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Isoperimitric problem perturbed by the potential of the reproduction kernel of a reproducing kernel Hilbert space

Idriss Ouskhnid* and Abdesselam Bouarich

ABSTRACT: The purpose of This paper is to study the minimizer of the isoperimetric perturbation problem, over measurable sets of fixed volume.The problem is perturbed by an addition of repulsive nonlocal potentials of kernel K, which is a reproduction kernel of a reproducing kernel Hilbert space. We establish the existence of a minimizer of this problem. Besides, we study the geometric shape of a minimizer.

Key Words: Isoperimetric problem, repulsive, non local potential, reproduction kernel, geometric, shape, minimizer.

Contents

1. Introduction

The study of isoperimetric problems for a set of finite perimeter and fixed volume has been a central topic in geometric analysis and the calculus of variations. Classical problems focus on minimizing the perimeter functional $P(E)$ over set E of fixed volume, leading to the well-known solution given by balls, as established by the isoperimetric inequality. When additional interaction terms, the problem, the problem more complicate and requires new techniques to understand the existence, regularity and some geometric properties of minimizers.

In this paper, we investigate the minimization of the functional:

$$\mathcal{G}(\Omega) := P(\Omega) + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_\Omega(x)\chi_\Omega(y)K(x-y)dxdy, \tag{1.1}$$

where Ω is a set of finite perimeter and prescribed volume $|\Omega| = v$, and K is the fundamental solution of the operator elliptic

$$\mathcal{L}(u) = -\text{div}(A\nabla(u)) + u,$$

with A is a symetry matrix. The first term in the $\mathcal{A}(\Omega)$ is a classical perimeter of the set Ω in the sense of De Giorgi (see [10]):

$$P(\Omega) := \sup\left\{\int_\Omega \text{div}(F(x))dx; \quad F \in C^1_c(\mathbb{R}^n,\mathbb{R}^n), \quad \|F\|_\infty \le 1\right\}$$

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and the second term in $\mathcal{A}(\Omega)$ is non local interaction energy, making the problem more complex than the isoperimetric problem,

$$\mathcal{P}_K(\Omega) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_\Omega(x)\chi_\Omega(y)K(x-y)dxdy,$$

where the kernel K satisfies the following hypothesises:

- (H_1) K is non-negative and radially symmetric.
- (H_2) $K(\cdot - x)$ is belonging $L^2(\mathbb{R}^n)$ for every $x \in \mathbb{R}^n$.

This non local interaction term arise in various physical and biological models, such as in the study of aggregation phenomena, where attractive and repulsive forces compete to determine the shape of optimal configurations. This work interests to study the minimization problem under volume constraint :

$$\min\{\mathcal{G}(\Omega) \ : \Omega \in D\} \tag{1.2}$$

In the reminder of this paper we denote by D the admissible class of sets and defined by:

$$D := \{\Omega \subset \mathbb{R}^n / \chi_\Omega \in BV(\mathbb{R}^n), |\Omega| = m\}$$

Note that the two components of G have opposing effects. the perimeter term works to concentrate the mass into a ball, while the term $\mathcal{P}_K$ pushes the mass outwards.

Our main results in this work are the following:

1. Existence of minimizers for every mass. This part, in subsection (2.2).
2. Study some properties of the operator $\mathcal{L}(u) = -\text{div}(A\nabla(u)) + u$, in (3).
3. Building a reproducing kernel Hilbert space, and we investigate this result to prove some estimation for energy functional $\mathcal{P}_K(\Omega)$. The analysis is conducted in (4) and the exact result is shown in theorem (4.1).
4. The difference $\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega)$ for every sets $\Omega \in D$ is controlled by the term $\inf_{y\in\Omega}\sqrt{|B_r(y) \triangle \Omega|}$ in section (5) and the precise result is stated in theorem (5.1).

This paper is laid out as follows: in section (1), I give a brief introduction to isoperimetric problems; in section (2), we study the existence of a minimizer in admissible class D; in section (3), I state some properties fundamental of operator elliptic $\mathcal{L} = \mathcal{L}(u) = -\text{div}(A\nabla(u)) + u$ where A is symmetric matrix; in section (4) I prove some lemma and the main results building the reproducing kernel Hilbert space; in section (5), I study shape of the minimizer of the optimization problem (1.2).

2. Study the minimizer of the optimization problem 1.2

2.1. The lower semi continuity of the functional $\mathcal{G}$

In this part, we aim to examine the lower semi continuity of the functional $\mathcal{P}_K$ and derive that the functional $\mathcal{G}$ is lower semi continuous, which plays a significant role in the analysis of the existence of a minimizer for the optimization problem 1.2.

The following proposition includes an auxiliary result that will be utilized frequently throughout the remainder of the paper.

