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# Oscillatory flow of Couple stress fluid flow past a fluid sphere with slip condition: Exact solution

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ABSTRACT: This article is about the usage of interfacial slip on the boundary and the exact solution for an oscillatory flow of couple stress fluid (CSF) beyond a fluid sphere filled with a (CSF) couple stress fluid. The stream functions and drag are computed analytically. Special cases are deduced for drag force which were observed and ratified with data in literature. The numerical results are tabulated and represented in graphs. It was observed that at a fixed value of a couple stress parameter value, there is a direct relation between slip parameter and real drag values and inverse relation among the imaginary drag values. In addition, oscillatory Newtonian flow beyond a viscous fluid sphere is analyzed and the oscillatory couple stress fluid (CSF) flow over a viscous fluid sphere and vice versa were also analyzed.

Key Words: Couple stress fluid (CSF), viscous fluid, slip condition, oscillatory flow, drag force, stream function.

## Nomenclature:

$p$ is hydro-static pressure at any position
$\bar{q}$ is the fluid velocity
$\bar{f}$ is body force per unit mass
$\alpha, \beta$ are roots of the equation for stream function
$\omega$ is oscillation frequency parameter
$t_1$ is time
$T$ is real drag coefficient
$\rho$ is the fluid density
$\mu$ is viscosity coefficient
$\eta, \eta'$ are couple stress viscosity parameters
$U_\infty$ is velocity at infinity
$e$ is couple stress parameter
$d_f$ is drag force
$T1$ is imaginary drag coefficient

## Contents



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## 1. Introduction

Oscillatory flow types are generally found in beaches due to forced vibrations acting on the flow. Mainly a couple stress fluid flows were observed in many of the applications like electrically conductive bio-technological liquids, hydro-magnetic control of blood flow in medical devices, oil drilling, towing operations, micro-polar fluids etc. as mentioned by Ramana Murthy and Nagaraju [1], Anwar Beg et al. [2].

Alassar and Badr [3] study describes the behavior of oscillating streams studied when time-dependent axisymmetric flow passes over a sphere. They used a series of truncation methods to obtain the solutions. Abiodun O. Ajibade and Stephen Ishaya [4] considered an oscillatory flow of a Newtonian non-compressible fluid passed over a horizontal channel made of porous material. The outcomes presented that, in sine and cosine oscillation forms as fluid velocity increases then Darcy number also increases. Boroujerdi and Esmaeili [5] worked on extracting the new relations between heat transfer and pressure drop evaluated by oscillatory flows. Sherief et al. [6] Considered the slow motion of a solid spherical particle immersed in a fluid with semi-infinite viscosity and surrounded by an impermeable plane wall. They studied the no-slip kinematic condition which was solved analytically. Jordi Ortin [7] considered the significance of oscillatory flows associated with various industrial exhaust flow problems. The work mainly focused on determining the dimensionless variables, which govern the flow of fluids in a steady state. Lakshmana Rao and Bhujanga Rao [8] detailed the rectilinear and rotating oscillatory flow behaviour on a sphere investigated along the diameter under no-slip conditions. The work also explored the effect of couple and drag in rectilinear and rotary types of oscillatory flow on a sphere with no-slip condition. Srinivasacharya and Iyengar [9] examined the effect of micro-polarity, frequency and geometric characteristics on drag as experienced by spheroidal shaped bodies and oscillatory flows on sphere. Lou et al. [10] reported the outcome of yaw angle on hydrodynamic forces observed in oscillatory flows when passed over a cylindrical structure. The above-mentioned studies focus on the findings on oscillating viscous flow over various geometries.

Tatsuo Sawada et al. [11] studied the impact of couple stress and other factors through flow rate of fluids having an oscillatory flow. The finding states that, if the rate of viscosity was small then the parameters like fluid polarity and micro-rotation become zero. Nasir Ali et al. [12] used the Homotopy Analysis Method to solve couple stress fluid problems in an unsteady flow condition by reducing to two coupled partial differential equations with dimensionless values. Suresh Babu et al. [13] workout the influence of ohmic and Newtonian dissipation effects on porous structured semi-infinite vertical permeable plate accomplished by an non-compressible CSF subjected to oscillatory flows. Nirmala Ratchagar et al. [14] investigated the effect of hall current on oscillatory flow in each channel and closed form solutions were obtained on different physical quantities, where blood is assumed to be a couple stress fluid. Govindarajan and Vijayalakshmi [15] in their work developed a mathematical model to analyze the Hall current effect in magneto hydrodynamic oscillatory couple stress dusty fluids when passed over a given permeable medium channel.

