Bol. Soc. Paran. Mat.
(3s.) **v. 2026 (44)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.78488

On Total Graphs Related to Cosingular Submodule

Amirmohammad Momeni Kohestani and Yahya Talebi

ABSTRACT: Let W be a unitary right A-module, where A is a ring with identity. In this paper, we will introduce a new total graph that is related to cosingular submodule. The cosingular submodule, denoted by $Z^*(W)$, is defined as the set of all elements $w \in W$ such that $wA \ll W$. The total graph of a module W with respect to $Z^*(W)$, denoted by $T(\Gamma(W))$, is defined as an undirect simple graph with all elements of W as vertices and two vertices w and k are adjacent, written as w adj k, if and only if $w + k \in Z^*(W)$. We investigate some results for connectivity, completeness and planarity of these graphs and their induced subgraphs.

Keywords: Cosingular submodule, total graph, singular submodule, small submodule.

Contents

1. Introduction

In this work, A denotes a (not necessarily commutative) ring with a unity element, and W represents an arbitrary unitary right A-module. In 1988, Istvan Beck established an important connection between graph theory and the algebraic structure of rings [12]. He presented the concept of zero divisor graphs of commutative rings, which was later improved in [8]. This concept has been revised several times as mentioned in [7]. Building upon Beck's influential idea, many connections have been established between graph theory and algebraic structures. Several notable examples of these graphs are presented in [1,5,13,17,20]. In [5], the concept of a total graph of a commutative ring was introduced. It is a simple graph with a vertex set of all elements of a ring and two vertices in this graph are adjacent if the sum of them is a zero divisor of the ring. In their research, the authors examined some attributes and examples of the total graph and its two induced subgraphs in two different ways. The first scenario involved when $Z(A)$ of A forming an ideal of A, while the second scenario, $Z(A)$, is not forming an ideal. Expanding upon previous research, Akbari et al. [4] delved deeper into this kind of graph. In [2], the authors explored the relationship between proper ideals of a ring and its total graph. Furthermore, in [5], the authors specifically examined these graphs, without considering the zero element.

In [9,14,21], fundamental attributes of total graphs, related to finite commutative rings were presented. This stimulated interest in exploring total graphs of modules. For example, in [3] the authors examined total graphs of proper submodules of modules, while S. Atani and S. Habibi [10] introduced the concept of total graph related to torsion elements of modules. These developments in total graph related to module theory build on the foundational work originally proposed by Anderson and Badawi [5].

In 2016, Goswami et al. [15] defined a new total graph of modules that related to their singular submodule $Z(W)$ and presented the idea of a connection between a module's singularity and total graphs. This paper will define the total graph of a module W that relates to its cosingular submodule $Z^*(W)$, discuss the cosingularity of a module over a ring, and look at some results on this type of total graph. The study enhances the understanding of the interplay between algebraic structures and graph theory through the lens of cosingular submodules, highlighting how these mathematical frameworks can inform and enrich one another. By establishing a clear connection between the properties of cosingular submodules and the structure of total graphs, the research contributes to a deeper comprehension of module theory and its

2020 *Mathematics Subject Classification:* 05C15, 05C25, 05C75.
Submitted August 19, 2025. Published March 19, 2026

graphical representations. Furthermore, the findings open avenues for future research on total graphs in various algebraic contexts, potentially leading to new insights in both fields. It is crucial to go over the following details before we begin our conversation. Let $\eta(A)$ be the collection of all of a ring A's essential ideals. The set $Z(W) = \{w \in W | wJ = 0$ for some $J \in \eta(A)\}$ is a submodule of W given any right A-module W. It is obvious that $Z(W)$ is a fully invariant submodule of W. A module W is called a singular module provided by $Z(W) = W$. At the other extreme, a module W is called nonsingular, provided $Z(W) = 0$. A submodule L of a module W is defined as superfluous or small if, for every submodule, K of W, $B + K = W$ implies $K = W$ and we write $B \ll W$. The set $Z^*(W) = \{w \in W | wA \ll W\}$ is also a submodule of W. We define $Z^*(W)$ as cosingular submodule of W. Let W and E be A-modules. For a right A-module W, the radical of W denoted by $Rad(W)$ is defined as a sum of all superfluous submodules in W. A nonzero A-module T is defined hollow, when every proper submodule of T will be superfluous. Any undefined terminology can be found in [11,16,18,19,22].

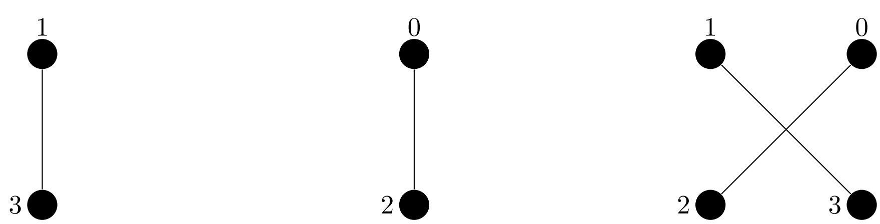
2. Preliminary Results

Let us begin this section by presenting our definitions and a simple example of $T(\Gamma(W))$. Our definition leads to the appearance of two induced subgraphs of $T(\Gamma(W))$ that we denote by $Z^*(\Gamma(W))$ and $\bar{Z}^*(\Gamma(W))$. We show that if ψ is an A-module homomorphism between two modules, W and K, then it preserves adjacency in the total graph. For instance, we prove that if ψ is an A-module isomorphism between two modules, then it is also an isomorphism between their total graph. At the end of this section we show that the induced subgraph $Z^*(\Gamma(W))$ is a complete subgraph.