Proposition 2.1 *For every pair of measurable set in admissible class D and for every $K \in L^2(\mathbb{R}^n)$, there is a constant $C_{v,K}$ such that*

$$|\mathcal{P}_K(E) - \mathcal{P}_K(F)| \leqslant C_{m,K}|E\Delta F|^{\frac{1}{2}} \tag{2.1}$$

In particular the functional $\mathcal{P}_K$ is continuous in the sense of characteristic function.

Proof: We have,

$$\begin{aligned}\mathcal{P}_K(E)-\mathcal{P}_K(F)=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\chi_E-\chi_F\right)(x)\chi_E(y)K(x-y)dxdy\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\chi_E-\chi_F\right)(y)\chi_E(x)K(x-y)dxdy\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\left(\chi_E-\chi_F\right)(x)\chi_E(y)\Big|^2dxdy&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\left(\chi_E-\chi_F\right)(x)\Big|^2\Big|\chi_E(y)\Big|^2dxdy\\&=\int_{\mathbb{R}^n}\Big|\chi_E(y)\Big|^2(y)dy\int_{\mathbb{R}^n}\Big|\left(\chi_E-\chi_F\right)(x)\Big|^2dx\\&=\int_{\mathbb{R}^n}\Big|\chi_E(y)\Big|(y)dy\int_{\mathbb{R}^n}\Big|\left(\chi_E-\chi_F\right)(x)\Big|dx\\&=|E||E\Delta F|\end{aligned}\tag{2.3}$$

Hence, $(x,y)\longmapsto\left(\chi_E-\chi_F\right)(x)\chi_E(y)$ belongs the space $L^2(\mathbb{R}^m\times\mathbb{R}^m)$.

Since the functional K is belonged to $L^2(\mathbb{R}^m\times\mathbb{R}^m)$, then by Hölder inequality

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\left(\chi_E-\chi_F\right)(x)\chi_E(y)\right]K(x-y)dxdy\Big|&\leqslant\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\left(\chi_E-\chi_F\right)(x)\chi_E(y)\Big|^2dxdy\right)^{\frac{1}{2}}\\&\quad\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|K(x-y)\Big|^2dxdy\right)^{\frac{1}{2}}\\&\leqslant|E|^{\frac{1}{2}}|E\Delta F|^{\frac{1}{2}}\|K\|_{L^2(\mathbb{R}^n)}\end{aligned}\tag{2.4}$$

Consequently,

$$|\mathcal{P}_K(E)-\mathcal{P}_K(F)|\leqslant 2v^{\frac{1}{2}}\|K\|_{L^2(\mathbb{R}^n)}|E\Delta F|^{\frac{1}{2}}$$

This implies that, the functional $\mathcal{P}_K$ is Hölderian . □

Since the perimeter is lower semi-continuous (see [4, Theorem 5.2 page 199]), then we have this significant finding.

Corollary 2.1 *The functional $\mathcal{G}$ is lower semi-continuous.*

2.2. The minimizers of the optimization problem (1.2) is reatched in admissible class D

Theorem 2.1 *There exists a set Ω^* in the admissible class D such that*

$$\inf_{\Omega\in D}\{\mathcal{G}(\Omega)\}=\mathcal{G}(\Omega^*)$$

Proof: The fact that the kernel K is positive, gives that the infimum of the energy functional $\mathcal{G}(\Omega)$ over set of finite perimeter and fixed volume is a finite reel.

Putting $M=\inf_{\Omega\in D}\mathcal{G}(\Omega)$. By the characterisation of the lower bound, there exists a minimising sequence (Ω_k) in D such that $\mathcal{G}(\Omega_k)$ converge to M as $k\longrightarrow+\infty$. Then there exists a reel constant $C>0$ such that $\mathcal{G}(\Omega_k)\leqslant C$ for all $k\in\mathbb{N}$.

We have $|\Omega_k|_{BV(\mathbb{R}^n)}=|\Omega_k|+P(\Omega_k)$. Then for every $k\in\mathbb{N}$,

$$\begin{aligned}|\Omega_k|_{BV(\mathbb{R}^n)}&=v+\mathcal{G}(\Omega_k)-\mathcal{P}_K(\Omega_k)\\&\leqslant v+C\end{aligned}$$

This implies that the sequence (Ω_k) is bounded in the space $BV(\mathbb{R}^n)$. So by the compactness in $BV(\mathbb{R}^n)$ there exists a set $\Omega^*\subset\mathbb{R}^n$ of the finite perimeter and a subsequence which note also (Ω_k) such that χ_{Ω_k} converge to χ_{Ω^*} in $L^1(\mathbb{R}^n)$.