Geetha Vani and Ravi Kanth [16] have evaluated the effect of stress tensor and couple stress tensor on rotary oscillatory flow of a non-compressible CSF when passed between two spherical bodies having the same diameter and frequency but with distinct angular speeds. The results also addressed that incompressible couple impacts on rotary oscillatory flow. Aparna et al. [17] obtained the stream function analytically for the oscillatory flow of a CSF past a permeable sphere using modified Bessel functions. The analytical results proved that the drag on permeable sphere for viscous fluid flow is less compared with CSF flow. Alsudais et al. [18] explored the effect of couple stress between two eccentric spherical bodies studied with no-slip condition. The numerical results showed that the drag force values are directly proportional to the couple stress viscosity parameter. The above-mentioned investigations have considered the studies on couple stress fluids and oscillatory flows on using boundary no-slip conditions. Hikmat Saad and Ashmawy [19] have developed an analytical expression to the behavior of fluid velocity in an unsteady flow of an incompressible couple stress fluid that is passed between two long parallel horizontal plates with slip over its boundary. Naga Lakshmi Devi and Phani Kumar [20, 21] have established an exact solution for uniform CSF flow over a fluid drop filled with a CSF employing slip and analytically showed the drag force, and a partially surfactant non-Newtonian fluid sphere with an interfacial slip condition have been derived respectively. Naga Lakshmi Devi and Phani Kumar [22] have the mentioned work addressed oscillation of a CSF flows on a contaminated fluid sphere with slip on boundary. The analytic method was used to obtain stream function and drag force. Vijaya Lakshmi and Phani Kumar [23, 24] have obtained an analytical solution for drag with a uniform flow over a contaminated fluid sphere and the fluid sphere with slip condition respectively.

From the available literature it is evident that most of the research has been limited on addressing the couple stress fluid problems with no-slip condition, and the oscillatory flow over different geometries except a fluid sphere and the oscillatory flow of a viscous fluid flow over another viscous fluid sphere has not yet been researched. These oscillatory flows have applications in the field of oil recovery, pumping etc., as mentioned in Jordi Ortin [7]. The problems addressed over spheroidal droplets arise in petroleum refining, fluidized beds etc., from Clift et al., [26].

In the present note, it mainly focuses on developing analytical solutions for an oscillatory flow at different conditions such as:

- Oscillatory flow of viscous fluid beyond a viscous fluid sphere,
- Oscillatory flow of viscous fluid beyond a CSF sphere and vice versa,
- Oscillatory flow of CSF over a CSF sphere by using slip condition.

## 2. Oscillatory flow of viscous fluid flow past a viscous fluid sphere

### 2.1. Fundamental equations of oscillatory flow on viscous fluid:

Consider an oscillatory flow over a fluid sphere filled with viscous fluid and fixed in a viscous fluid stream. The flow is presumed non-compressible and axi-symmetric. The geometry of a problem is shown in Figure 1.

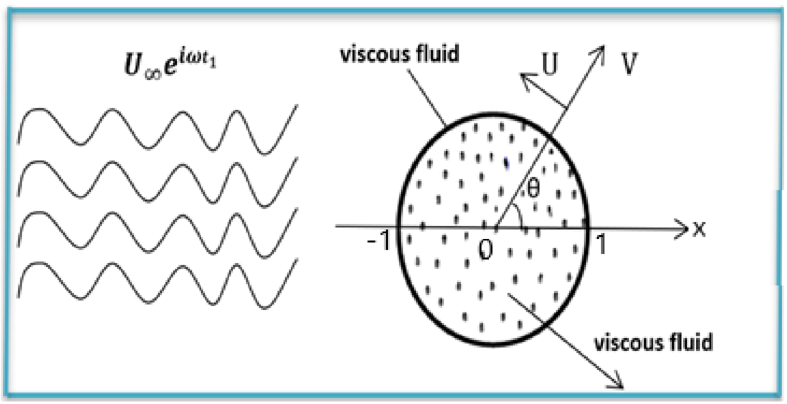


Figure 1: Geometry of oscillatory viscous flow on a viscous fluid sphere

As the flow is axisymmetric, select the velocity vector of the form as $\bar{q} = U\,(R,\ \theta)\,\bar{e}_r +\ V\,(R,\ \theta)\,\bar{e}_\theta$. The basic equations governing the flow of an in-compressible viscous fluid are

$$\nabla.\bar{q} = 0. \tag{2.1}$$

The field equations that determine viscous fluid flow are as follows:

$$\rho\frac{d\bar{q}}{dt} = -\nabla \mathrm{P} + \mu\,(\nabla\times\nabla\times\bar{q})\,. \tag{2.2}$$