Definition 2.1 *The total graph related to $Z^*(W)$, denoted by $T(\Gamma(W))$, is defined as an undirected simple graph where all elements of W serve as vertices. For any two distinct elements w and v in W, we say that the vertices w and v are connected (written as w adj v) if and only if $w + v$ is an element of $Z^*(W)$.*

Definition 2.2 *An induced subgraph of $T(\Gamma(W))$ is denoted by $Z^*(\Gamma(W))$ and defined as a subgraph that the set of all vertices is the elements of $Z^*(W)$. Another induced subgraph is denoted by $\bar{Z}^*(\Gamma(W))$ and it is the induced subgraph of $T(\Gamma(W))$ with the vertex set $\bar{Z}^*(W) = W \backslash Z^*(W)$.*

Example 2.1 *If $A = \mathbb{Z}_8$ and $W = \mathbb{Z}_4$, then vertex set of $T(\Gamma(W))$ is the elements of $\mathbb{Z}_4$ i.e $\{0, 1, 2, 3\}$. In this case $Z^*(W) = \{0, 2\}$ and hence $\bar{Z}^*(W) = \{1, 3\}$. As we see in this example, we have two subgraphs, that both of them are complete and connected. In this case $T(\Gamma(W))$ is not complete and it is disconnected graph and $\gamma(T(\Gamma(W))) = 2$ and $\Delta(T(\Gamma(W))) = \delta(T(\Gamma(W))) = 1$.*



(a) $\bar{Z}^*(\Gamma(W))$ (b) $Z^*(\Gamma(W))$ (c) $T(\Gamma(W))$

Lemma 2.1 *Let $\psi : W_1 \longrightarrow W_2$ be an A-module homomorphism. Then for every $a, b \in W_1$, if a adjacent to b in $T(\Gamma(W_1))$ then $\psi(a)$ adjacent to $\psi(b)$ in $T(\Gamma(W_2))$.*

Proof: Inasmuch as a adjacent to b we conclude that $a + b \in Z^*(W)$, so $(a + b)A \ll W_1$. Thus $\psi((a+b)R) \ll W_2$ and we have $\psi((a+b)A) = \psi(aA+bA) = (\psi(x)+\psi(y))A \ll W_2$, so by definitions we have $\psi(a) + \psi(b) \in Z^*(W_2)$, thus $\psi(a)$ adjacent to $\psi(b)$. □

Theorem 2.1 *Let $\psi : W_1 \longrightarrow W_2$ be an A-module homomorphism. If $T(\Gamma(W))$ is a complete graph, then $T(\Gamma(\psi(W_1)))$ is also a complete graph.*

Proof: Let $T(\Gamma(W_1))$ be a complete graph. Consider two arbitrary elements $b_1, b_2 \in \psi(W_1)$. So we have $a_1, a_2 \in W_1$ such that $y_1 = \psi(a_1), b_2 = \psi(a_2)$. Inasmuch as $T(\Gamma(W_1))$ is a complete graph, a_1 adjacent to a_2 and by previous lemma, $\psi(a_1)$ adjacent to $\psi(a_2)$. Thus, $T(\Gamma(\psi(W_1)))$ is a complete graph. □

Theorem 2.2 *Let $\psi : W_1 \longrightarrow W_2$ be an A-module isomorphism. Then ψ is an isomorphism between $T(\Gamma(W_1))$ and $\psi(\Gamma(W_2))$.*

Proof: To illustrate that there is an isomorphism between two graphs, we must show that ψ preserves edge relations, i.e. suppose for all $a, b \in W_1$ that a adjacent to b, we must show that $\psi(a)$ is adjacent to $\psi(b)$. Inasmuch as $(a+b) \in Z^*(W_1)$ we have $(a+b)A \ll W_1$, hence $\psi((a+b)A) \ll W_2$, so $\psi(a) + \psi(b) \in Z^*(W_2)$ and we conclude that $\psi(a)$ adjacent to $\psi(b)$. □

Theorem 2.3 *For all $a, b \in \bar{Z}^*(W)$, a adjacent to b if and only if every element of $a + Z^*(W)$ adjacent to every element of $b + Z^*(W)$.*

Proof: Let $x = a + z_1 \in Z^*(W), y = b + z_2 \in Z^*(W)$ with $z_1, z_2 \in Z^*(W)$. Assume that a adjacent to y i.e $(a + b) \in Z^*(W)$. Thus, we have $a = x - z_1, b = y - z_2$ and then $((x - z_1) + (y - z_2)) \in Z^*(W)$ hence $((x + y) - (z_1 + z_2)) \in Z^*(W)$, since $Z^*(W)$ is a submodule of W, $x + y \in Z^*(W)$ hence x adjacent to y. Conversely, let x adjacent to y, then $x + y \in Z^*(W)$ i.e $((a + z_1) + (b + z_2)) \in Z^*(W)$, so $((a + b) + (z_1 + z_2)) \in Z^*(W)$, hence $a + b \in Z^*(W)$ i.e. a adjacent to b. □

Theorem 2.4 *The following statements hold:*
(i) $Z^*(\Gamma(W))$ is complete, and it is not connected to $\bar{Z}^*(\Gamma(W))$.
(ii) Assume that K is a submodule of W, then $T(\Gamma(K))$ will be an induced subgraph of $T(\Gamma(W))$.

Proof: (i) According to the definitions, we know that $Z^*(\Gamma(W))$ is an induced subgraph of $T(\Gamma(W))$. Therefore, it is sufficient to prove that this induced subgraph is complete. To do so, consider a and b are two distinct elements in $Z^*(W)$. Since $Z^*(W)$ is a submodule of W, we can conclude that $a+b \in Z^*(W)$, this indicates that a is adjacent to b. Hence, $Z^*(\Gamma(W))$ is indeed an induced subgraph. Furthermore, since the vertices of $Z^*(\Gamma(W))$ are from $Z^*(W)$ and the vertices of $\bar{Z}^*(\Gamma(W))$ are from $\bar{Z}^*(W)$, so $Z^*(\Gamma(W))$ and $\bar{Z}^*(\Gamma(W))$ are disjoint.

