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On $h\alpha$-Open Sets In Topological Spaces

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ABSTRACT: In 2022, Abdullah et.al [10] introduced a new class of open sets in topological space called $h\alpha$-open sets. In this paper, we have introduced and study topological properties of $h\alpha$-interior, $h\alpha$-interior points, $h\alpha$-neighbourhood, $h\alpha$-closure, $h\alpha$-exterior, $h\alpha$-limit points, $h\alpha$-derived, $h\alpha$-border, $h\alpha$-frontier by using the concept of $h\alpha$-open sets. Also, we have presented the notion of almost $h\alpha$-continuous, $h\alpha$-contra continuous, almost $h\alpha$-contra continuous, and strongly $h\alpha$-continuous functions. Some properties, counter examples and theorems are established. Furthermore, we have shown that every strongly $h\alpha$-continuous function is continuous and the composition of an $h\alpha$-continuous and a strongly $h\alpha$-continuous function is $h\alpha$-irresolute.

Key Words: $h\alpha$-open sets, $h\alpha$-derived sets, almost $h\alpha$-continuous functions, $h\alpha$-contra continuous functions, strongly $h\alpha$-continuous functions.

Contents

1. Introduction

Open sets are crucial to topology because they form the basis of topological spaces, which in turn provides a way to the development of concepts like continuity, connectedness, compactness, separation axioms, and convergence etc. In order to better understand the structure and characteristics of topological spaces, numerous investigations from time to time have been conducted on open sets. The standard definition of open sets is crucial for many classical results, topologist have long been exploring more generalized and refine version of open sets to understand spaces with additional structure or weakened conditions. There are various generalizations of open sets, (see [7,22]). One of the earliest advancements came in 1963 when Levine [11] proposed the concept of semi-open sets along with semi-continuity, which provided a framework for studying weaker forms of openness and continuity in topological spaces. Later, in 1970 Levine [15] proposed the idea of generalized open sets as an extension of the traditional idea of open sets. Andrijevic [12] in 1996, establisehed the concept of b-open sets, and various forms of continuity related to this idea have been explored in [3,1,19,23,31,30]. In 2001, Sundaram and Pushpalatha [26] extended the study of generalized open set and proposed the idea of strongly generalized open sets. Generalized open sets introduction has allowed for broader definition of continuity such as generalized continuity which is defined as, a function $f : (X, \tau) \to (Y, \tau)$ is termed as generalized continuous, if $f^{-1}(G)$ is generalized open in (X, τ) for each generalized open set G in (Y, τ). Generalized continuity has enabled the study of mappings that do not strictly preserve classical open sets but still retain a weaker form of continuity. Chawalit [8,9] further study generalized continuous mapping and introduced some relationships between generalized continuous and continuous functions. Njasted [25] in 1965, initially

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proposed the idea of α-open sets. In 2021, Abbas [13] gave the idea of h-open sets. By using these notions, Abdullah et al. [10] in 2022 presented the idea of $h\alpha$-open sets. Our aim of this paper is to present and investigate the topological characteristics of $h\alpha$-interior, $h\alpha$-interior points, $h\alpha$-neighbourhood, $h\alpha$-closure, $h\alpha$-exterior, and $h\alpha$-limit points, $h\alpha$-derived $h\alpha$-border, $h\alpha$-frontier in the context of [13], by using the idea of $h\alpha$-open sets.
In 1996, Dontchev [15] introduced the notion of contra-continuous function. Baker [11] defined almost contra-continuous and contra almost-β-continuous functions. A new class of functions known as regular set connected functions was given by Dontchev et al. [16]. In [19], Singh and Singal presented the class of almost continuous functions. These functions have also been the subject of some research in recent years, see [14,20]. In 1960, Levine [23] proposed the idea of strong continuity in topological spaces. Few decades earlier, many researchers contributed in this, see [21,6,27]. Recently Sharma et al. [28] established the notion of strongly h-continuous function. In this paper, we introduced the notion of almost $h\alpha$-continuous, $h\alpha$-contra-continuous, almost $h\alpha$-contra-continuous, strongly $h\alpha$-continuous functions and investigate some of their properties. Also we show that every strongly $h\alpha$-continuous function is continuous and the composition of an $h\alpha$-continuous and a strongly $h\alpha$-continuous function is $h\alpha$-irresolute.

2. Preliminaries

This section deals with some basic definitions and results which will be used in the our next sections. Throughout this paper, X, Y and Z stands for topological spaces with no separation axioms assumed, unless otherwise stated. For a subset A of X , the closure of A and the interior of A will be a denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 2.1 *A subset A of a topological space (X, τ) is said to be:*

1. *h-open [13]: If for each non-empty set U in $X, U \neq X$ and $U \in \tau$, as a result $A \subseteq int(A \cup U)$.*
2. *α-open [25]: If $A \subseteq int(Cl(int(A)))$ and the complement of α-open is called α-closed.*
3. *$h\alpha$-open [10]: If for each non-empty set U in X, $U \neq X$ and $U \in \tau^{\alpha}$, as a result $A \subseteq int(A \cup U)$ and the complement of $h\alpha$-open set is called $h\alpha$-closed.*
4. *regular open [18]: If $A = int(Cl(A))$.*
5. *regular closed [18]: If $A = Cl(int(A))$.*
   The family of $h\alpha$-open(resp. α-open) is denoted by $\tau^{h\alpha}(resp.\tau^{\alpha})$.

Lemma 2.1 *[25] Every open set in a topological space is α-open.*

Lemma 2.2 *[10] Every open set in a topological space is $h\alpha$-open.*

Proposition 2.1 *[10] Each continuous mapping is $h\alpha$-continuous mapping.*

Definition 2.2 *Let (X, τ) and (Y, σ) be two topological spaces. Then a function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *$h\alpha$-continuous [10]: If $f^{-1}(G)$ is $h\alpha$-open set in X for every open set G in Y.*
2. *$h\alpha$-irresolute [10]: If $f^{-1}(G)$ is $h\alpha$-open set in X for every $h\alpha$-open set G in Y.*
3. *$h\alpha$-totally continuous [10]: If $f^{-1}(G)$ is clopen in X for each $h\alpha$-open set G in Y.*
4. *almost continuous [19]: If $f^{-1}(G)$ is open in X for every regular open set G in Y.*
5. *contra-continuous [15]: If $f^{-1}(G)$ is closed in X for every open set G of Y.*
6. *regular set connected [16]: If $f^{-1}(G)$ is clopen in X for every regular open set G of Y.*
7. *almost contra-continuous [11]: If $f^{-1}(G)$ is closed in X for every regular open set V of Y.*
8. *strongly continuous [23]: If $f^{-1}(G)$ is clopen in X for every open set G in Y.*

3. $h\alpha$-open sets

In this section, we define and study the topological properties of $h\alpha$-interior, $h\alpha$-interior points, $h\alpha$-neighbourhood, $h\alpha$-closure, $h\alpha$-exterior, $h\alpha$-limit points, $h\alpha$-derived, $h\alpha$-border, $h\alpha$-frontier by using the concept of $h\alpha$-open sets and established some important results. Also we have shown that the family of $h\alpha$-open sets form a generalized topology in the sense of Császár [4].