″ Due to the geometrical shape of the present problem, we choose spherical coordinate system for reference. The system's scale factors are $h_1 = 1,\ h_2 =\ R,\ h_3 = R\sin\theta$. The spherical coordinate system is used, with the origin in the central of the sphere and the Z-axis running along the flow direction″. (Naga Lakshmi Devi Parasa, Phani Kumar Meduri [22]).

The stream function can be used as (Happel and Brenner [25])

$$U\left(R,\ \theta\right)=\frac{1}{R^{2}sin\theta}\frac{\partial\Psi}{\partial\theta},\quad V\left(R,\ \theta\right)=\ \frac{-1}{R\ sin\theta}\frac{\partial\Psi}{\partial R}.$$

The velocity of the field acceptable for this oscillating flow is analyzed in the following form,

$$\bar{q}=\ \nabla\times\left(\frac{\Psi\bar{e}_{\Phi}}{h_{3}}\right)e^{i\omega t_{1}}=\left(\frac{1}{R^{2}sin\theta}\frac{\partial\Psi}{\partial\theta}\bar{e}_{r}-\frac{-1}{R\ sin\theta}\frac{\partial\Psi}{\partial R}\bar{e}_{\theta}\right)e^{i\omega t_{1}}. \tag{2.3}$$

$$\nabla\times\bar{q}=-\left(\frac{E_{0}^{2}\Psi}{h_{3}}\right)\bar{e}_{\Phi}e^{i\omega t_{1}}. \tag{2.4}$$

where $E_0^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{cot\theta}{r^2}\frac{\partial}{\partial\theta}$
Substituting (2.3), (2.4) and eliminating pressure $P$ in eq. (2.2) gives momentum equation as

$$E_{0}^{2}\left[E_{0}^{2}-\frac{\alpha^{2}}{a^{2}}\right]\Psi=0.\quad \text{where}\quad \alpha^{2}=\frac{i\rho\omega}{\mu} \tag{2.5}$$

Using the non-dimensional approach is applied.
$R=ar;\ \Psi=\psi U_\infty a^2;\ P=\frac{pU_\infty\mu}{a};\ E_0^2=\frac{E^2}{a^2};\ U=uU_\infty;\ V=vV_\infty.$
Equation. (2.5) reduces to

$$E^{2}\left[E^{2}-\alpha^{2}\right]\psi=0. \tag{2.6}$$

### 2.2. Finding the problem:

The solutions of Eqn.(2.6) which are systematic for external flow of viscous fluid ($\psi_e$) and for internal flow of viscous fluid ($\psi_i$) regions are given by (Aparna et al. [17]):

$$\psi_{e}=\left[r^{2}+\frac{B_{1}}{r}+C_{1}\sqrt{r}\ \ K_{\frac{3}{2}}\left(\alpha_{e}r\right)\right]G_{2}\left(x\right),\qquad \text{and} \tag{2.7}$$

$$\psi_{i}=\left[B_{2}\ r^{2}+C_{2}\sqrt{r}\,I_{\frac{3}{2}}\left(\alpha_{i}r\right)\right]\ G_{2}\left(x\right), \tag{2.8}$$

The modified Bessel's functions $K_{\frac{3}{2}}(x)$ *and* $I_{\frac{3}{2}}(x)$ are of order $\frac{3}{2}$, while $G_2(x)=\frac{1}{2}(1-x^2)$ is a Gegenbauer polynomial of order 2 with $x=cos\theta$. The parameters "$B_1, C_1, B_2, C_2,$" are determined by applying the following B.C's on Eqn. (2.7) and Eqn. (2.8).
(i) Regularity conditions:

$$\left.\begin{aligned}\lim_{r\to\infty}\psi_{e}&=\frac{1}{2}r^{2}sin^{2}\theta\quad(\text{outer region})\\ \lim_{r\to0}\psi_{i}&=\text{finite}\quad(\text{inner region})\end{aligned}\right\} \tag{2.9}$$

(ii). Impermeability condition i.e., no mass transfer occurs at the fluid-sphere interface.