(ii) This fact is obvious from the definitions. □

3. Main Results

In this section we obtain some properties such as connectedness and completeness, and discuss Hamilton cycles in $T(\Gamma(W))$. We introduce some modules whose total graphs are complete and study the conditions under which the total graph is planar.

Theorem 3.1 *Let W be an A-module. Then the following holds:*

(i) If H is a subgraph of $\bar{Z}^*(\Gamma(W))$ and a and b are two distinct vertices in H such that they are connected by a path, then there exists a path in H of length 2 between a and b. Also, if $Z^*(\Gamma(W))$ is connected, then $diam(\bar{Z}^*(W)) \leq 2$.

(ii) If a and b are two distinct vertices of $\bar{Z}^*(\Gamma(W))$, they are connected together by a path. If we have $a + b \notin Z^*(W)$, then we have 2 paths of length 2 between a and b, $a - (-a) - b$ and $a - (-b) - b$.

Proof: (i)It is sufficient to demonstrate that if a_1, a_2, a_3, a_4 are distinct vertices in Γ and there exists a path $a_1 - a_2 - a_3 - a_4$ (with a length exceeding 2) connecting a_1 to a_4, then a_1 and a_4 must be adjacent. Analyzing the path reveals that $a_1 + a_2 \in Z^*(W)$, $a_2 + a_3 \in Z^*(W)$, and $a_3 + a_4 \in Z^*(W)$. Consequently, since $a_1 + a_4 = (a_1 + a_2) - (a_2 + a_3) + (a_3 + a_4) \in Z^*(W)$, we can deduce that a_1 and a_4 are adjacent. Thus, if $\bar{Z}^*(\Gamma(W))$ is connected, it follows that $diam(\bar{Z}^*(\Gamma(W))) \leq 2$.

(ii) According to the stated hypothesis, there exists an element $z \in Z^*(\Gamma(W))$ such that $a - z - b$ constitutes a path of length 2. This implies that $(a+b) \in Z^*(W)$ and $(b+z) \in Z^*(W)$. Therefore, it can be inferred that $a - b = (a+z) - (z+b) \in Z^*(W)$. Furthermore, given that $a + b \notin Z^*(W)$, we can conclude that $a \neq -a$ and $b \neq -b$. This indicates that $a - (-a) - b$ and $a - (-b) - b$ are two distinct paths that have a length 2 between a and b in $\bar{Z}^*(\Gamma(W))$. □

Theorem 3.2 *The following conditions are equivalent.*
(i) $\bar{Z}^*(\Gamma(W))$ is connected.
(ii) Either $a + b \in Z^*(W)$ or $a - b \in Z^*(W)$ for all $a, b \in \bar{Z}^*(W)$.
(iii) Either $a + b \in Z^*(W)$ or $a + 2b \in Z^*(W)$ for all $a, b \in \bar{Z}^*(W)$. In particular, either $2a \in Z^*(W)$ or $3a \in Z^*(W)$ (only one of them) for every $a \in \bar{Z}^*(W)$.

Proof: (i) ⇒ (ii) Let $\bar{Z}^*(\Gamma(W))$ be a connected set. Consider $a, b \in \bar{Z}^*(W)$ such that $a + b \notin Z^*(W)$. We need to show that $a - b \in Z^*(W)$. Since $a + b \notin Z^*(W)$, we can conclude that a and b are not adjacent. By the previous theorem, we know that $a - (-b) - b$ is a path of length 2, i.e. $a - b \in Z^*(W)$. Conversely, if $a - b \notin Z^*(W)$, then by Theorem 3.1, both $a - (-a) - (-b)$ and $a - (-b) - b$ are paths of length 2 between a and b in $\bar{Z}^*(\Gamma(W))$. This implies that $-a - b \in Z^*(W)$ and since $Z^*(W)$ is a submodule, we have $a + b \in Z^*(W)$. Additionally, by the second path, we have $a + b \in Z^*(W)$.

(ii) ⇒ (iii) Consider $a, b \in \bar{Z}^*(W)$ such that $a + b \notin Z^*(W)$. By (i), we have $(a+b) - b = a \notin Z^*(W)$, thus $a + 2b = (a+b) + b \in Z^*(W)$. Conversely, suppose that $a + 2b \notin Z^*(W)$, so $(a+b) + b \notin Z^*(W)$. If $a + b \notin Z^*(W)$, then it implies $y \notin Z^*(W)$. By (ii), we have $a + b - b \in Z^*(W)$, which is a contradiction. In particular, if $a \in \bar{Z}^*(W)$, then either $2a \in Z^*(W)$ or $3a \in Z^*(W)$. However, both of these cannot be in $Z^*(W)$ because we would have $a = 3a - 2a \in Z^*(W)$, which is a contradiction.