Definition 3.1 *Let (X, τ) be a topological space and $A \subseteq X$.Then $h\alpha$-interior of A is defined as the union of all $h\alpha$-open sets contained in A and we denote it by $int_{h\alpha}(A)$.It is obvious the $int_{h\alpha}(A)$ is $h\alpha$-open set for any subset A of X.*

Definition 3.2 *Let (X, τ) be a topological space and $A \subseteq X$.Then A is said to be $h\alpha$-neighbourhood of a point $x \in X$ if there exist $W \in \tau^{h\alpha}(X, x)$ such that $W \subseteq A$.The family of all $h\alpha$-neighbourhood of a point $x \in X$ is denoted by $N_{h\alpha}(x)$ and is called the $h\alpha$-nbhd system of x.*

Theorem 3.1 *Let (X, τ) be a topological space. Then the union of any collection of $h\alpha$-open set is $h\alpha$-open.*

Proof: Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of $h\alpha$-open sets, and let $A = \bigcup_{\lambda\in\Lambda} A_\lambda$. Then for any non-empty $U \neq X, U \in \tau^\alpha$, each A_λ satisfies $A_\lambda \subseteq int(A_\lambda \cup U)$. Since $A \cup U = \bigcup_{\lambda\in\Lambda}(A_\lambda \cup U)$. Then

$$int(A \cup U) = int\left(\bigcup_{\lambda\in\Lambda}(A_\lambda \cup U)\right) \supseteq \bigcup_{\lambda\in\Lambda} int(A_\lambda \cup U) \supseteq \bigcup_{\lambda\in\Lambda} A_\lambda = A.$$

Thus, $A \subseteq int(A \cup U)$, therefore A is $h\alpha$-open. □

Theorem 3.2 *The Finite intersection of $h\alpha$-open sets need not be $h\alpha$-open*

Example 3.1 *Consider the set of real numbers ($\mathbb{R}$) with the usual topology τ. Let $A = (0, 1) \cup (1, 2)$ and $B = (1, 2) \cup (2, 3)$. Then both A and B are $h\alpha$-open in τ, but their intersection $A \cap B = (1, 2)$ is not $h\alpha$-open because if we take an α-open set $U = (0, 1)$. Then the set $A \cap B = (1, 2)$ will not satisfy the condition of $h\alpha$-open set.*

Remark 3.1 *In general, the family of $h\alpha$-open sets does not form a topology. But it forms a generalized topology in the sense of Császár [4].*

Proposition 3.1 *If A is a subset of X.Then $int_{h\alpha}(A) = \bigcup\{W : W$ is $h\alpha$-open, $W \subseteq A\}$*

Proof: Let A be a subset of X and

$$\begin{aligned} \text{let } x \in int_{h\alpha}(A) &\Leftrightarrow x \text{ is an } h\alpha\text{-interior point of} A. \\ &\Leftrightarrow A \text{ is an } h\alpha\text{-nbhd of point } x. \\ &\Leftrightarrow \text{there exist an } h\alpha\text{-open set } W \text{ suchthat } x \in W \subseteq A. \\ &\Leftrightarrow x \in \bigcup\{W : W \; is \; h\alpha\text{-open}, W \subseteq A\}. \end{aligned}$$

Hence $int_{h\alpha}(A) = \bigcup\{W : W$ is $h\alpha$-open, $W \subseteq A\}$.

□

Theorem 3.3 *Let (X, τ) be a topological space and A, B are subsets of X.Then following holds:*

1. *If $A \subseteq B$, then $int_{h\alpha}(A) \subseteq int_{h\alpha}(B)$.*
2. *$int_{h\alpha}(A) \subseteq A$.*

3. *If B is an $h\alpha$-open set contained in A, then $B \subseteq int_{h\alpha}(A)$.*

4. *If A is $h\alpha$-open set contained in A, then $B \subseteq int_{h\alpha}(A)$.*

5. *$int_{h\alpha}(X) = X$ and $int_{h\alpha}(\phi) = \phi$.*

Proof: (1) Let A and B be any two subsets of X such that $A \subseteq B$. Let $x \in int_{h\alpha}(A)$. Then x is an $h\alpha$-interior point of A and so A is an $h\alpha$-nbhd of x. Also B an $h\alpha$-nbhd of $x \Rightarrow x \in int_{h\alpha}(B)$. Thus $int_{h\alpha}(A) \subseteq int_{h\alpha}B$.
(2) Let $x \in int_{h\alpha}(A) \Rightarrow A$ is an $h\alpha$-nbhd of $x \Rightarrow x \in A$.Thus $int_{h\alpha}(A) \subseteq A$.
(3) Let B be any $h\alpha$-open set suchthat $B \subseteq A$. Let $x \in B \subseteq A$. Since B is $h\alpha$-open set contained in $A \Rightarrow A$ is an $h\alpha$-nbhd of x and consequently x is an $h\alpha$-interior point of A. Hence $x \in int_{h\alpha}(A)$. Thus $B \subseteq int_{h\alpha}(A)$.

Note 1 *$int_{h\alpha}(A)$ is the largest $h\alpha$-open set contained in B.*

(4) Let A be an $h\alpha$-open subset of X. Then by (2), we have $int_{h\alpha}(A) \subseteq A$. Also A is $h\alpha$-open set contained in A. Then by (3), we have $A \subseteq int_{h\alpha}(A)$.Hence $A = int_{h\alpha}(A)$.
(5) Since X and ϕ are $h\alpha$-open sets. Then by Proposition 3.1, we have $int_{h\alpha}(X) = \bigcup\{W : W$ is $h\alpha$-open,$W \subseteq X\} = X$ and since ϕ is the only $h\alpha$-open set contained in ϕ. Thus $int_{h\alpha}(\phi) = \phi$. Therefore $int_{h\alpha}(X) = X$ and $int_{h\alpha}(\phi) = \phi$. □

Theorem 3.4 *If A and B are subsets of a topological space X. Then $int_{h\alpha}(A) \cup int_{h\alpha}(B) \subseteq int_{h\alpha}(A \cup B)$.*

Proof: We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then by Theorem 3.3, $int_{h\alpha}(A) \subseteq int_{h\alpha}(A \cup B)$ and $int_{h\alpha}(B) \subseteq int_{h\alpha}(A \cup B) \Rightarrow int_{h\alpha}(A) \cup int_{h\alpha}(B) \subseteq int_{h\alpha}(A \cup B)$. □

Theorem 3.5 *If A and B are subsets of a topological space X. Then $int_{h\alpha}(A \cap B) = int_{h\alpha}(A) \cap int_{h\alpha}(B)$.*

Proof: Since we know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then by Theorem 3.3, we have $int_{h\alpha}(A \cap B) \subseteq int_{h\alpha}(A)$ and $int_{h\alpha}(A \cap B) \subseteq int_{h\alpha}(B) \Rightarrow int_{h\alpha}(A \cap B) \subseteq int_{h\alpha}(A) \cap int_{h\alpha}(B)$
Again, let $x \in int_{h\alpha}(A) \cap int_{h\alpha}(B)$.Then $x \in int_{h\alpha}(A)$ and $x \in int_{h\alpha}(B)$. Hence x is an $h\alpha$-interior point of A and B.It follows that A and B are $h\alpha$-nbhd of x, so that their intersection $A \cap B$ is also $h\alpha$-nbhds of x. Hence $x \in int_{h\alpha}(A \cap B)$.Thus $x \in int_{h\alpha}(A) \cap int_{h\alpha}(B) \Rightarrow x \in int_{h\alpha}(A \cap B)$. Therefore $int_{h\alpha}(A) \cap int_{h\alpha}(B) \subseteq int_{h\alpha}(A \cap B)$. Thus we have $int_{h\alpha}(A \cap B) = int_{h\alpha}(A) \cap int_{h\alpha}(B)$. □

Proposition 3.2 *Let (X, τ) be a topological space and A be a subset of X. Then*

1. *$int_{h\alpha}(int_{h\alpha}(A)) = int_{h\alpha}(A)$.*

2. *$int_{h\alpha}(int(A)) = int(A)$.*

Proof: (1) Since $int_{h\alpha}(A) \in \tau^{h\alpha}$. Then by Theorem 3.3(4), we have $int_{h\alpha}(A) = int_{h\alpha}(int_{h\alpha}(A))$.
(2) Since $int(A)$ is an open set.Then $int(A)$ is $h\alpha$-open set. By Theorem 3.3(4), we have $int(A) = int_{h\alpha}(int(A))$. □