We have,

$$\Big||\Omega_k| - |\Omega^*|\Big| = \Big|\int_{\mathbb{R}^n} (\chi_{\Omega_k} - \chi_{\Omega^*})(x)dx\Big|$$
$$\leqslant \int_{\mathbb{R}^n} \Big|\chi_{\Omega_k}(x) - \chi_{\Omega^*}(x)\Big|dx$$
$$= \|\chi_{\Omega_k} - \chi_{\Omega^*}\|_{L^1(\mathbb{R}^n)}$$

Thus, $|\Omega_k| \to |\Omega^*|$ as $k \to \infty$. Since $|\Omega_k| = v$ for all k, then $|\Omega^*| = \lim_{k\to\infty} |\Omega_k| = v$. Consequently the set Ω^* belongs to D_v. Hence by the lower semi continuous of the functional $\mathcal{G}$ (see the corollary 2.1), one has

$$\mathcal{G}(\Omega^*) \leqslant \liminf_{n\longrightarrow+\infty} \mathcal{G}(\Omega_k)$$

Since that the sequence$(\mathcal{G}(\Omega_n))$ converge to M as $n \longrightarrow +\infty$ then $\liminf_{n\longrightarrow+\infty} \mathcal{G}(\Omega_n) = M$. This implies that $\mathcal{G}(\Omega^*) \leqslant M$.

The fact that the set Ω^* belongs to the admissible class $\mathcal{D}$, gives
$M \leqslant \mathcal{G}(\Omega^*)$. Consequently,

$$\inf_{\Omega\in D} \mathcal{G}(\Omega) = \mathcal{G}(\Omega^*)$$

□

3. The fundamental properties of the operator $\mathcal{L}$

In this section, we examine the principal properties of the operator $\mathcal{L}$, which will be used subsequently to ascertain the configuration of the minimizer Ω^* of the optimization problem (1.1).

Let $A : \mathbb{R}^n \to \mathbf{M}(n, \mathbb{R})$ be a map which send a point $x \in \mathbb{R}^n$ on a symmetric matrix $A(x) = (a_{i,j}(x))$, where the entries $a_{i,j}(x)$ will be supposed differential bounded functions on $\mathbb{R}^n$, that there is $M > 0$ such that $|a_{i,j}(x)| \leq M$ for all i, j. Moreover, in the sequel we suppose that this hypotheses :

$$\exists\lambda_0 > 0, \lambda_1 > 0, \forall x, z \in \mathbb{R}^n, \quad \lambda_0\|z\|^2 \leq \langle A(x)z, z\rangle \leq \lambda_1\|z\|^2$$

Thunks to the field of matrices $A(x)$ we define on the Sobolev space $H^1(\mathbb{R}^n)$ a bilinear form by,

$$\forall u, v \in H^1(\mathbb{R}^n), \qquad (u, v)_A := \int_\Omega uvdx + \int_\Omega \langle A\nabla u, \nabla v\rangle dx$$

Proposition 3.1 *The bilinear form, $(\cdot,\cdot)_A : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$, define an inner product such that,*

$$\forall u \in H^1(\mathbb{R}^n), \qquad (u, u)_A \geq \min(\lambda_0, 1)(\|u\|_{H^1(\mathbb{R}^n)})^2$$

Consequently, the norm $\sqrt{(u, u)_A}$ is equivalent to the standard Sobolev norm on $H^1(\mathbb{R}^n)$.

Proof: Let u and v be elements in $H^1(\mathbb{R}^n)$.

we have,

$$\Big|\int_{\mathbb{R}^n} A(x)\nabla u(x)\cdot\nabla v(x)dx\Big| \leqslant \int_{\mathbb{R}^n} \Big|A(x)\nabla u(x)\cdot\nabla v(x)\Big|dx$$
$$\leqslant \int_{\mathbb{R}^n} \| A(x)\nabla u(x) \|\| \nabla v(x) \| dx$$
$$\leqslant nM \| \nabla u(x) \|_2\| \nabla v(x) \|_2$$
$$\leqslant nM\|u\|_{H^1(\mathbb{R}^n)}\|v\|_{H^1(\mathbb{R}^n)}$$

By the Hölder inequality, we have

$$\int_{\mathbb{R}^n} \Big|u(x)v(x)\Big|dx \leqslant \|u\|_{H^1(\mathbb{R}^n)}\|v\|_{H^1(\mathbb{R}^n)}$$

Moreover, $|(u, v)_A| \leqslant \max(1, nM)\|u\|_{H^1(\mathbb{R}^n)}\|v\|_{H^1(\mathbb{R}^n)}$.