(iii) ⇒ (i) Consider distinct elements $a, b \in \bar{Z}^*(W)$ such that $a + b \notin Z^*(W)$. By (iii), we have $a + 2b \in Z^*(W)$. Hence, $2b \notin Z^*(W)$, so $3b \in Z^*(W)$. Since $a + b \notin Z^*(W)$ and $3b \in Z^*(W)$, we can conclude that $a \neq 2b$. Therefore, $a - 2b - b$ is a path between a and b in $\bar{Z}^*(W)$. □

Example 3.1 *Let $A = \mathbb{Z}_4$ and $W = \mathbb{Z}_8$, then the vertex set of $T(\Gamma(W))$ is the elements of $\mathbb{Z}_8$. In this case, $Z^*(W) = \{0,1,2,3,4,5,6,7\} = \mathbb{Z}_8$. As we see in this example, the graph is complete, but if we consider $\mathbb{Z}$ as a ring, we see that the graph will not be complete, because $\mathbb{Z}_8$ as $\mathbb{Z}$-module is a cyclic module, but as $\mathbb{Z}_4$-module is not cyclic. In the following results we show that if a module is cyclic, then the total graph of it can not be complete.*

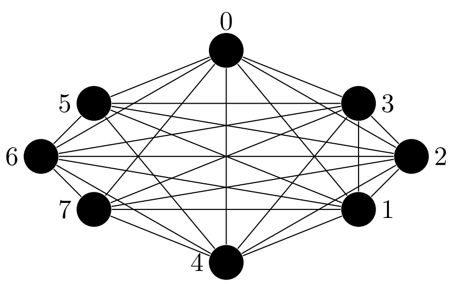


Figure 2: $T(\Gamma(\mathbb{Z}_8))(\mathbb{Z}_4 - module)$

Proposition 3.1 *For a noncyclic module W, $T(\Gamma(W))$ is a complete graph if and only if $W = Z^*(W)$.*

Proof: First, suppose that $T(\Gamma(W))$ is a complete graph. So for any $a, b \in W$ we have $a + b \in Z^*(W)$, i.e $(a+b)R \ll W$. Now let $b = 0$ then $a \in Z^*(W)$, hence $W = Z^*(W)$. Conversely, suppose that $W = Z^*(W)$. Now for any $a \in W$, we have $a \in Z^*(W)$. Let $b \in W$ then $b \in Z^*(W)$, so $a + b \in Z^*(W)$. We conclude that for every element $a, b \in W$, we have $a + b \in Z^*(W)$ i.e. $T(\Gamma^*(W))$ is a complete graph. □

Corollary 3.1 *Let W be a noncyclic hollow module. Then $T(\Gamma(W))$ is a complete graph.*

Proof: Since W is a hollow module, every submodule of W is small. So $Z^*(W) = W$. By Proposition 3.1 we conclude that $T(\Gamma(W))$ is a complete graph. □

Remark 3.1 *The condition that a hollow module W must be non-cyclic to $T(\Gamma(W))$ be complete is necessary. For example, the reader can clearly examine that $T(\Gamma(\mathbb{Z}_4))$ as $\mathbb{Z}$-module is not complete. Because that $\mathbb{Z}_4$ as $\mathbb{Z}$-module is cyclic and 1 is the generator of $\mathbb{Z}_4$. So $1 \notin Z^*(W)$ and by Proposition 1.3, $T(\Gamma(W))$ is not complete.*

Theorem 3.3 *Let A be a commutative ring and W a cyclic A-module and w be the generator of W. Suppose A as A-module. If $T(\Gamma(A))$ is connected, then $T(\Gamma(W))$ is connected as well.*

Proof: Suppose A as A-module and $T(\Gamma(A))$ is connected. We know that 0 is not adjacent to 1 in R since, in this case, we must have $(0+1)A \ll A$ i.e. $A \ll A$ contradiction. So in $T(\Gamma(W))$, 0 is not adjacent to 1. But since, by assumption $T(\Gamma(W))$ is connected, we must have a path from 0 to 1. Assume that the path is like this: $0 - a_1 - a_2 - a_3 - ... - a_k - 1$. So we have $(0+a_1)A \ll A, (a_1+a_2)A \ll A, (a_2+a_3)A \ll A, ..., (a_{k-1}+a_k)A \ll A$. By multiplying the path by w we have $(0+a_1w)A \ll Aw, (a_1w+a_2w)A \ll Aw, (a_2w+a_3w)A \ll Aw, ..., (a_{k-1}w+a_kw)A \ll Aw$ i.e. $a_1wA \ll W, (a_1w+a_2w)A \ll W, (a_2w+a_3w)A \ll W, ..., (a_{k-1}w+a_kw)A \ll W$. That means we have this path in $T(\Gamma(W))$, $0 - a_1w - a_2w - a_3w - ... - a_kw - 1$. So $T(\Gamma(W))$ is connected. □

Theorem 3.4 *Let A be a ring with identity and suppose A as an A-module. Then $T(\Gamma(A))$ can't be a complete graph.*

Proof: For every ring with an identity like A as A-module, $T(\Gamma(A))$ can't be a complete graph because in this graph always 0 can't be adjacent to 1, since in this case $(0+1)A \ll A = A \ll A$, contradiction. □

Example 3.2 *For every prime number, $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow and $Z^*(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$, so $T(\Gamma(\mathbb{Z}_{p^\infty}))$ is complete graph.*

Theorem 3.5 *Let W be an A-module and $Z^*(W) = 0$ and let a be a vertex of $T(\Gamma(W))$. Then the following holds:*

(i) If $a = -a$, then $deg(a) = 0$.

(ii) If $a \neq -a$, then $deg(a) = 1$.

Proof: (i) Since $Z^*(W) = 0$, two vertices a and b are adjacent if and only if $a + b \in Z^*(W)$ i.e $a + b = 0$ i.e. $a = -b$. If $x = -x$, then since $T(\Gamma(W))$ is a simple graph, so $deg(a) = 0$.