Theorem 3.6 *If A is a subset of a topological space X.Then $int(A) \subseteq int_{h\alpha}(A)$.*

Proof: Let A be a subset of X.
let $x \in int(A) \Rightarrow x \in \bigcup\{W : W$ is open,$W \subseteq A\}$
$\Rightarrow$ there exist an open set W suchthat $x \in W \subseteq A$.
$\Rightarrow$ there exist an $h\alpha$-open set W suchthat $x \in W \subseteq A$.
$\Rightarrow x \in \bigcup\{W : W$ is $h\alpha$-open,$W \subset A\}$.
$\Rightarrow x \in int_{h\alpha}(A)$.Hence $int(A) \subseteq int_{h\alpha}(A)$. □

Theorem 3.7 *A subset A of a topological space X is $h\alpha$-open iff A is an $h\alpha$-nbhd of each of its points.*

Proof: Suppose A is $h\alpha$-open set.Let $x \in A$ then by Definition 3.2, an $h\alpha$-nbhd of x is any subset of X containing an $h\alpha$-open set which itself contain x. But A is itself an $h\alpha$-open set which itself contain x, so A is a subset of A which itself contains an $h\alpha$-open set which itself contains x. Thus for all points $x \in X$, A is an $h\alpha$-nbhd of x.
Conversely, suppose A is an $h\alpha$-nbhd of each of its points. Then for each $x \in A$, there is an $h\alpha$-open set W containing x(denoted by W_x) suchthat $x \in W_x \subseteq A$. Clearly $A = \bigcup\limits_{x\in A} W_x$, since each W_x is $h\alpha$-open set and we know that union of $h\alpha$-open set is $h\alpha$-open. Thus A is $h\alpha$-open set. □

Corollary 3.1 *Let (X, τ) be a topological space and $A \subseteq X$. If A is $h\alpha$-closed and $x \in A^c$. Then there is an $h\alpha$-nbhd N of x suchthat $N \cap A = \phi$.*

Proof: Since A is an $h\alpha$-closed set, then A^c is an $h\alpha$-open set. By Theorem 3.7, A^c contains $h\alpha$-nbhd of each of its points. Thus there exist an $h\alpha$-nbhd N of x such that $N \subseteq A^c$ i.e; $N \cap A = \phi$. □

Theorem 3.8 *Let (X, τ) be a topological space and $x \in X$ be arbitrary. Then*

1. *If M is a superset of an $h\alpha$-nbhd N of x. Then M is also an $h\alpha$-nbhd of x.*
2. *Is N_1 and N_2 be two $h\alpha$-nbhd of x, then $N_1 \cap N_2$ is also $h\alpha$-nbhd of x.*
3. *there is atleast one $h\alpha$-nbhd of x.*
4. *for each $h\alpha$-nbhd N of x, $x \in N$.*

Proof: (1) Let N be an $h\alpha$-nbhd of x, then there exist $W \in \tau^{h\alpha}(X, x)$ such that $W \subseteq N$. Let M be a superset of N such that $x \in W \subseteq N \subseteq M \Rightarrow x \in W \subseteq M \Rightarrow M$ is an $h\alpha$-nbhd of x.
(2) Let N_1 and N_2 be two $h\alpha$-nbhd of x, then there exist W_1, $W_2 \in \tau^{h\alpha}(X, x)$ such that $W_1 \subseteq N_1$ and $W_2 \subseteq N_2$.Since $x \in W_1$, $x \in W_2 \Rightarrow x \in W_1 \cap W_2$.Also $W_1 \subseteq N_1$, $W_2 \subseteq N_2 \Rightarrow W_1 \cap W_2 \subseteq N_1 \cap N_2$ and W_1,$W_2 \in \tau^{h\alpha}(X) \Rightarrow W_1 \cap W_2 \in \tau^{h\alpha}(X)$.Therefore we have, there exist $W_1 \cap W_2 \in \tau^{h\alpha}(X, x)$ suchthat $W_1 \cap W_2 \subseteq N_1 \cap N_2$. Thus $N_1 \cap N_2$ is an $h\alpha$-nbhd of x.
(3) Since $x \in X \subseteq X \in \tau^{h\alpha}$. By definition 3.7, X is an $h\alpha$-nbhd of x. Hence there exist at least one $h\alpha$-nbhd of x.
(4) Let N be an $h\alpha$-nbhd of x, then there exist $W \in \tau^{h\alpha}(X, x)$ such that $W \subseteq N$. Consequently $x \in N$. □

Definition 3.3 *Let (X, τ) be a topological space and $A \subseteq X$. Then $h\alpha$-closure of A is defined as the intersection of all $h\alpha$-closed sets in X containing A. It is denoted by $Cl_{h\alpha}(A)$.*

Note 2 (1)*$Cl_{h\alpha}(A)$ is $h\alpha$-closed for any subset A of X.*
(2)*$Cl_{h\alpha}(A)$ is the smallest closed set containing A.*

Theorem 3.9 *Let (X, τ) be a topological space and A and B are subsets of X. Then*

1. *$Cl_{h\alpha}(X) = X$ and $Cl_{h\alpha}(\phi) = \phi$.*
2. *$A \subseteq Cl_{h\alpha}(A)$.*
3. *If $A \subseteq B$, then $Cl_{h\alpha}(A) \subseteq Cl_{h\alpha}(B)$.*
4. *If B is an $h\alpha$-closed set containing A. Then $Cl_{h\alpha}(A) \subseteq B$.*
5. *A is $h\alpha$-closed iff $A = Cl_{h\alpha}(A)$.*

Proof: (1) By definition of $h\alpha$-closure, X is the only $h\alpha$-closed set containing X. Therefore $Cl_{h\alpha}(X) = \bigcap\{X\} = X$. Hence $Cl_{h\alpha}(X) = X$ also by definition of $h\alpha$-closure, $Cl_{h\alpha}(\phi) = \bigcap\{\phi\}$. Hence $Cl_{h\alpha}(\phi) = \phi$.
(2) By definition of $h\alpha$-closure of A, it is obvious that $A \subseteq Cl_{h\alpha}(A)$.
(3) Let A and B be subset of X such that $A \subseteq B$, by (2) $B \subseteq Cl_{h\alpha}(B)$, Since $A \subseteq B$, we have $A \subseteq Cl_{h\alpha}(B)$.But $Cl_{h\alpha}(B)$ is closed set.Thus $Cl_{h\alpha}(B)$ is a closed set containing A.Since we know that $Cl_{h\alpha}(A)$ is the smallest closed set containing A, we have $Cl_{h\alpha}(A) \subseteq Cl_{h\alpha}(B)$.
(4) Let B be any $h\alpha$-closed set containing A. Since $Cl_{h\alpha}(A)$ is the intersection of all $h\alpha$-closed set containing A, $Cl_{h\alpha}(A)$ is contained in every $h\alpha$-closed set containing A. Thus in particular $Cl_{h\alpha}(A) \subseteq B$.
(5) If A is $h\alpha$-closed then A itself is the smallest $h\alpha$-closed set containing A and hence $Cl_{h\alpha}(A) = A$. Conversely, suppose $Cl_{h\alpha}(A) = A$, we know that $Cl_{h\alpha}(A)$ is the smallest $h\alpha$-closed set containing A and so A is $h\alpha$-closed. □

Theorem 3.10 *If A and B are subsets of a topological space X. Then* $Cl_{h\alpha}(A \cap B) \subseteq Cl_{h\alpha}(A) \cap Cl_{h\alpha}(B)$.