Hence the bilinear form $(.,.)_A$ is well defined on $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$.
The fact that the matrix A is symmetric, gives the bilinear form $(.,.)_A$ is symmetric.
We have,

$$A(x)\nabla u(x)\cdot\nabla u(x)\geqslant\lambda_0\parallel\nabla u(x)\parallel^2 .$$

Then,

$$\int_{\mathbb{R}^n}A(x)\nabla u(x)\cdot\nabla u(x)dx\geqslant\lambda_0\int_{\mathbb{R}^n}\parallel\nabla u(x)\parallel^2 dx$$

Thus,

$$(u,u)_A\geqslant\min(\lambda_0,1)\left(\int_{\mathbb{R}^n}\parallel\nabla u(x)\parallel^2 dx+\int_{\mathbb{R}^n}u(x)^2dx\right)=\min(\lambda_0,1)\parallel u\parallel^2_{H^1(\mathbb{R}^n)}$$

This gives that if $(u,u)_A=0$, then $\|u\|_{H^1(\mathbb{R}^n)}=0$, which gives that $u=0$.
Consequently the bilinear form $(.,.)_A$ is a inner product on $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$ □

Note that, by the mean of the field of matrices $x\in\mathbb{R}^m\to A(x)\in\mathbf{M}(m,\mathbb{R})$ we can define a differential operator:

$$\begin{array}{rcccc}\mathcal{L}=-\mathrm{div}(A\nabla)+id & : & H^1(\mathbb{R}^m) & \to & H^{-1}(\mathbb{R}^n)\\ & & u & \mapsto & -\mathrm{div}(A\nabla u)+u\end{array}$$

which acts naturally as a distribution on $\mathbb{R}^n$.

Proposition 3.2 *The operator $\mathcal{L}:H^1(\mathbb{R}^n)\to H^{-1}(\mathbb{R}^n)$ is bijective, and its inverse $\mathcal{L}^{-1}$ is continuous.*

Proof: Let f be an element in $H^{-1}(\mathbb{R}^n)$. We prove that the equation

$$\mathcal{L}(u)=f \tag{3.1}$$

has a unique solution in $H^1(\mathbb{R}^n)$.
For every $v\in H^1(\mathbb{R}^n)$, we have in the sense of distribution :

$$\langle\mathcal{L}(u),v\rangle_{H^{-1}(\mathbb{R}^n),H^1(\mathbb{R}^n)}=\langle f,v\rangle_{H^{-1}(\mathbb{R}^m),H^1(\mathbb{R}^n)}$$

Note that for every u and v in $H^1(\mathbb{R}^n)$, one has

$$\begin{aligned}(u,v)_A&=\int_{\mathbb{R}^m}A(\nabla u)\cdot\nabla v dx+\int_{\mathbb{R}^n}u(x)v(x)dx\\&=\langle A(\nabla u),\nabla v\rangle_{\mathcal{D}'(\mathbb{R}^n)}+\langle u,v\rangle_{\mathcal{D}'(\mathbb{R}^n)}\\&=\langle-\mathrm{div}(A(\nabla u)),v\rangle_{\mathcal{D}'(\mathbb{R}^n)}+\langle u,v\rangle_{\mathcal{D}'(\mathbb{R}^n)}\\&=\langle-\mathrm{div}(A(\nabla u))+u,v\rangle_{\mathcal{D}'(\mathbb{R}^n)}\\&=\langle\mathcal{L}(u),v\rangle_{\mathcal{D}'(\mathbb{R}^n)}\end{aligned}$$

By above we can deduce that to establish the equation (3.1) has a unique solution is equivalent to proving that the formulation variational

$$a(u,v)=F(v)$$

has a unique solution in $H^1(\mathbb{R}^n)$, where $a(u,v)=(u,v)_A$ and
$F(v)=\langle f,v\rangle_{H^{-1}(\mathbb{R}^m),H^1(\mathbb{R}^n)}$.
By the proposition (3.1) the bilinear form $a(.,.)$ is bounded and coercive in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. The fact that f is an element of the space $H^{-1}(\mathbb{R}^n)$, gives that the linear form F is bounded.
Thanks to the Lax-Milgram's theorem, there exists unique element u_f in $H^1(\mathbb{R}^n)$ such that for every v in $H^1(\mathbb{R}^n)$

$$a(u_f,v)=F(v)$$

Consequently, for every $f\in H^{-1}(\mathbb{R}^n)$ there exists unique $u_f\in H^1(\mathbb{R}^n)$ such that $\mathcal{L}(u_f)=f$. This implies that the operator $\mathcal{L}$ is bijective.

By the proposition (3.1) we have, $\min(1,\lambda_0) \parallel u_f \parallel^2 \leq (u_f, u_f)_A$.