$$\psi_{e}=\psi_{i}=0\ \text{on}\quad r=1. \tag{2.10}$$

(iii). Slip condition: The tangential stress applied at a point on the surface determines the tangential velocity of the liquid relative to the solid. (Happel and Brenner [25]) i.e.,

$$\tau_{r\theta e}=\ \delta(q_{\theta}-V_{\theta i}). \tag{2.11}$$

Where $\delta$ is coefficient of sliding friction.
(iv).Shear stress is continuous on the fluid sphere's interface i.e.,

$$\tau_{r\theta e}=\ \tau_{r\theta i}. \tag{2.12}$$

Using the boundary conditions of (2.9) - (2.12) in Eqns. (2.7) and (2.8), the following system of equations are derived,
$B_1+\ C_1'=-1,$

$B_2 + \ C_2' = 0,$
$B_1\left(s + 2 - \alpha_e^2 - (2+s)\,\Delta_1(\alpha_e)\right) + B_2 s\,(2 + \Delta_2(\alpha_i)) = 2s + 4 + \alpha_e^2 + (2+s)\,\Delta_1(\alpha_e)\,,$
$B_1\left(2 - \alpha_e^2 - 2\Delta_1(\alpha_e)\right) + B_2\mu\left(4 + \alpha_i^2 + 2\Delta_2(\alpha_i)\right) = 4 + \alpha_e^2 + 2\Delta_1(\alpha_e)\,,$
where, $C_1' = C_1\ K_{\frac{3}{2}}(\alpha_e)\,,\ \ C_2' = C_2\ I_{\frac{3}{2}}(\alpha_i)\,,$ slip parameter $s = \frac{\delta a}{\mu}$, viscosity ratio $\mu = \frac{\mu_i}{\mu_e}$.
Solving the above equations analytically, resulted to

$$C_1' = -1 - B_1, \quad C_2' = -B_2,$$
$$B_1 = \frac{\left(2s + 4 + \alpha_e^2 + (2+s)\,\Delta_1(\alpha_e)\right) g_2' - \left(4 + \alpha_e^2 + 2\Delta_1(\alpha_e)\right) k_2'}{\pi'}, \tag{2.13}$$
$$B_2 = \frac{\left(4 + \alpha_e^2 + 2\Delta_1(\alpha_e)\right) k_1' - \left(2s + 4 + \alpha_e^2 + (2+s)\,\Delta_1(\alpha_e)\right) g_1'}{\pi'}.$$

Here $\pi' \ = \ k_1' g_2' - k_2' g_1'$,
where $k_1' = \left[s + 2 - \alpha_e^2 - (2+s)\,\Delta_1(\alpha_e)\right]$,
$k_2' = s\,[2 + \Delta_2(\alpha_i)]\,,$
$g_1' = \left[2 - \alpha_e^2 - 2\Delta_1(\alpha_e)\right],$
$g_2' = \ \mu\left[4 + \alpha_i^2 + 2\Delta_2(\alpha_i)\right].$
Thus, external and internal flow stream functions Eqn. (2.7) and Eqn. (2.8) are derived.

### 2.3. Drag force on a sphere

The limit form of drag force on a body which is placed in an oscillatory flow by Srinivasa charya and Iyengar [9] is

$$D_f = i\rho\omega U V_0 + 4\pi i\rho\omega \lim_{r\to\infty}\left[\frac{r\,(\psi_e^* - \psi_\infty^*)}{sin^{\ 2}\ \theta}\right], \tag{2.14}$$

where $\psi_\infty^*$ denoting the stream function correlate to the fluid motion at infinity, then the stream function $(\psi_e^* - \psi_\infty^*)$ gives a state of rest at infinity by Happel and Brenner [25]. Here $V_0$ is the volume of the body, which is $V_0 = \frac{4}{3}\pi r^3$.
Substituting Eqn. (2.7), Eqn. (2.9), $V_0$ in Eqn. (2.14) and after simplifying, we get

$$D_f = \frac{4}{3}\pi i\rho\omega e^{i\omega t_1}\left(1 + B_1 + \ C_1'\right) + 2\pi i\rho\omega e^{i\omega t_1} B_1, \tag{2.15}$$

$$D_f = 2\pi i\rho\omega e^{i\omega t_1} B_1 (\text{using Eqn. (2.13)}), \tag{2.16}$$

$$D_f = (T + iT1). \tag{2.17}$$

where $T$ (Real drag) $= -2\pi\rho\omega sin\omega t_1 B_1$, $T1$ (Imaginary drag) $= 2\pi\rho\omega cos\omega t_1 B_1$
The real drag $T$ and imaginary drag $T1$ are calculated for distinct values and slip parameter $(s)$.
As $\mu \to \infty,\ s \to \infty$ then it changes to oscillatory viscous flow on the solid sphere with no-slip condition, which matches to the drag force evaluated by (Lakshmana Rao and Bhujanga Rao [8]).
In addition, we have attained solutions of oscillatory flow beyond a viscous fluid sphere over a CSF sphere and vice versa. The oscillatory flow past a CSF sphere over a CSF sphere is studied in continuation. These three are presented in part 3, part 4 and part 5 as follows:

## 3. Oscillatory flow of viscous fluid flow beyond a couple stress fluid sphere

### 3.1. Fundamental equation:

The theory of CSFs was studied by Stokes [27] for the flows of polar fluids past bodies with impermeable surfaces. The velocity and pressure expressions for a CSF over a sphere are mentioned in Stokes [28]. It was noticed that there is a significant rise in viscosity with the effect of coupling stress.
As the flow is axisymmetric, choose the velocity vector in the form of $\bar{q} = U(R,\ \theta)\,\bar{e}_r + \ V(R,\ \theta)\,\bar{e}_\theta$.
The field equations that determine non-Newtonian fluid flow are as follows:

$$\rho\frac{d\bar{q}}{dt} = -\nabla \mathrm{P} + \ \mu(\nabla\times\nabla\times\bar{q}) - \eta(\nabla\times\nabla\times\nabla\times\nabla\times\bar{q}). \tag{3.1}$$

Here $\mu$ is the viscosity coefficient. Couple stress parameter $e = \frac{\eta'}{\eta}$, where the couple stress viscosity parameters are $\eta$ and $\eta'$.
The velocity of oscillating flow is considered as,

$$\bar{q} = \nabla \times \left(\frac{\Psi \bar{e}_\Phi}{h_3}\right) e^{i\omega t_1} = \left(\frac{1}{R^2 sin\theta}\frac{\partial \Psi}{\partial \theta}\bar{e}_r - \frac{-1}{R\sin\theta}\frac{\partial \Psi}{\partial R}\bar{e}_\theta\right) e^{i\omega t_1}. \tag{3.2}$$

$$\therefore \nabla \times \bar{q} = -\left(\frac{E_0^2 \Psi}{h_3}\right)\bar{e}_\phi e^{i\omega t_1},\ (\nabla \times \nabla \times \nabla \times \bar{q}) = -\left(\frac{E_0^4 \Psi}{h_3}\right)\bar{e}_\phi e^{i\omega t_1}. \tag{3.3}$$

Substituting Eqn. (3.2), (3.3) in (3.1) and eliminating the pressure $P$, from it, we get

$$E_0^2\left[E_0^2 - \frac{\alpha^2}{a^2}\right]\left[E_0^2 - \frac{\beta^2}{a^2}\right]\Psi = 0. \tag{3.4}$$

The following non-dimensional approach is used to determine the equation of motion.

$$R = ar; \Psi = \psi U_\infty a^2; P = \frac{pU_\infty \mu}{a}; E_0^2 = \frac{E^2}{a^2}; U = uU_\infty; V = vV_\infty.$$

The momentum equation in non-dimensional approach is

$$E^2\left[E^2 - \alpha^2\right]\left[E^2 - \beta^2\right]\psi = 0. \tag{3.5}$$

where $\alpha^2 + \beta^2 = \frac{\mu a^2}{\eta}$; $\alpha^2\beta^2 = \frac{i\rho\omega a^4}{\eta}$.

### 3.2. Solution of the problem

The solutions of Eqn. (3.5) which are regular for external flow of viscous fluid ($\psi_e$) and for internal flow of CSF ($\psi_i$) regions are given by (Aparna et al. [17])

$$\psi_e = \left[r^2 + \frac{B_1}{r} + C_1\sqrt{r}\ \ K_{\frac{3}{2}}(\alpha_e r)\right] G_2(x), \quad \text{and} \tag{3.6}$$

$$\psi_i = \left[B_2\ r^2 + C_2\sqrt{r}\, I_{\frac{3}{2}}(\alpha_i r) + D_2\sqrt{r}\, I_{\frac{3}{2}}(\ \beta_i r)\right]\ G_2(x). \tag{3.7}$$

The Parameters $B_1, C_1, B_2, C_2, D_2$ in Eqn. (3.6) - Eqn. (3.7) are evaluated by using boundary conditions (2.9) - (2.12) along with type A condition i.e., vanishing of couple stress on the boundary.