Proof: Let A and B be subsets of X. Clearly $A \cap B \subseteq A$ and $A \cap B \subseteq B$.Then by Theorem 3.8, we have $Cl_{h\alpha}(A \cap B) \subseteq Cl_{h\alpha}(A)$ and $Cl_{h\alpha}(A \cap B) \subseteq Cl_{h\alpha}(B) \Rightarrow Cl_{h\alpha}(A \cap B) \subseteq Cl_{h\alpha}(A) \cap Cl_{h\alpha}(B)$. □

Corollary 3.2 *If A and B are subsets of a topological space X. If A is $h\alpha$-closed set, then* $Cl_{h\alpha}(A \cap B) \subseteq A \cap Cl_{h\alpha}(B)$.

Proof: Proof follows from Theorem 3.10 and 3.9(5). □

Theorem 3.11 *For an $x \in X$, $x \in Cl_{h\alpha}(A)$ iff $V \cap A \neq \phi$ for every $h\alpha$-open set V containing x.*

Proof: Let $x \in X$ and $x \in Cl_{h\alpha}(A)$. Suppose there exist an $h\alpha$-open set V containing x suchthat $V \cap A = \phi$. Then $A \subset X \setminus V$ and $X \setminus V$ is $h\alpha$-closed , we have $Cl_{h\alpha}(A) \subset X \setminus A$. This shows that $x \notin Cl_{h\alpha}(A)$, a contradiction. Thus $V \cap A \neq \phi$ for every $h\alpha$-open set V containing x. Conversely suppose that $V \cap A \neq \phi$ for every $h\alpha$-open set V containing x. To prove $x \in Cl_{h\alpha}(A)$, let $x \notin Cl_{h\alpha}(A)$ then there exist an $h\alpha$-closed set B containing A such that $x \notin B$. Then $x \in X \setminus B$ and $X \setminus B$ is $h\alpha$-open. Also $(X \setminus B) \cap A = \phi$, which is a contradiction. Thus $x \in Cl_{h\alpha}(A)$. □

Corollary 3.3 *Let (X, τ) be a topological space and $A \subseteq X$. Then* $Cl_{h\alpha}(Cl_{h\alpha}(A)) = Cl_{h\alpha}(A)$.

Proof: Since $Cl_{h\alpha}(A)$ is $h\alpha$-closed set. Then by Theorem 3.9, we have $Cl_{h\alpha}(Cl_{h\alpha}(A)) = Cl_{h\alpha}(A)$. □

Definition 3.4 *Let (X, τ) be a topological space and $A \subseteq X$. Then $h\alpha$-exterior of A is denoted by $ext_{h\alpha}(A)$ and is defined as* $ext_{h\alpha}(A) = int_{h\alpha}(X \setminus A)$.

Example 3.2 *Let* $X = \{x, y, z\}$, $\tau = \{\phi, X, \{y\}, \{y, z\}\}$ *and* $\tau^{h\alpha} = \{\phi, X, \{y\}, \{y, z\}, \{z\}, \{x, z\}$
If $A = \{x, y\}$, *then* $ext_{h\alpha}(A) = \{z\}$.

Theorem 3.12 *Let (X, τ) be a topological space and A,B are subsets of X. Then*

1. *If* $A \subseteq B$, *then* $ext_{h\alpha}(A) \subseteq ext_{h\alpha}(B)$.
2. $ext_{h\alpha}(A \cup B) = ext_{h\alpha}(A) \cap ext_{h\alpha}(B)$.
3. $ext_{h\alpha}(A \cap B) \supseteq ext_{h\alpha}(A) \cup ext_{h\alpha}(B)$.
4. $ext_{h\alpha}(X) = \phi$ *and* $ext_{h\alpha}(\phi) = X$.
5. $ext_{h\alpha}(A) = ext_{h\alpha}(X \setminus ext_{h\alpha}(A))$.
6. $ext_{h\alpha}(A)$ *is $h\alpha$-open set.*

7. $ext_{h\alpha}(A) = X \setminus Cl_{h\alpha}(A)$

8. $ext_{h\alpha}(ext_{h\alpha}(A)) = int_{h\alpha}(Cl_{h\alpha}(A))$.

Proof: (1) Since $A \subseteq B$, then $ext_{h\alpha}(B) = int_{h\alpha}(X \setminus B) \subseteq int_{h\alpha}(X \setminus A) = ext_{h\alpha}(A)$.Hence $ext_{h\alpha}(B) \subseteq ext_{h\alpha}(A)$.
(2) $ext_{h\alpha}(A \cup B) = int_{h\alpha}(X \setminus (A \cup B)) = int_{h\alpha}((X \setminus A) \cap (X \setminus B)) = int_{h\alpha}(X \setminus A) \cap int_{h\alpha}(X \setminus B) = ext_{h\alpha}(A) \cap ext_{h\alpha}(B)$.
(3)$ext_{h\alpha}(A \cap B) = int_{h\alpha}(X \setminus (A \cap B)) = int_{h\alpha}((X \setminus A) \cup (X \setminus B)) \supseteq int_{h\alpha}(X \setminus A) \cup int_{h\alpha}(X \setminus B) = ext_{h\alpha}(A) \cup ext_{h\alpha}(B)$.
(4) $ext_{h\alpha}(X) = int_{h\alpha}(X \setminus X) = int_{h\alpha}(\phi) = \phi$ and $ext_{h\alpha}(\phi) = int_{h\alpha}(X \setminus \phi) = int_{h\alpha}(X) = X$.
(5) $ext_{h\alpha}(X \setminus ext_{h\alpha}(A)) = ext_{h\alpha}(X \setminus (int_{h\alpha}(X \setminus A))) = int_{h\alpha}(X \setminus A) = ext_{h\alpha}(A)$.
(6) and (7) straight forward.
(8) $ext_{h\alpha}(ext_{h\alpha}(A)) = ext_{h\alpha}(X \setminus Cl_{h\alpha}(A)) = int_{h\alpha}(X \setminus (X \setminus Cl_{h\alpha}(A)) = int_{h\alpha}(Cl_{h\alpha}(A))$. □

Theorem 3.13 *Let (X, τ) be a topological space and A is a subset of X. Then*

1. $ext_{h\alpha}(A) \subseteq A^c$.

2. $int_{h\alpha}(A) \subseteq ext_{h\alpha}(ext_{h\alpha}(A))$.

Proof: (1) $ext_{h\alpha}(A) = int_{h\alpha}(A^c) \subseteq A^c$, by Theorem 3.3(2).
(2) By (1), we have $ext_{h\alpha}(A) \subseteq A^c$. Then by Theorem 3.12(1), we have $ext_{h\alpha}(A^c) \subseteq ext_{h\alpha}(ext_{h\alpha}(A))$ but $ext_{h\alpha}(A^c) = int_{h\alpha}(A)$. Thus $int_{h\alpha}(A) \subseteq ext_{h\alpha}(ext_{h\alpha}(A))$. □

Theorem 3.14 *Let (X, τ) be a topological space and A be a subset of X. Then $ext_{h\alpha}(A) = \bigcup\{W : W$ is $h\alpha$-open, $W \subseteq A^c\}$.*

Proof: By Definition 3.4, $ext_{h\alpha}(A) = int_{h\alpha}(X \setminus A)$ i.e; $ext_{h\alpha}(A) = int_{h\alpha}(A^c)$. But by Theorem 3.1 we have $int_{h\alpha}(A^c) = \bigcup\{W : W$ is $h\alpha$-open, $W \subseteq A^c\}$. Thus $ext_{h\alpha}(A) = \bigcup\{W : W$ is $h\alpha$-open, $W \subseteq A^c\}$. □

Definition 3.5 *Let (X, τ) be a topological space and $A \subseteq X$.A point $x \in X$is said to be $h\alpha$-limit point of A if every $h\alpha$-open set W containing x contains atleast one point of A different from x i.e; $\forall\ W \in \tau^{h\alpha}(X, x) \Rightarrow W \bigcap (A \setminus \{x\}) \neq \phi$. The set of all $h\alpha$-limit point of A is called $h\alpha$-derived set of A and is denoted by $D_{h\alpha}(A)$.*