(ii) With the above explanation and by hypothesis, we must have a adjacent to $-a$, because $a+(-a) = 0$ i.e. $deg(a) = 1$. □

Example 3.3 *Consider $W = \mathbb{Z}_5$ as $\mathbb{Z} - module$. Then $Z^*(W) = 0$ and in $T(\Gamma(\mathbb{Z}_5))$, $deg(0) = 0$ and $deg(1) = deg(2) = deg(3) = deg(4)$.*

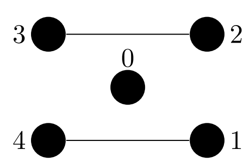


Figure 3: $T(\Gamma(\mathbb{Z}_5))$

Corollary 3.2 *If W is a nonzero module and $Z^*(W) = \{0\}$, then $T(\Gamma(W))$ is disconnected.*

Proof: With the assumption and by Theorem 3.5, in $T(\Gamma(W))$ we always have at least one isolated vertex, that is 0. So the result. □

Remark 3.2 *For every module W we have $Z^*(W) \subset Rad(W)$. Since for any semisimple module we have $Rad(W) = \{0\}$, we conclude that in the total graph of any semisimple module with respect to a consingular submodule, two elements h and h' are adjacent if and only if $h + h' = 0$. In particular, in $T(\Gamma(\mathbb{Z}_6)$, 4 adjacent to 2 and 1 adjacent to 5.*

Remark 3.3 *(i) The only empty total graph with respect to cosingular submodule is $T(\Gamma(\mathbb{Z}_2))$($\mathbb{Z}_2$ as $\mathbb{Z}$-module). In this case, $Z^*(\mathbb{Z}_2) = \{0\}$ and we do not have any edge.*

(ii)If a module W is cyclic, then $T(\Gamma(W))$ can't be complete. If it is complete and w is the generator of W, then since $0 \in Z^*(W)$, w must adjacent 0 i.e. $a \in Z^*(W)$, so $wA \ll W$ i.e. $W \ll W$ contradiction.

Proposition 3.2 *Suppose that W is a nonzero semisimple A-module. Then the following holds:*
(i) Vertex 0 is an isolated vertex in $T(\Gamma(W))$.
(i) $T(\Gamma(W))$ is disconnected.
(ii) $\gamma(T(\Gamma(W))) > 1$.
(iv) $\Delta(T(\Gamma(W))) = 1$ and $\delta(T(\Gamma(W))) = 0$.

Proof: (i) Since W is a semisimple A-module, we have $Z^*(W) = 0$. Hence, for every nonzero vertex $x \in W$, we have $x + 0 \neq 0$, and it shows that x not adjacent 0. So, in this case, the element 0 is an isolated vertex.

(ii) By (i), we see that 0 is an isolated vertex, so $T(\Gamma(W))$ is disconnected.

(iii) By the above descriptions it is obvious.

(iv) Assume that x is a nonzero element in W such that $deg(x) > 1$. It means there exist at least two elements like y and z such that $x + y = 0$ and $x + z = 0$. So $x + y = x + z$ and we conclude that $y = z$. Hence, $\Delta(T(\Gamma(W))) = 1$ and $\delta(T(\Gamma(W))) = 0$. □

Example 3.4 *Let $A = \mathbb{Z}$ and $W = \mathbb{Z}_8$, then the set of all vertices $T(\Gamma(W))$ is the elements of $\mathbb{Z}_8$ i.e $\{0, 1, 2, 3, 4, 5, 6, 7\}$. In this case, $Z^*(W) = \{0, 2, 4, 6\} = Z_8$.*

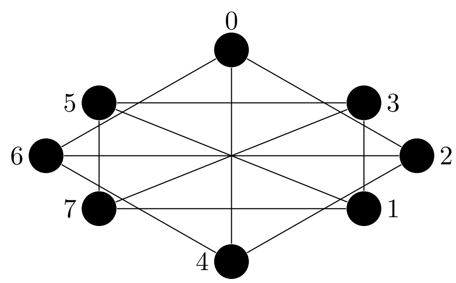


Figure 4: $T(\Gamma(\mathbb{Z}_8))$

Theorem 3.6 *Let W be a noncyclic A-module, then $T(\Gamma(W))$ is complete if and only if 0 adjacent to all vertices.*

Proof: First suppose that 0 adjacent to all vertices in W, then for all elements $a \in W$, we have $0 + a = a \in Z^*(W)$, thus $W = Z^*(W)$, and by Theorem 3.1 we conclude that $T(\Gamma(W))$ is complete. Conversely, it is obvious because if $T(\Gamma(W))$ is complete, then 0 must be adjacent to all vertices. □

Theorem 3.7 *Let $|Z^*(W)| = \theta$ and $|W/Z^*(W)| = \lambda$.*
(i) In the case where $2 \in Z^*(W)$, it follows that $\bar{Z}^*(\Gamma(W))$ can be expressed as the union of $\lambda - 1$ disjoint sets of the form K^{θ}.
(ii) Conversely, if $2 \notin Z^*(W)$, then $\bar{Z}^*(\Gamma(W))$ is represented as the union of $\frac{\lambda-1}{2}$ disjoint sets of the type $K^{\theta,\theta}$.

Proof: (i) First, we will prove that if $2 \in Z^*(W)$, then for every $a \in W$, $2a \in Z^*(W)$. Let x be an arbitrary element of W. Since $2 \in Z^*(W)$, we have $2A \ll W$, which means that $< 2 > \ll W$, or in other words, for all $a \in W$, $2aA \ll W$, which implies that $2a \in Z^*(W)$. Additionally, it is clear that $a + Z^*(W) \subset \bar{Z}^*(W)$ for every $a \notin Z^*(W)$. Now, let $a + a_1, x + a_2 \in x + Z^*(W)$, where $a_1, a_2 \in Z^*(W)$. This means that the coset $a + Z^*(W)$ is a complete subgraph of $\bar{Z}^*(\Gamma(W))$. Furthermore, all pairwise distinct cosets generate disjoint subgraphs of $\bar{Z}^*(\Gamma(W))$. To see this, suppose $a + a_1$ is adjacent to $b + a_2$ for some $a, b \in \bar{Z}^*(W)$ and $a_1, a_2 \in Z^*(W)$. Then, we have $a - b = (a+b) - 2b \in Z^*(W)$ since $Z^*(W)$ is a submodule of W and $2b \in Z^*(W)$. However, this leads to the contradiction that $a + Z^*(W) = y + Z^*(W)$. Therefore, $\bar{Z}^*(\Gamma(W))$ is a union of $\lambda - 1$ disjoint induced subgraphs $a + Z^*(W)$, each of which is a K^{θ}, where $\theta = |Z^*(W)| = |a + Z^*(W)|$.