So,

$$\parallel u_f \parallel^2_{H^1(\mathbb{R}^n)} \leq \frac{1}{\min(1,\lambda_0)} a(u_f, u_f)$$

This implies that, $\parallel u_f \parallel^2_{H^1(\mathbb{R}^n)} \leqslant \frac{1}{\min(1,\lambda_0)} \langle f, u_f \rangle_{H^{-1}(\mathbb{R}^m), H^1(\mathbb{R}^n)}$. The fact that $f \in H^{-1}(\mathbb{R}^n)$, gives that

$$\langle f, u_f \rangle_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} \leq \parallel f \parallel_{H^{-1}(\mathbb{R}^n)} \parallel u_f \parallel_{H^1(\mathbb{R}^n)}$$

this and above implies that $\parallel u_f \parallel_{H^1(\mathbb{R}^n)} \leq \frac{1}{\min(1,\lambda_0)} \parallel f \parallel_{H^{-1}(\mathbb{R}^n)}$.

Therefore,

$$\parallel \mathcal{L}^{-1}(f) \parallel_{H^1(\mathbb{R}^n)} \leq \frac{1}{\min(1,\lambda_0)} \parallel f \parallel_{H^{-1}(\mathbb{R}^n)}$$

Consequently the operator $\mathcal{L}^{-1} : H^{-1}(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ is continuous. □

Proposition 3.3 *For every measurable set $E \subset \mathbb{R}^n$, the characteristic function χ_E belongs to the space $H^{-1}(\mathbb{R}^n)$ and $\|\chi_E\|_{H^{-1}(\mathbb{R}^n)} \leqslant |E|^{\frac{1}{2}}$.*

Proof: Let u be an element in the space $\mathrm{H}^1(\mathbb{R}^n)$.

we have

$$\begin{aligned}
|\langle \chi_E, u \rangle| &= \Big| \int_{\mathbb{R}^n} \chi_E(x) u(x) dx \Big| \\
&\leq \Big(\int_{\mathbb{R}^n} |\chi_E(x)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} |u(x)|^2 dx \Big)^{\frac{1}{2}} \\
&\leq |E|^{\frac{1}{2}} \parallel u \parallel_{L^2(\mathbb{R}^n)} \\
&\leq |E|^{\frac{1}{2}} \parallel u \parallel_{\mathrm{H}^1(\mathbb{R}^n)}
\end{aligned}$$

Which entails that $\chi_E \in \mathrm{H}^{-1}(\mathbb{R}^n)$ and $\parallel \chi_E \parallel_{\mathrm{H}^{-1}(\mathbb{R}^n)} \leq |E|^{\frac{1}{2}}$ □

4. Build a reproducing kernel Hilbert space

This section is mainly devoted to the construction of a reproducing kernel space Hilbert, which denoted by $\mathbb{H}(\mathbb{R}^n)$ and which will allow us to establish an estimation for the energy functional $\mathcal{P}_K$.

We consider the application $j : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ define by :

$$\forall f \in L^2(\mathbb{R}^n), j(f)(x) = \int_{\mathbb{R}^n} f(z) K(z - x) dz$$

Lemma 4.1 *The application $j : L^2(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ is continuous.*

Proof: Let (f_k) be a sequence of the function which converge to 0 in $L^2(\mathbb{R}^n)$.

For every $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
|j(f_k)(x)| &= \Big| \int_{\mathbb{R}^n} f_k(y) U(y - x) dy \Big| \\
&\leqslant \int_{\mathbb{R}^n} |f_k(y)||K(y - x)| dy \\
&\leqslant \|f_k\|_{L^2(\mathbb{R}^n)} \|K(\cdot - x)\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

The fact that the sequence (f_k) converge to 0 in $L^2(\mathbb{R}^n)$, and $K(\cdot - x) \in L^2(\mathbb{R}^n)$, give that the sequence $(j(f_k)(x))$ converge to 0 for all $x \in \mathbb{R}^n$ on $\mathbb{R}$. Hence the sequence $(j(f_k))$ converge to 0 in $\mathcal{F}(\mathbb{R}^n, \mathbb{R})$. Consequently the application j is continuous. □

In the following we can interest in a building the reproducing kernel Hilbert space, which will noted by $\mathbb{H}(\mathbb{R}^n)$ and defined as $\mathbb{H}(\mathbb{R}^n) = j(L^2(\mathbb{R}^n))$, which we equip with the inner product ,

$$\forall u, v \in L^2(\mathbb{R}^n) \ :< j(u), j(v) >_{\mathbb{H}(\mathbb{R}^n)} :=< u, v >_{L^2(\mathbb{R}^n)}$$

Theorem 4.1 *The Hilbert space* $(\mathbb{H}(\mathbb{R}^n); < .,. >_{\mathbb{H}})$ *is a reproducing kernel Hilbert space and its kernel* $\mathcal{K}$ *is given by:*

$$\mathcal{K}(x,y) = \int_{\mathbb{R}^n} K(z-x)K(z-y)dz$$

where K *is the fundamental solution of the differential operator*

$$\mathcal{L} = -div(A\nabla) + id$$

Proof: Let g be an element in $\mathbb{H}(\mathbb{R}^n)$.