Example 3.3 *Let $X = \{a, b, c, d\}$, $A = \{a, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\tau^{h\alpha} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}$.Then $D_{h\alpha}(A) = \{d\}$.*

Theorem 3.15 *Let (X, τ) be a topological space and $A \subseteq X$. Then a point $x \in X$ is an $h\alpha$-exterior point of A iff x is not an $h\alpha$-limit point of A i.e; iff $x \in Cl_{h\alpha}(A^c)$.*

Proof: Let x be an $h\alpha$-exterior point of A, then x is an $h\alpha$-interior point of A^c, so A^c is an $h\alpha$-nbhd of x containing no point of A. It follows that x is not an $h\alpha$-limit point of A i.e; $x \in Cl_{h\alpha}(A^c)$. Conversely suppose that x is not an $h\alpha$-limit point of A, then there exist an $h\alpha$-nbhd N of x which contains no point of A implies that $x \in N \subseteq A^c$. It follows that x is an $h\alpha$-exterior point of A. □

Theorem 3.16 *Let (X, τ) be a topological space and A be a subset of X. Then following are equivalent:*

1. $\forall\ W \in \tau^{h\alpha}$, $x \in W \Rightarrow A \cap W \neq \phi$.

2. $x \in Cl_{h\alpha}(A)$.

Proof: (1) $\Rightarrow$ (2): Suppose (1) holds, suppose $x \notin Cl_{h\alpha}(A)$ then there exist an $h\alpha$-closed set M such that $A \subseteq M$ and $x \notin M$. Thus $W = X \setminus M$ is an $h\alpha$-open set such that $x \in W$ and $W \cap A = \phi$, which is a contradiction.
(2) $\Rightarrow$ (1):Straight forward. □

Theorem 3.17 *Let (X, τ) be a topological space and $A \subseteq X$. Then following holds:*

1. *$Cl_{h\alpha}(A) = A \cup D_{h\alpha}(A)$.*
2. *A is $h\alpha$-closed iff $D_{h\alpha}(A) \subseteq A$.*

Proof: (1) Let $x \notin Cl_{h\alpha}(A)$ then there exist an $h\alpha$-closed set M suchthat $A \subseteq M$ and $x \notin M$. Hence $W = X \setminus M$ is an $h\alpha$-open set such that $x \in W$ and $W \cap A = \phi$.Therefore $x \notin A$ and $x \notin D_{h\alpha}(A)$ then $x \notin A \cup D_{h\alpha}(A)$. Thus $A \cup D_{h\alpha}(A) \subseteq Cl_{h\alpha}(A)$. On the other hand suppose $x \notin A \cup D_{h\alpha}(A)$ implies that there exist an $h\alpha$-open set W in X such that $x \in W$ and $W \cap A = \phi$. Hence $M = X \setminus W$ is $h\alpha$-closed set in X such that $A \subseteq M$ and $x \notin M$ and thus $x \notin Cl_{h\alpha}(A)$. Hence $Cl_{h\alpha}(A) \subseteq A \cup D_{h\alpha}(A)$.
(2) Let A be a an $h\alpha$-closed set, then A^c is $h\alpha$-open so for each $x \in A^c$ there exist an $h\alpha$-nbhd N of x such that $N \subseteq A^c$. Since $A \cap A^c = \phi$.Then $h\alpha$-nbhd N of x contains no point of A and so x is not an $h\alpha$-limit point of A. Thus no point of A can be a $h\alpha$-limit point of A i.e; A contains all its limits points.Hence $D_{h\alpha}(A) \subseteq A$. Conversely, Let $D_{h\alpha}(A) \subseteq A$ and let $x \in A^c$ then $x \notin A$. Since $D_{h\alpha}(A) \subseteq A$, $x \notin D_{h\alpha}(A)$ hence there exist an $h\alpha$-nbhd N of x such that $N \cap A = \phi$, so $N \subseteq A^c$. Thus A^c contains an $h\alpha$-nbhd of each of its points and so A^c is $h\alpha$-open i.e; A is $h\alpha$-closed. □

Theorem 3.18 *If A is a subset of a discrete topological space (X, τ).Then $D_{h\alpha}(A) = \phi$.*

Proof: Let $x \in X$ and since every subset of X is open and so $h\alpha$-open. In particular the singleton $W = \{x\}$ is $h\alpha$-open. But $x \in W$ and $W \cap A = \{x\} \cap A \subseteq \{x\}$. Hence x is not an interior point of A and so $D_{h\alpha}(A) = \phi$. □

Theorem 3.19 *Let τ_1 and τ_2 be two topologies on X such that $\tau_1^{h\alpha} \subseteq \tau_2^{h\alpha}$.For any subset A of X every $h\alpha$-limit point of A with respect to τ_2 is an $h\alpha$-limit point of A with respect to τ_1.*

Proof: Let x be an $h\alpha$-limit point of A with respect to τ_2. Then $W \cap (A \setminus \{x\}) \neq \phi$ for every $W \in \tau_2^{h\alpha}$ such that $x \in W$.But $\tau_1^{h\alpha} \subseteq \tau_2^{h\alpha}$ so, in particular $W \cap (A \setminus \{x\}) \neq \phi$ for every $W \in \tau_1^{h\alpha}$ such that $x \in W$. Hence x is an $h\alpha$-limit point of A with respect to τ_1. □

Remark 3.2 *The converse of the above theorem need not be true by the following example.*

Example 3.4 *Let $X = \{1, 2, 3\}$, $\tau_1 = \{\phi, X, \{1\}\}$, $\tau_2 = \{\phi, X, \{1\}, \{1, 2\}\}$, $\tau_1^{h\alpha} = \{\phi, X, \{1\}, \{2, 3\}\}$ and $\tau_2^{h\alpha} = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}$. Then $\tau_1^{h\alpha} \subseteq \tau_2^{h\alpha}$ and $\{2\}$ is an $h\alpha$-limit point of $A = \{1, 3\}$ with respect to τ_1 while $\{2\}$ is not an $h\alpha$-limit point of $A = \{1, 3\}$ with respect to τ_2.*

Definition 3.6 *Let (X, τ) be a topological space and $A \subseteq X$. Then $h\alpha$-border of A is denoted by $b_{h\alpha}(A)$ and is defined as $b_{h\alpha}(A) = A \setminus int_{h\alpha}(A)$ and the set $Fr_{h\alpha}(A) = Cl_{h\alpha}(A) \setminus int_{h\alpha}(A)$ is called the $h\alpha$-frontier of A.*

Note 3 *If A is $h\alpha$-closed subset of X, then $b_{h\alpha}(A) = Fr_{h\alpha}(A)$.*

Example 3.5 Let $X = \{1, 2, 3\}$,$A = \{1, 2\}$, $\tau = \{\phi, X, \{2\}, \{2, 3\}$, and $\tau^{h\alpha} = \{\phi, X, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}$. Then $int_{h\alpha}(A) = \{2\}$, $b_{h\alpha}(A) = \{1\}$, $Cl_{h\alpha}(A) = \{1, 2\}$ and $Fr_{h\alpha}(A) = \{1\}$.

Theorem 3.20 *Let (X, τ) be a topological space and $A \subseteq X$. Then following holds:*

1. *$A = int_{h\alpha}(A) \cup b_{h\alpha}(A)$.*

2. $int_{h\alpha}(A) \cap b_{h\alpha}(A) = \phi$.
3. A is $h\alpha$-open iff $b_{h\alpha}(A) = \phi$.
4. $b_{h\alpha}(int_{h\alpha}(A)) = \phi$.
5. $int_{h\alpha}(b_{h\alpha}(A)) = \phi$.
6. $b_{h\alpha}(b_{h\alpha}(A)) = b_{h\alpha}(A)$.
7. $Cl_{h\alpha}(A) = int_{h\alpha}(A) \cup Fr_{h\alpha}(A)$.