(ii) Let us examine the scenario in which $a \in \bar{Z}^*(W)$ and $2 \notin Z^*(W)$. In this situation, it follows that no two distinct elements of the set, $a + Z^*(W)$ can be adjacent. This is due to the fact that if $a + a_1$ were adjacent to $a + a_2$, where $a_1, a_2 \in Z^*(W)$, it would imply that $2a \in Z^*(W)$. This conclusion contradicts our initial assumption that $2 \notin Z^*(W)$, as established in part (i). Furthermore, since $2a$ does not belong to $Z^*(W)$, the cosets $a + Z^*(W)$ and $-x + Z^*(W)$ are disjointed. It is also readily apparent that any vertex of $a + Z^*(W)$ is adjacent to every element of $-a + Z^*(W)$. Consequently, the union $(a + Z^*(W)) \cup (-a + Z^*(W))$ generates a complete bipartite induced subgraph of $\bar{Z}^*(\Gamma(W))$. Moreover, if $a + a_1$ is adjacent to $b + a_2$ for some $a, b \in \bar{Z}^*(W)$ and $a_1, a_2 \in Z^*(W)$, it follows that $a + b \in Z^*(W) - 0$, leading to the conclusion that $a + Z^*(W) = -b + Z^*(W)$. Therefore, $\bar{Z}^*(\Gamma(W))$ can be expressed as the union of $\frac{\lambda-1}{2}$ disjoint induced subgraphs $(a + Z^*(W)) = (-b + Z^*(W))$, that all of them are $K^{\theta,\theta}$ graph, where $\theta = |Z^*(W)| = |a + Z^*(W)|$. □

To prove the following results, we will need to refer to a celebrated theorem by Kuratowski. It is important to note that a graph Γ is considered planar if none of its edges intersect each other.

Theorem 3.8 *[11] A graph is classified as planar if and only if it does not possess a subdivision of either K^5 or $K^{3,3}$.*

Theorem 3.9 *Let W be an A-module. Then the following conditions hold:*
(i) if $2 \in Z^*(W)$ and $|Z^*(W)| = 5$, then $T(\Gamma(W))$ is not planar.
(ii) if $2 \notin Z^*(W)$ and $|Z^*(W)| = 3$, then $T(\Gamma(W))$ is not planar.

Proof: (i) Since $|Z^*(W)| = 5$ and $2 \in Z^*(W)$ By Theorem 3.4 we conclude that $\bar{Z}^*(\Gamma(W))$ is the union $|W/Z^*(W)|$ disjoint $K^{|Z^*(W)|} = K^5$. Hence, $\bar{Z}^*(\Gamma(W))$ contains a subdivision of K^5 and by Kuratowski theorem, $\bar{Z}^*(\Gamma(W))$ is not planar. So $T(\Gamma(W))$ is not planar.

(ii) Since $|Z^*(W)| = 3$ and $2 \notin Z^*(W)$ By Theorem 3.4 we conclude that $\bar{Z}^*(\Gamma(W))$ is the union $|\frac{(W/Z^*(W))-1}{2}|$ disjoint $K^{|Z^*(W)|,|Z^*(W)|} = K^{3,3}$. Hence, $\bar{Z}^*(\Gamma(W))$ contains a subdivision of $K^{3,3}$ and, by Kuratowski theorem, $\bar{Z}^*(\Gamma(W))$ is not planar. So $T(\Gamma(W))$ is not planar. □

Assume that h and h' are two distinct vertices in a graph H. The vertices h and h' are considered orthogonal, represented as $h \perp h'$, if h adjacent h' and we do not have any vertex h in H that is adjacent to both h and h'. A graph H is referred to as complemented if, for every vertex h in H, there exists a vertex h' in H such that h and h' are orthogonal to each other. With these definitions and last theorems, we find the following corollary.

Corollary 3.3 *Let W be an A-module with the graph $T(\Gamma(W))$. Then the following conditions hold:*
(i) if $2 \in Z^*(W)$ and $|Z^*(W)| = 5$, then $T(\Gamma(W))$ is not complemented.
(ii) if $2 \notin Z^*(W)$ and $|Z^*(W)| = 3$, then $T(\Gamma(W))$ is not complemented.

Proof: (i) By Theorem 3.6, $\bar{Z}^*(\Gamma(W))$ contains a subdivision of K^5. So we can find two vertices that are adjacent and a vertex that is adjacent to both of them. Hence, $T(\Gamma(W))$ is not complemented.

(ii) By Theorem 3.6, $\bar{Z}^*(\Gamma(W))$ contains a subdivision of $K^{3,3}$. So we can find two vertices that are adjacent and a vertex that is adjacent to both of them. Hence, $T(\Gamma(W))$ is not complemented. □

The length of the shortest cycle that exists within the graph H denoted as $gr(H)$ is called the girth of H. If H does not contain any cycles, then $gr(H) = \infty$. [16]

Theorem 3.10 *Let W be and A-module with the graph $T(\Gamma(W))$ and $|Z^*(W)| \geq 3$. Then the following holds:*
(i) if $2 \in Z^*(W)$, then $gr(\bar{Z}^*(\Gamma(W)) = 3$.
(ii) if $2 \notin Z^*(W)$, then $gr(\bar{Z}^*(\Gamma)) = 4$.