Then there exists an element G in $L^2(\mathbb{R}^n)$ such that : $g = j(G)$.

We have for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |g(x)| &= \Big| \int_{\mathbb{R}^n} G(z)K(z-x)dz \Big| \\ &\leq \| G \|_{L^2} \| K(.-x) \|_{L^2} \end{aligned}$$

The fact that $\| G \|^2_{L^2} =< G, G >_{L^2} =< j(G), j(G) >_{\mathbb{H}} =< g, g >_{\mathbb{H}}$, gives that for every x in $\mathbb{R}^n$

$$|g(x)| \leq \| g \|_{\mathbb{H}} \| K(.-x) \|_{L^2}$$

.

This implies that the evaluation functional

$$\begin{array}{rccc} E_x & : & \mathbb{H}(\mathbb{R}^n) & \to & \mathbb{R} \\ & & g & \mapsto & g(x) \end{array}$$

is bounded on $\mathbb{H}(\mathbb{R}^n)$.

Now we show that the space $\mathbb{H}(\mathbb{R}^n)$ is complete.

Let (f_k) be a Cauchy sequence in $\mathbb{H}(\mathbb{R}^n)$. Then there is a Cauchy sequence (v_k) in $L^2(\mathbb{R}^n)$ such that $j(v_k) = f_k$.

The fact that the space $L^2(\mathbb{R}^m)$ is complete, gives that there exists $v \in L^2(\mathbb{R}^n)$ such that the sequence (v_k) converge to v. Since that the function j is continuous, then $j(v_k)$ converge to $j(v)$. Therefore the Cauchy sequence f_k converge to $j(v)$ in $\mathbb{H}(\mathbb{R}^n)$. Finally the space $(\mathbb{H}(\mathbb{R}^n), < .,. >_{\mathbb{H}(\mathbb{R}^n)})$ is a Hilbert space.

This implies that the space $(\mathbb{H}(\mathbb{R}^n), < \cdot, \cdot >_{\mathbb{H}})$ is a reproducing kernel Hilbert space.

Now we are passing to determine the reproducing kernel of the space $\mathbb{H}(\mathbb{R}^n)$.

Let k_x be the reproducing kernel at the point x and $N_x \in L^2(\mathbb{R}^n)$ such that $j(N_x) = k_x$.

We consider g an element in $\mathbb{H}(\mathbb{R}^n)$, then there exists an element $G \in \mathrm{L}^2(\mathbb{R}^n)$ such that $g = j(G)$.

So for all $x \in \mathbb{R}^n$ we have,

$$\begin{aligned} g(x) &=< g, k_x >_{\mathbb{H}} \\ &=< G, N_x >_{L^2} \\ &= \int_{\mathbb{R}^n} G(z)N_x(z)dz \end{aligned}$$

In the other hands, we have $g(x) = j(G)(x) = \int_{\mathbb{R}^n} G(z)K(z-x)dz$, then this and above implies that:

$$\forall z \in \mathbb{R}^n : N_x(z) = K(z-x)$$

For every $y \in \mathbb{R}^n$ we have,

$$\begin{aligned} k_x(y) &= j(N_x)(y) \\ &= j(K(.-x))(y) \\ &= \int_{\mathbb{R}^n} K(z-x)K(z-y)dz \end{aligned}$$

Which entails that the reproducing kernel $\mathcal{K}$ for the reproducing kernel Hilbert space $\mathbb{H}$ is given by the following :

$$\mathcal{K}(x,y) = \int_{\mathbb{R}^n} K(z-x)K(z-y)dz \quad \text{for all} \quad x,y \in \mathbb{R}^n$$

□

Proposition 4.1 *The application* $\begin{array}{rcl} \mathcal{L}^{-1} : L^2(\mathbb{R}^n) & \to & \mathbb{H}(\mathbb{R}^n) \\ f & \mapsto & \mathcal{L}^{-1}(f) := j(f) \end{array}$ *is bijective.*