Proof: (1) and (2) directly follows from Definition 3.6.
(3) Since $int_{h\alpha}(A) \subseteq A$, by Theorem 3.3 A is $h\alpha$-open iff $A = int_{h\alpha}(A)$ iff $b_{h\alpha}(A) = A \setminus int_{h\alpha}(A) = \phi$.
(4) Since $int_{h\alpha}(A)$ is $h\alpha$-open, it follows from (3) that $b_{h\alpha}(int_{h\alpha}(A)) = \phi$.
(5) If $x \in int_{h\alpha}(b_{h\alpha}(A))$, then $x \in b_{h\alpha}(A) \subseteq A$ and $x \in int_{h\alpha}(A)$. Since $int_{h\alpha}(b_{h\alpha}(A)) \subseteq int_{h\alpha}(A)$.Thus $x \in b_{h\alpha}(A) \cap int_{h\alpha}(A) = \phi$, which is a contradiction to (2).
(6) We have $b_{h\alpha}(b_{h\alpha}(A)) = b_{h\alpha}(A) \setminus int_{h\alpha}(b_{h\alpha}(A)) = b_{h\alpha}(A)$.
(7) $int_{h\alpha}(A) \cup Fr_{h\alpha}(A) = int_{h\alpha}(A) \cup (Cl_{h\alpha}(A) \setminus int_{h\alpha}(A)) = Cl_{h\alpha}(A)$. □

Theorem 3.21 *Let (X, τ) be a topological spaces and $A \subseteq X$. Then A is $h\alpha$-closed iff $Fr_{h\alpha}(A) \subseteq A$.*

Proof: Suppose A is $h\alpha$-closed. Then $Fr_{h\alpha}(A) = b_{h\alpha}(A) \subseteq A \Rightarrow Fr_{h\alpha}(A) \subseteq A$.Conversely, suppose $Fr_{h\alpha}(A) \subseteq A$. Then $Cl_{h\alpha}(A) \setminus int_{h\alpha}(A) \subseteq A$ and so $Cl_{h\alpha}(A) \subseteq A$. By Theorem 3.9, $A \subseteq Cl_{h\alpha}(A)$. Hence A is $h\alpha$-closed. □

Corollary 3.4 *Let (X, τ) be a topological space and $A \subseteq X$. Then following holds:*

1. $int_{h\alpha}(A) \cap Fr_{h\alpha}(A) = \phi$.
2. $Fr_{h\alpha}(A) = Cl_{h\alpha}(A) \cap Cl_{h\alpha}(X \setminus A)$.
3. $int_{h\alpha}(A) = A \setminus Fr_{h\alpha}(A)$.
4. $b_{h\alpha}(A) = A \cap Cl_{h\alpha}(X \setminus A)$.

4. Almost $h\alpha$-continuous functions, $h\alpha$-contra continuous functions, Almost $h\alpha$-contra continuous functions and Strongly $h\alpha$-continuous functions

In this section we have introduced and discussed the notion of Almost $h\alpha$-continuous, $h\alpha$-contra continuous, Almost $h\alpha$-contra continuous, and strongly $h\alpha$-continuous functions. Some properties, counter examples and theorems are established.

Definition 4.1 *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be almost $h\alpha$-continuous if $f^{-1}(V)$ is $h\alpha$-open set in X for every regular open set V in Y.*

Example 4.1 *Let $X = \{a, b, c\} = Y$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$, $\sigma = \{\phi, Y, \{b, c\}, \{b\}, \{a.c\}, \{c\}\}$ and $\tau^{h\alpha} = \{\phi, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.*
Clearly the identity function $f : (X, \tau) \to (Y, \sigma)$ is almost $h\alpha$-continuous function.

Definition 4.2 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $h\alpha$-contra continuous if $f^{-1}(V)$ is $h\alpha$-closed in X for every open set V in Y.*

Example 4.2 *Let $X = \{a, b, c\} = Y$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a\}\}$ and $\tau^{h\alpha} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}$.*
Clearly the identity function $f : (X, \tau) \to (Y, \sigma)$ is $h\alpha$-contra-continuous function.

Definition 4.3 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost $h\alpha$-contra continuous, if $f^{-1}(V)$ is $h\alpha$-closed set in X for every regular open set V in Y.*

Example 4.3 *Let* $X = \{a, b, c, d\} = Y$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}$, $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ *and* $\tau^{h\alpha} = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}\}$.
Define a function $f : (X, \tau) \to (Y, \sigma)$ *by* $f(d) = b$ *and* $f(b) = a$. *Then* f *is almost* $h\alpha$*-contra-continuous function.*

Theorem 4.1 *Every* $h\alpha$*-continuous function is almost* $h\alpha$*-continuous.*

Theorem 4.2 *Every* $h\alpha$*-contra continuous function is almost* $h\alpha$*-contra continuous.*

Theorem 4.3 *Every* $h\alpha$*-continuous function is regular set connected function.*

Theorem 4.4 *Every* $h\alpha$*-contra continuous function is regular set connected function.*

Theorem 4.5 *Every almost continuous function is almost* $h\alpha$*-continuous function.*

Theorem 4.6 *Every totally continuous function is* $h\alpha$*-contra continuous function.*

Note 4 *The proof of Theorems 4.1-4.6 are straight forward.*

Theorem 4.7 *Every contra-continuous function is* $h\alpha$*-contra-continuous function.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ be a contra continuous function. Let V be an open set in Y. Since f is contra-continuous, we have $f^{-1}(V)$ is closed in $X \Rightarrow f^{-1}(V)$ is $h\alpha$-closed in X. □

Remark 4.1 *The converse of the above theorem need not be true by the following example.*

Example 4.4 *In Example 4.2, the identity function is* $h\alpha$*-contra-continuous but not contra-continuous.*

Theorem 4.8 *If* $f : (X, \tau) \to (Y, \sigma)$ *is* $h\alpha$*-continuous and* $g : (Y, \sigma) \to (Z, \rho)$ *is almost continuous. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is almost* $h\alpha$*-continuous function.*

Proof: Let V be a regular open subset of Z. Since g is almost continuous, so $g^{-1}(V)$ is open subset of Y. Since f is $h\alpha$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $h\alpha$-open subset in X. Therefore $g \circ f : (X, \tau) \to (Z, \rho)$ is almost $h\alpha$-continuous function. □

Corollary 4.1 *If* $f : (X, \tau) \to (Y, \sigma)$ *is* $h\alpha$*-contra continuous and* $g : (Y, \sigma) \to (Z, \rho)$ *is almost continuous. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is almost* $h\alpha$*-contra continuous.*

Theorem 4.9 *If* $f : (X, \tau) \to (Y, \sigma)$ *is continuous and open and* $g : (Y, \sigma) \to (Z, \rho)$ *is almost* $h\alpha$*-continuous. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is almost* $h\alpha$*-continuous function.*

Proof: Let V be any regular open subset of Z. Since g is almost $h\alpha$-continuous, we have $g^{-1}(V)$ is $h\alpha$-open subset of Y. Also f is continuous and open, we have $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $h\alpha$-open set in X. Therefore $g \circ f : (X, \tau) \to (Z, \rho)$ is almost $h\alpha$-continuous function. □

Theorem 4.10 *If* $f : (X, \tau) \to (Y, \sigma)$ *is* $h\alpha$*-contra continuous and* $g : (Y, \sigma) \to (Z, \rho)$ *is* $h\alpha$*-totally continuous. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is* $h\alpha$*-contra-continuous function.*

Proof: Let V be an open set in Z which implies V is $h\alpha$-open set. Since g is $h\alpha$-totally continuous, then $g^{-1}(V)$ is clopen in Y. Also f is $h\alpha$-contra-continuous, so $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $h\alpha$-closed set in X. Thus $g \circ f : (X, \tau) \to (Z, \rho)$ is $h\alpha$-contra-continuous-function. □

Definition 4.4 *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be strongly* $h\alpha$*-continuous if* $f^{-1}(V)$ *is open set in* X *for every* $h\alpha$*-open set* V *in* Y.