Proof: (i) If $w \in Z^*(W)$ and $|Z^*(W)| \geq 3$, then $\bar{Z}^*(W)$ is the union of $|W/Z^*(W)|$ disjoint $K^{|Z^*(W)|}$ by theorem 3.5. So we have at least one K^3 subdivision in $\bar{Z}^*(\Gamma(W))$. Hence, $gr(\bar{Z}^*(\Gamma(W)) = 3$.

(ii) If $w \notin Z^*(W)$ and $|Z^*(W)| \geq 3$, then $\bar{Z}^*(W)$ is the union of $\frac{|W/Z^*(W)|-1}{2}$ disjoint $K^{|Z^*(W)|,|Z^*(W)|}$ by theorem 3.5. So we have at least one K^3 subdivision in $\bar{Z}^*(\Gamma(W))$. Hence, $gr(barZ^*(\Gamma(W)) = 4$. □

Theorem 3.11 *Let $\bar{Z}^*(W) \neq \varnothing$. Then the following statements hold:*
(i) If $\bar{Z}^*(\Gamma(W))$ is complete, then $|W/Z^*(W)| = 2$ or $|W/Z^*(W)| = |W| = 3$.
(ii) If $\bar{Z}^*(\Gamma(W))$ is connected, then $|W/Z^*(W)| = 2$ or $|W/Z^*(W)| = |W| = 3$.
(iii) Let $\bar{Z}^*(\Gamma(W))$ (and therefore $Z^*(\Gamma(W))$ and $T(\Gamma(W))$) be totally(completely) disconnected, then either $Z^*(W) = 0$ or $2 \in Z^*(W)$.

Proof: Let $|W/Z^*(W)| = \lambda$ and $|Z^*(W)| = \theta$.

(i) Assume that $\bar{Z}^*(\Gamma(W))$ is complete. According to Theorem 3.2, this condition indicates that $\bar{Z}^*(\Gamma(W))$ corresponds to either a single K^θ or $K^{1,1}$. If $2 \in Z^*(W)$, then it follows that $\beta - 1 = 1$, leading to $\lambda = 2$ and consequently $|W/Z^*(W)| = 2$. Conversely, if $2 \notin Z^*(W)$, we find $\alpha = 1$ and $(\beta - 1)/2 = 1$. This results in $Z^*(W) = 0$ and $\lambda = 3$, thus $|W/Z^*(W)| = |W| = \lambda = 3$.

(ii) Now, consider the case where $\bar{Z}^*(\Gamma(W))$ is connected. By Theorem 3.2, this indicates that $\bar{Z}^*(\Gamma(W))$ is represented as only one K^θ or $K^{\theta,\theta}$. If $2 \in Z^*(W)$, then $\lambda - 1 = 1$, which gives $\lambda = 2$ and thus $|W/Z^*(W)| = 2$. On the other hand, if $2 \notin Z^*(W)$, we have $\theta = 1$ and $(\beta - 1)/2 = 1$, resulting in $\lambda = 3$ and therefore $|W/Z^*(W)| = 3$.

(iii) The condition that $\bar{Z}^*(\Gamma(W))$ is totally disconnected is equivalent to it being a disjoint union of K^1 components. Consequently, by Theorem 3.2, we conclude that $|Z^*(W)| = 1$ and $|W/Z^*(W)| = 1$, and hence the theorem is proofed. □

Theorem 3.12 *Let a be an arbitrary vertex of $T(\Gamma(W))$. Then if $2 \in Z^*(W)$ and $a \in Z^*(W)$, then $deg(a) = |Z^*(W)| - 1$. Otherwise, we have $deg(a) = |Z^*(W)|$.*

Proof: If $a_i \in Z^*(W)$, the vertex $a \in W$ is adjacent to vertices $a_i - a$, because $a + (a_i - a) = a_i \in Z^*(W)$. Then $deg(a) = 1$ if and only if $a = a_i - a$ for some $a_i \in Z^*(W)$, i.e. if and only if $2a \in Z^*(W)$. If $2a \notin Z^*(W)$, then $deg(a) = |Z^*(W)|$. If $2 \in Z^*(W)$, then $2a \in Z^*(W)$ for all $a \in W$, thus $deg(a) = |Z^*(W)| - 1$. Again, if $2 \notin Z^*(W)$, then we have 2 cases. First suppose that $a \in Z^*(W)$, then $deg(a) = |Z^*(W)| - 1$. Second, suppose that $a \notin Z^*(W)$, then $deg(a) = |Z^*(W)|$. □

Example 3.5 *In Example 3.1 and 3.4, since $2 \in Z^*(W)$, we see that for all vertices $h \in V(T(\Gamma(W)))$, we have $deg(h) = |Z^*(W)| - 1$.*

Theorem 3.13 *Consider two finite modules, W_1 and W_2, over a finite ring A. The following statements hold:*
(i) If $T(\Gamma(W_1))$ is a Hamiltonian graph, then $T(\Gamma(W_1 \times W_2))$ is also Hamiltonian.
(ii) If $\bar{Z}(\Gamma(W_1))$ is a Hamiltonian graph, then $\bar{Z}(\Gamma(W_1 \times W_2))$ is also Hamiltonian.