Proof:

$$\begin{aligned} f \in \ker(\mathcal{L}^{-1}) &\Leftrightarrow \forall x \in \mathbb{R}^n; \mathcal{L}^{-1}(f)(x) = 0 \\ &\Leftrightarrow \forall y \in \mathbb{R}^n; \int_{\mathbb{R}^n} \mathcal{L}^{-1}(f)(x)U(x-y)dx = 0 \\ &\Leftrightarrow \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)U(x-z)U(x-y)dzdx = 0 \\ &\Leftrightarrow \int_{\mathbb{R}^n} f(z)\Big(\int_{\mathbb{R}^n} U(x-z)U(x-y)dx\Big)dz = 0 \\ &\Leftrightarrow \int_{\mathbb{R}^n} f(z)\mathbf{K}(z,y)dz = 0 \\ &\Leftrightarrow < f;\mathbf{K}(.,y) >_{L^2} = 0 \\ &\Leftrightarrow f \in \text{span}\{\mathbf{K}(.,y); y \in \mathbb{R}^n\}^{\perp} \end{aligned}$$

Since the subspace $\{\mathbf{K}(.,y)/y \in \mathbb{R}^n\}$ is dense in $\mathbb{H}(\mathbb{R}^n)$, then

$$\text{span}\{\mathbf{K}(.,y); y \in \mathbb{R}^m\}^{\perp} = \{0\}$$

This implies that the application $\mathcal{L}^{-1}$ is injective.

By the above and The fact that $\mathcal{L}^{-1}(L^2(\mathbb{R}^n)) = \mathbb{H}(\mathbb{R}^n))$, give that the application $\mathcal{L}^{-1}$ is bijective.

□

Corollary 4.1 *The application* $\mathcal{L}^{-1}$ *realize an isometric.*

5. The geometric shape of the minimizer of the optimization problem (1.2)

In this section we give an estimation which compare $\mathcal{P}_K(B_r)$ with $\mathcal{P}_K(\Omega)$ for all set Ω in the admissible class D.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be a set in the admissible class D. Then there exist a constant $C_0 > 0$ such that for every set Ω in admissible class D,*

$$\mathcal{P}_K(B_r) \leqslant \mathcal{P}_K(\Omega) + C_0 \inf_{y\in\Omega} \sqrt{|B_r(y) \triangle \Omega|}$$

Proof: By organizing the term $\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega)$ we have

$$\begin{aligned} \mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{B_r}(y)\Big(\chi_{B_r} - \chi_\Omega\Big)(x)K(x-y)dxdy \\ - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\chi_{B_r} - \chi_E\Big)(x)\Big(\chi_{B_r} - \chi_\Omega\Big)(y)K(x-y)dxdy \end{aligned} \tag{5.1}$$

Define $\mathcal{V}(x) = \int_{\mathbb{R}^n} \Big(\chi_{B_r} - \chi_\Omega\Big)(x)K(x-y)dy$ i.e $\mathcal{V}(x) = (\chi_{B_r} - \chi_\Omega) * K(x)$.

The fact that the kernel K is a fundamental solution of the operator $\mathcal{L}$, gives

$$\mathcal{L}\Big(K(x-y)\Big) = \delta_x \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)$$

where δ_x is the Dirac distribution at the x.

Formally, we compute the $\mathcal{L}(\mathcal{V})(y)$ as follows. We have

$$\begin{aligned}\mathcal{L}(\mathcal{V})(y) &= \int_{\mathbb{R}^n} \mathcal{L}(K(y-x))(\chi_{B_r} - \chi_\Omega)(x)dx \\ &= \int_{\mathbb{R}^n} \delta_y(\chi_{B_r} - \chi_\Omega)(x)dx \\ &= (\chi_{B_r} - \chi_\Omega)(y)\end{aligned}$$

which implies that the function $\mathcal{V}$ can be satisfied as the solution to the equation:

$$\mathcal{L}(\mathcal{V}) = \chi_{B_r} - \chi_\Omega$$

We have,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\chi_{B_r} - \chi_\Omega\Big)(x)\Big(\chi_{B_r} - \chi_\Omega\Big)(y)K(x-y)dxdy &= \int_{\mathbb{R}^n} \Big(\chi_{B_r} - \chi_\Omega\Big)(y)\mathcal{V}(y)dy \\ &= \int_{\mathbb{R}^n} \mathcal{L}(\mathcal{V}(y))\mathcal{V}(y)dy \\ &= \int_{\mathbb{R}^n} A(y)\nabla\mathcal{V}(y)\cdot\nabla\mathcal{V}(y)dy + \int_{\mathbb{R}^n}\mathcal{V}^2(y) \\ &\geq \lambda_0\int_{\mathbb{R}^n} \|\nabla\mathcal{V}(y)\|^2dy + \int_{\mathbb{R}^n}\mathcal{V}^2(y)dy \\ &\geq 0\end{aligned}$$

This and (5.1) yield

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega)$$

$$\begin{aligned}\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) &\leq 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{B_r}(y)\Big(\chi_{B_r} - \chi_E\Big)(x)K(x-y)dxdy \\ &= 2\int_{B_r}\mathcal{V}(y)dy\end{aligned} \tag{5.2}$$

We have $\mathcal{L}(\mathcal{V})(y) = (\chi_{B_r} - \chi_\Omega)(y)$. Then $\mathcal{L}(\mathcal{V})(y) \geqslant 0$ for all y in B_r.