Example 4.5 *(1) Let* $X = \{1,2,3\} = Y$, $\tau = \{\phi, \{1\}, \{2\}, \{1,2\}, \{3\}, X\}$, $\sigma = \{\phi, \{1\}, Y\}$, $\sigma^{h\alpha} = \{\phi, \{1\}, \{2,3\}, Y\}$. *Define a function* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(1) = 3$, $f(3) = 1$ *and* $f(2) = 2$. *Then* f *is strongly* $h\alpha$*-continuous function.*
(2) Let $X = \{1,2,3,4\} = Y$, $\tau = \{\phi, \{1,2,3\}, \{4\}, X\}$, $\sigma = \{\phi, \{4\}, Y\}$, $\sigma^{h\alpha} = \{\phi, \{4\}, \{1,2,3\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is strongly* $h\alpha$*-continuous function.*

Theorem 4.11 *Every* $h\alpha$*-totally continuous function is strongly* $h\alpha$*-continuous function.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ is $h\alpha$-totally continuous function. Then for every $h\alpha$-open set V in Y, we have $f^{-1}(V)$ is clopen in $X \Rightarrow f^{-1}(V)$ is open in X for every $h\alpha$-open set V in Y. □

Remark 4.2 *The converse of the above theorem need not be true by the following example.*

Example 4.6 *Let* $X = \{1,2,3\} = Y$, $\tau = \{\phi, \{1\}, \{3\}, \{1,3\}, X\}$, $\sigma = \{\phi, \{1\}, \{3\}, \{1,3\}, Y\}$ *and* $\sigma^{h\alpha} = \{\phi, \{1\}, \{3\}, \{1,3\}, Y\}$. *Then the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is strongly* $h\alpha$*-continuous function but not* $h\alpha$*-totally continuous function.*

Theorem 4.12 *Every strongly* $h\alpha$*-continuous function is* $h\alpha$*-irresolute.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ be a strongly $h\alpha$-continuous function. Let V be an $h\alpha$-open set in Y. Since f is strongly $h\alpha$-continuous, we have $f^{-1}(V)$ is open set in $X \Rightarrow f^{-1}(V)$ is $h\alpha$-open set in X. □

Remark 4.3 *The converse of the above theorem need not be true by the following example.*

Example 4.7 *Let* $X = \{a, b, c\} = Y$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, $\sigma = \{\phi, \{a\}, Y\}$, $\sigma^{h\alpha} = \{\phi, \{a\}, \{b, c\}, Y\}$, *and* $\tau^{h\alpha} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$.
Then clearly the identity function $f : (X, \tau) \to (Y, \sigma)$ *is an* $h\alpha$*-irresolute function but not strongly* $h\alpha$*-continuous because* $g^{-1}(\{b, c\}) = \{b, c\}$ *is not open in* X.

Theorem 4.13 *Every strongly* $h\alpha$*-continuous function is* $h\alpha$*-continuous.*

Proof: Let V be an open set in Y. Then V is an $h\alpha$-open set in Y. Since f is strongly $h\alpha$-continuous, we have $f^{-1}(V)$ is open set in $X \Rightarrow f^{-1}(V)$ is open set in X for every open set V in Y. □

Remark 4.4 *The converse of the above theorem need not be true by the following example.*

Example 4.8 *In Example 4.7, the identity function* $f : (X, \tau) \to (Y, \sigma)$ *is* $h\alpha$*-continuous but not strongly* $h\alpha$*-continuous.*

Theorem 4.14 *If a function* $f : (X, \tau) \to (Y, \sigma)$ *is strongly* $h\alpha$*-continuous and* $g : (Y, \sigma) \to (Z, \rho)$ *is* $h\alpha$*-totally continuous. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is strongly* $h\alpha$*-continuous function.*

Proof: Let V be an $h\alpha$-open set in Z.Since g is $h\alpha$-totally continuous function, we have $g^{-1}(V)$ is clopen in $Y \Rightarrow g^{-1}(V)$ is $h\alpha$-open set in Y. Also f is strongly $h\alpha$-continuous, then $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open set in X. Thus $g \circ f : (X, \tau) \to (Z, \rho)$ is strongly $h\alpha$-continuous function. □

Theorem 4.15 *If* $g : (X, \tau) \to (Y, \sigma)$ *is* $h\alpha$*-continuous and* $f : (Y, \sigma) \to (Z, \rho)$ *is strongly* $h\alpha$*-continuous. Then* $f \circ g : (X, \tau) \to (Z, \rho)$ *is* $h\alpha$*-irresolute.*

Proof: Suppose $g : (X, \tau) \to (Y, \sigma)$ is $h\alpha$-continuous and $f : (Y, \sigma) \to (Z, \rho)$ is strongly $h\alpha$-continuous function. Let V be an open set in $Z \Rightarrow V$ is an $h\alpha$-open set in Z, then we have $f^{-1}(V)$ is open in Y.Also g is $h\alpha$-continuous, then $g^{-1}(f^{-1}(V)) = (f \circ g)^{-1}(V)$ is $h\alpha$-open set in X for every open set V. □

Corollary 4.2 *If $g : (x, \tau) \to (Y, \sigma)$ is strongly $h\alpha$-continuous and $f : (Y, \sigma) \to (Z, \rho)$ is $h\alpha$-irresolute. Then $f \circ g : (X, \tau) \to (Z, \rho)$ is strongly $h\alpha$-continuous.*

Theorem 4.16 *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ are strongly $h\alpha$-continuous functions. Then $g \circ f : (X, \tau) \to (Z, \rho)$ is also strongly $h\alpha$-continuous function.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ are strongly $h\alpha$-continuous functions. Let V be an $h\alpha$-open set in Z. Since g is strongly $h\alpha$-continuous function, we have $g^{-1}(V)$ is open in $Y \Rightarrow g^{-1}(V)$ is $h\alpha$-open set in Y.Also f is strongly $h\alpha$-continuous function $\Rightarrow f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open in X. □

Remark 4.5 *The converse of above theorem need not be true.*

Example 4.9 *Let $X = Y = Z = \{1, 2, 3, 4\}$, $\tau = \{\phi, \{1, 2, 3\}, X\}$, $\sigma = \{\phi, \{1, 2\}, \{3, 4\}, Y\}$, $\rho = \{\phi, \{4\}, Z\}$, $\sigma^{h\alpha} = \{\phi, \{1, 2\}, \{3, 4\}, Y\}$ and $\rho^{h\alpha} = \{\phi, \{4\}, \{1, 2, 3\}, Z\}$.*
Consider the identity function $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$. Then the identity function $g \circ f : (X, \tau) \to (Z, \rho)$ is strongly $h\alpha$-continuous but neither g is strongly $h\alpha$-continuous nor f is strongly $h\alpha$-continuous because $f^{-1}(\{1, 2\}) = \{1, 2\}$ which is not open in X and $g^{-1}(\{4\}) = \{4\}$ is not open in Y.

