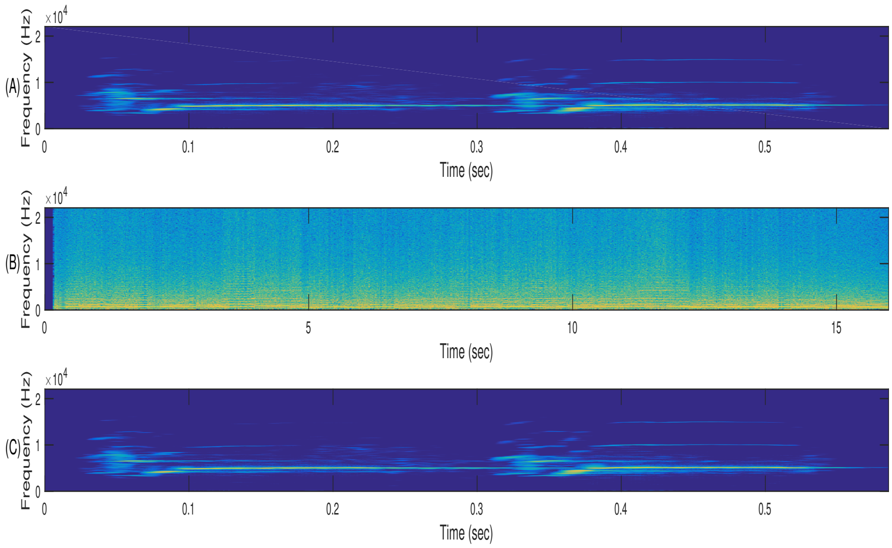

Figure 7: Spectrogram of digital audio signals: (A) original $\mathbb{M}$ (B) encrypted $\mathbb{E}$ and (C) decrypted $\mathbb{D}$

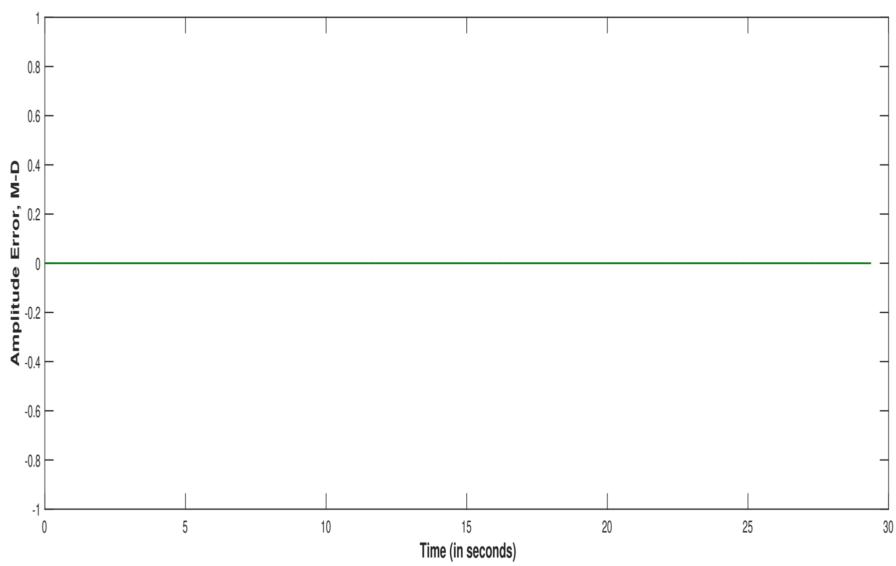

Figure 8: The error between the original $\mathbb{M}$ and decrypted $\mathbb{D}$ digital audio signals

Remark 5.2 *In the proposed algorithm, the receiver decrypted the exact original message without loss of any digital data information. Evidently, the error between original and decrypted digital audio signals is zero, which is depicted in Figure 8.*

6. Conclusion

The dynamical behavior of a five dimensional fractional order non-Hamiltonian conservative hyperchaotic system is proposed and investigated in the paper. We determine that the system is conservative but not Hamiltonian. An integral sliding mode control approach has been applied to synchronize fractional order non-Hamiltonian conservative hyperchaotic systems. Numerical simulations supports the theoretical conclusions and confirms the validity and effectiveness of the proposed control technique. We develop a novel safe digital data transmission method using synchronized fractional order non-Hamiltonian conservative hyperchaotic systems. The effectiveness of the suggested algorithm is confirmed by numerical simulations.

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