Hence, $\mathcal{V}$ is $\mathcal{L}$ super harmonic in B_r. By mean value property, we can deduce that

$$\int_{B_r}\mathcal{V}(y)dy \leq |B_r|\mathcal{V}(0)$$

This and (5.2) gives that

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) \leqslant 2|B_r|\mathcal{V}(0) \tag{5.3}$$

We have $\mathcal{V}(x) = (\chi_{B_r} - \chi_\Omega) * K(x)$. Then $\mathcal{V}(x) = j(\chi_{B_r} - \chi_\Omega)(x)$ for all x in $\mathbb{R}^n$.

The fact that $(\chi_{B_r} - \chi_\Omega) \in L^2(\mathbb{R}^n)$ and $j(L^2(\mathbb{R}^n)) = \mathbb{H}(\mathbb{R}^n)$ gives
$j(\chi_{B_r} - \chi_\Omega) \in \mathbb{H}(\mathbb{R}^n)$.

Since the space $\mathbb{H}(\mathbb{R}^n)$ is a reproducing kernel Hilbert space, which implies that

$$\mathcal{V}(x) \leqslant \| k_x \|_{\mathbb{H}(\mathbb{R}^n)} \| \mathcal{V} \|_{\mathbb{H}(\mathbb{R}^n)} \text{ for all } \quad x\in\mathbb{R}^n$$

This and (5.3) yield

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) \leqslant 2|B_r| \, \| k_0 \|_{\mathbb{H}(\mathbb{R}^n)} \| \mathcal{V} \|_{\mathbb{H}(\mathbb{R}^n)} \tag{5.4}$$

The fact that the application j is bounded, gives

$$\| \mathcal{V} \|_{\mathbb{H}(\mathbb{R}^n)} \leqslant \| j \|_{\mathcal{L}(L^2,\mathbb{H}(\mathbb{R}^n))} \sqrt{| B_r \triangle \Omega |} \tag{5.5}$$

Inserting (5.5) in (5.4) we obtain

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) \leqslant 2|B_r| \| k_0 \|_{\mathbb{H}(\mathbb{R}^n)} \| j \|_{\mathcal{L}(L^2,\mathbb{H}(\mathbb{R}^n))} \sqrt{| B_r \triangle \Omega |}$$

The fact that the functional is invariance by translation, gives that

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) \leqslant 2|B_r| \| k_0 \|_{\mathbb{H}(\mathbb{R}^n)} \| j \|_{\mathcal{L}(L^2,\mathbb{H}(\mathbb{R}^n))} \inf_{y\in\Omega} \sqrt{| B_r(y) \triangle \Omega |}$$

Then

$$\mathcal{P}_K(B_r) - \mathcal{P}_K(\Omega) \leqslant C_0 \inf_{y\in\Omega} \sqrt{| B_r(y) \triangle \Omega |}$$

where $C_0 := 2|B_r| \| k_0 \|_{\mathbb{H}(\mathbb{R}^n)} \| j \|_{\mathcal{L}(L^2,\mathbb{H}(\mathbb{R}^n))}$ □ □

Proposition 5.1 *There is a constant C_0 such that for every set Ω in the admissible class D, one has*

$$\mathcal{G}(B_r) \leqslant \mathcal{G}(\Omega) + C_0 \inf_{y\in\Omega} \sqrt{| B_r(y) \triangle \Omega |}$$

Proof: Thanks to the theorem (5.1), we have

$$\mathcal{P}_K(B_r) \leqslant \mathcal{P}_K(\Omega) + C_0 \inf_{y\in\Omega} \sqrt{|B_r(y) \triangle \Omega|}$$

for every set Ω in admissible class D.

By the isoperimetric inequality, the perimeter is minimized by balls.

So,

$$P(B_r) \leqslant P(\Omega)$$

Thus,

$$\mathcal{G}(B_r) \leqslant \mathcal{G}(\Omega) + C_0 \inf_{y\in\Omega} \sqrt{|B_r(y) \triangle \Omega|}$$

□

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Conflict of interest There are neither conflicts of interest nor competing interests.

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Idriss Ouskhnid and Abdesselam Bouarich,

Department of Mathematics and Applications,

Faculty of Sciences and Techniques, Sultan Moulay Slimane University,

Morocco.

E-mail address: idriss.ouskhnid@usms.ma

E-mail address: a.bouarich3@gmail.com