Theorem 4.17 *If a function $f : (X, \tau) \to (Y, \sigma)$ is strongly $h\alpha$-continuous and $g : (Y, \sigma) \to (Z, \rho)$ is $h\alpha$-continuous. Then $g \circ f : (X, \tau) \to (Z, \rho)$ is continuous.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ is strongly $h\alpha$-continuous and $g : (Y, \sigma) \to (Z, \rho)$ is $h\alpha$-continuous. Let V be an open set in Z. Then $g^{-1}(V)$ is $h\alpha$-open set in Y. Since f is strongly $h\alpha$-continuous, we have $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open set in X. Therefore $g \circ f : (X, \tau) \to (Z, \rho)$ is continuous. □

Proposition 4.1 *A function $f : (X, \tau) \to (Y, \sigma)$ is strongly $h\alpha$-continuous iff $f^{-1}(V)$ is closed in X for every $h\alpha$-closed set V in Y.*

Proof: Suppose f is strongly $h\alpha$-continuous function. Let V be an $h\alpha$-closed set in Y, then V^c is $h\alpha$-open set in Y. Since f is strongly $h\alpha$-continuous, we have $f^{-1}(V^c)$ is open set in X. But $f^{-1}(V^c) = X \backslash f^{-1}(V)$. Hence $f^{-1}(V)$ is closed in X. Conversely, suppose $f^{-1}(V)$ is closed set in X for every $h\alpha$-closed set V in Y. Let V be any $h\alpha$-open set in Y, then V^c is $h\alpha$-closed set in Y. By assumption $f^{-1}(V^c)$ is closed set in X. But $f^{-1}(V^c) = X \setminus f^{-1}(V)$ and $f^{-1}(V)$ is open set in X. Hence f is strongly $h\alpha$-continuous. □

Theorem 4.18 *Every strongly $h\alpha$-continuous function is continuous.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ be a strongly $h\alpha$-continuous function.Let V be an open set in $Y \Rightarrow V$ is an $h\alpha$-open set in Y. Since f is strongly $h\alpha$-continuous function, we have $f^{-1}(V)$ is open in X. □

Remark 4.6 *Converse of the above theorem need not be true by the following example.*

Example 4.10 *In Example 4.7, the identity function $f : (X, \tau) \to (Y, \sigma)$ is continuous but not strongly $h\alpha$-continuous function.*

Theorem 4.19 *Every strongly continuous function is strongly $h\alpha$-continuous function.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ be a strongly continuous function. Let V be an open set in Y.Since f is strongly continuous, we have $f^{-1}(V)$ is clopen$\Rightarrow f^{-1}(V)$ is $h\alpha$-open in X. □

Remark 4.7 *Converse of the above theorem need not be true.*

Example 4.11 *Example 4.6 works here.*

Theorem 4.20 *Let X be a discrete topological space and Y be any topological space and $f : X \to Y$ be a function. Then following are equivalent:*

1. *f is totally $h\alpha$-continuous.*
2. *f is strongly $h\alpha$-continuous.*

Proof: (1) $\Rightarrow$ (2) Proof is obvious.
(2) $\Rightarrow$ (1) Let V be an $h\alpha$-open set in Y. Then $f^{-1}(V)$ is open in X. Since X is discrete space which implies $f^{-1}(V)$ is also closed in X. Thus f is totally $h\alpha$-continuous function. □

Theorem 4.21 *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ are strongly $h\alpha$-continuous. Then $g \circ f : (X, \tau) \to (Z, \rho)$ is $h\alpha$-irresolute.*

Proof: Let V be an $h\alpha$-open set in Z. Since g is strongly $h\alpha$-continuous, $g^{-1}(V)$ is open set in $Y \Rightarrow g^{-1}(V)$ is $h\alpha$-open set in Y.Also f is strongly $h\alpha$-continuous function, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is open set in X. Hence $(g \circ f)^{-1}(V)$ is $h\alpha$-open set in X. □

Theorem 4.22 *If $f : (X, \tau) \to (Y, \sigma)$ is $h\alpha$-continuous and $g : (Y, \sigma) \to (Z, \rho)$ is strongly $h\alpha$-continuous. Then $g \circ f : (X, \tau) \to (Z, \rho)$ is $h\alpha$-irresolute.*

Proof: Suppose $f : (X, \tau) \to (Y, \sigma)$ is $h\alpha$-continuous and $g : (Y, \sigma) \to (Z, \rho)$ is strongly $h\alpha$-continuous. Let V be an $h\alpha$-open set in Z.Since g is strongly $h\alpha$-continuous, $g^{-1}(V)$ is open set in Y. Also f is $h\alpha$-continuous function, we have $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $h\alpha$-open set in X. □

Corollary 4.3 *If $f : (X, \tau) \to (Y, \sigma)$ is continuous and $g : (Y, \sigma) \to (Z, \rho)$ is strongly $h\alpha$-continuous. Then $g \circ f : (X, \tau) \to (Z, \rho)$ is strongly $h\alpha$-continuous function.*

Theorem 4.23 *Let (X, τ) and (Y, σ) be two topological spaces. Then the following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

1. *f is almost contra $h\alpha$-continuous.*
2. *$f^{-1}(V)$ is $h\alpha$-open set of X for every regular closed set V of Y.*
3. *for each $x \in X$ and each regular closed set V of Y containing $f(x)$ there exist an $h\alpha$-open set U containing x such that $f(U) \subset V$.*
4. *for each $x \in X$ and each regular open set V of Y not containing $f(x)$ there exist an $h\alpha$-closed set K not containing x such that $f^{-1}(V) \subset K$.*

Proof: (1) $\Rightarrow$ (2): Let V be a regular closed set of Y. Then $Y \setminus V$ is regular open set in Y then by (1), $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $h\alpha$-closed in X which gives $f^{-1}(V)$ is $h\alpha$-open set in X.
(2) $\Rightarrow$ (1): Let V be a regular open set in Y. Then $Y \setminus V$ is regular closed set in Y, by (2) we have $f^{-1}(Y \setminus V)$ is $h\alpha$-open set in $X \Rightarrow X \setminus f^{-1}(V)$ is $h\alpha$-open set in $X \Rightarrow f^{-1}(V)$ is $h\alpha$-closed set in X.
(2) $\Rightarrow$ (3): Let V be a regular closed set of Y containing $f(x)$ which implies $x \in f^{-1}(V)$. Then by (2), $f^{-1}(V)$ is $h\alpha$-open set in X containing x. Set $U = f^{-1}(V) \Rightarrow U$ is $h\alpha$-open set in X containing x and $f(U) = f(f^{-1}(V)) \subset V \Rightarrow f(U) \subset V$.
(3) $\Rightarrow$ (2): Let V be a regular closed set in Y containing $f(x)$ which implies $x \in f^{-1}(V)$. Then by (3), there exist an $h\alpha$-open set U_x in X containing x such that $f(U) \subset V$ i.e; $U \subset f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}$ which is union of $h\alpha$-open sets. Therefore $f^{-1}(V)$ is $h\alpha$-open set of X.
(3) $\Rightarrow$ (4): Let V be a regular open set of Y not containing $f(x)$. Then $Y \setminus V$ is regular closed set in Y containing $f(x)$, by (2) there exist an $h\alpha$-open set U in X containing x such that $f(U) \subset Y \setminus V \Rightarrow$

$U \subset f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Hence $f^{-1}(V) \subset X \setminus U$. Set $X \setminus U = K$, then K is $h\alpha$-closed set not containing x in X such that $f^{-1}(V) \subset K$.
(4) $\Rightarrow$ (3): Let V be a regular closed set in Y containing $f(x)$. Then $Y \setminus V$ is regular open set in Y not containing $f(x)$. From (4) there exists $h\alpha$-closed set K in X not containing x such that $f^{-1}(Y \setminus V) \subset K \Rightarrow X \setminus f^{-1}(V) \subset K$. Hence $Y \setminus K \subset f^{-1}(V)$ i.e; $f(X \setminus K) \subset V$. Set $U = X \setminus K$, then U is $h\alpha$-open set containing x in X such that $f(U) \subset V$. □

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6. Conflict of Interest

The authors declare there is no conflicts of interest.

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