Proof: (i) Let $W_1 = \{w_1, w_2, ..., w_s\}$ and $W_2 = \{w'_1, w'_2, ..., w'_t\}$ be such that the sequences $w_1, w_2, ..., w_s$ and $w'_1, w'_2, ..., w'_t$ form Hamiltonian cycles. Then $w_1 + w_s \in \mathbb{Z}^*(W_1)$. This means that we can obtain a Hamiltonian cycle in $T(\Gamma(W_1 \times W_2))$ as follows:

$$(w_1, w'_1), (w_2, w'_1), ..., (w_s, w'_1), (w_1, w'_2), ..., (w_s, w'_2), ..., (w_1, w'_t), ..., (w_s, w'_t)$$

(ii) Let $\bar{Z}^*(W_1) = \{w_1, w_2, ..., w_s\}$ and $\bar{Z}^*(W_2) = \{w'_1, w'_2, ..., w'_t\}$. The Hamiltonian cycle described above is also for $\bar{Z}(\Gamma(W_1 \times W_2))$. □

Theorem 3.14 *Let $W = W_1 \times W_2$ be finite modules. Then $\mathcal{K}(T(\Gamma(W))) \geq |W_1| + |W_2| - 4$.*

Proof: Suppose that (u, v) and (u', v') represent two distinct elements of the set W. Assuming $u \neq u'$, $v' \neq \pm v$, and $\gamma \notin \{v, -v, v', -v'\}$, we can analyze the paths defined as follows: $(u, v)–(-u, \gamma)–(-u', -\gamma)–(-u', -v')$ for $\gamma \in W_2$. If $\zeta \in W_1$ and the conditions $(u, v) \neq (-v)$ and $(u', v') \neq (-\zeta, -v')$ hold, we can consider the paths $(u, v), (\eta, -v), (-\zeta, -v'), (u', v')$. In the case where $(u, v) \neq (\zeta, -v)$ and $(u', v') = (-\zeta, -v')$, the paths $(u, v), (\zeta, -v), (u', v')$ are relevant. Furthermore, if $(u, v) = (\zeta, -v)$ and $(u', v') = (-\zeta, -v')$ for some ζ, we can examine the paths $(u, v), (u', v')$ and $(u, v), (\zeta, -v), (u', v')$ for a ζ that is distinct from v. Consequently, there exist at least $|W_1| + |W_2| - 4$ disjoint paths from (u, v) to (u', v').Now, let us consider the case where $x \neq u'$, $v' \neq y$, and $v' = v$. The paths $(u, v), (-u, \gamma), (-u', -\gamma), (x', -y)$ can be analyzed for $\gamma \in W_2 - \{\pm b\}$, alongside the paths $(u, v), (\zeta, -v), (-\zeta, v), (u', -v)$ for $\zeta \in W_1 - \{-u, u'\}$. If $u = u'$, since (u, v) and (u', v') are distinct, then $v \neq v'$ and the proof is identical to the case $u \neq u'$ and $v = v'$. □

References

1. Ahmadzadeh, R., Moniri Hamzekolaee, A.R. A specific intersection graph of a module applying semisimple property. Acta Math. Hungar. (2025). https://doi.org/10.1007/s10474-025-01545-z.
2. A. Abbasi, S. Habibi, On the total graph of a commutative ring with respect to proper ideals, J. Korean Math. Soc. 49 (2012) 85–98. 1.
3. A. Abbasi, S. Habibi, On the total graph of a module over a commutative ring with respect to proper submodules, J. Algebra Appl. 11 (3) (2012).
4. S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, J. Pure Appl. Algebr. 213 (2009) 2224–2228.
5. D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008) 2706–2719.
6. D.F. Anderson, A. Badawi, On the total graph of a commutative ring without the zero element, J. Algebra Appl. 11 (4) (2012) 1250074. (18 pages). http://dx.doi.org/10.1142/S0219498812500740.
7. D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra 217 (1999) 434–447.
8. D.D. Anderson, M. Naseer, Beck's coloring of a commutative ring, J. Algebra 159 (1993) 500–514.
9. J.H. Asalool, M. Ashraf, The dot total graph of a commutative ring without the zero element, Boletim da Sociedade Paranaense de Matemática. v. 2025 (43), 1-12.
10. S.E. Atani, S. Habibi, The total torsion element graph of a module over a commutative ring, An. St. Univ. Ovidius Constant 19 (1) (2011) 23–34.
11. R. Balkrishnan, K. Ranganathan, A Text Book of Graph Theory, Springer-verlag New York, Inc., 2008, Reprint.
12. I. Beck, Coloring of commutative rings, J. Algebra 116 (1988) 208–226.
13. I. Chakrabarty, S. Ghosh, T.K. Mukherjee, M.K. Sen, Intersection graphs of ideals of rings, Discrete Math. 309 (17) (2009) 5381–5392
14. F. Esmaeili khalil saraei, E. Navidinia, A note on the extended total graph of commutative rings, Journal of Algebra and Related Topics. Vol. 6, No 1, (2018), pp 25-33.
15. Goswami, Jituparna. Total Graph of a Module with Respect to Singular Submodule. Arab Journal of Mathematical Sciences, (2016) 242-249.
16. F. Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Mass, 1969.
17. Hamzekolaee, A.R.M., Khoramdel, M., Alikhani, S. et al. A study of a kind of an intersection graph. Indian J Pure Appl Math (2025). https://doi.org/10.1007/s13226-025-00847-w.

18. I. Kaplansky, Commutative Rings, Revised ed., University of Chicago Press, Chicago, 1974.

19. J. Lambeck, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, Toronto, London, 1966.

20. P.K. Sharma, S.M. Bhatwadekar, A note on graphical representation of rings, J. Algebra 176 (1995) 124–127.

21. M.H. Shekarriz, M.H.S. Haghighi, H. Sharif, On the total graph of a finite commutative ring, Comm. Algebra 40 (2012) 2798–2807.

22. R. Wisbauer, Foundations of module and ring theory, german ed., vol. 3 of Algebra, Logic and Applications. Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

Amirmohammad Momeni Kohestani,
Department of Mathematics,
University of Mazandaran,
Iran.
E-mail address: `a.momeni04@umail.umz.ac.ir`

and

Yahya Talebi,
Department of Mathematics,
University of Mazandaran,
Iran.
E-mail address: `talebi@umz.ac.ir`