$$\frac{\partial[E^2\psi]}{\partial r} = \left(\frac{1}{r} + e\right)E^2\psi, \quad \text{where, couple stress parameter} \quad e = \frac{\eta'}{\eta}. \tag{3.8}$$

By using B.Cs (2.9) - (2.12) and adding Type A condition Eqn. (3.8) also, We obtain the following system of equations:
$B_1 + C_1' = -1,$
$B_2 + C_2' + D_2' = 0,$
$(4+s)\,B_1 + C_1'\left(\alpha_e^2 + 2 + (2+s)\,\Delta_1(\alpha_e)\right) + 2sB_2 - s\ C_2'\Delta_2(\alpha_i) - D_2's\ \Delta_2(\beta_i) = 2 + 2s,$
$6B_1 + C_1'\left(\alpha_e^2 + 4 + 2\Delta_1(\alpha_e)\right) - \mu C_2'\left(\alpha_i^2 + 4 + 2\Delta_2(\alpha_i) - \frac{{\alpha_i}^4}{\lambda^2}\right) + D_2'\left({\beta_i}^2 + 4 + 2\Delta_2(\beta_i) - \frac{{\beta_i}^4}{\lambda^2}\right) = 0,$
$C_2'{\alpha_i}^2\left(\Delta_2(\alpha_i) + (1+e)\right) + D_2'{\beta_i}^2\left(\Delta_2(\beta_i) + (1+e)\right) = 0,$
where, $C_1' = C_1\ K_{\frac{3}{2}}(\alpha_e)$, $C_2' = C_2\ K_{\frac{3}{2}}(\alpha_i)$, $D_2' = D_2\ I_{\frac{3}{2}}(\beta_i)$, slip parameter $s = \frac{\delta a}{\mu}$, viscosity ratio $\mu = \frac{\mu_i}{\mu_e}$, $e = \frac{\eta'}{\eta}$,
solving the preceding equations analytically, resulted to

$$B_1 = -1 - C_1', \quad C_1' = \frac{(3s+6)\,g_3' - 6k_3'}{\pi_1'}, \quad \xi_2' = \frac{\beta_i^2[\Delta_2(\beta_i) + (1+e)]}{\alpha_i^2[\Delta_2(\alpha_i) + (1+e)]}$$

$$B_2 = (\xi_2' - 1\ )\,D_2', \quad C_2' = \ -\xi_2' D_2', \quad \text{and} \quad D_2' = \frac{-\,(3s+6)\,g_1' + 6k_1'}{\pi_1'}.$$

Here, $\pi_1' = k_1'g_3' - k_3'g_1'$ where
$k_1' = \left[2 + s - \alpha_e^2 - (2+s)\,\Delta_1(\alpha_e)\right],$
$k_3' = s\;\left[\xi_2'\left(-2 - \Delta_2(\alpha_i)\right) + \Delta_2(\beta_i) + 2\right],$
$g_1' = \left[2 - \alpha_e^2 - 2\Delta_1(\alpha_e)\right],$
$g_3' = \;\mu\left[\xi_2'\left(-\alpha_i^2 - 4 - 2\Delta_2(\alpha_i) + \frac{{\alpha_i}^4}{\lambda^2}\right) + {\beta_i}^2 + 4 + 2\Delta_2(\beta_i) - \frac{{\beta_i}^4}{\lambda^2}\right].$
Thus, exterior and interior flow stream functions Eqn. (3.6) and Eqn. (3.7) are derived.

## 4. Oscillatory flow of CSF flow beyond a viscous fluid sphere

### 4.1. Solution of the problem:

The solutions of Eqn. (3.5) which are regular for external flow of couple stress fluid ($\psi_e$) and for internal flow of viscous fluid ($\psi_i$) regions are given by:

$$\psi_e = \left[r^2 + \frac{B_1}{r} + C_1\sqrt{r}\;\;K_{\frac{3}{2}}(\alpha_e r) + D_1\sqrt{r}\;\;K_{\frac{3}{2}}(\beta_e r)\right]G_2(x),\quad \text{and} \tag{4.1}$$

$$\psi_i = \left[B_2\;r^2 + C_2\sqrt{r}\,I_{\frac{3}{2}}(\alpha_i r)\right]\;G_2(x). \tag{4.2}$$

The parameters $B_1, C_1, D_1, B_2, C_2$ in Eqns. (4.1) - (4.2) are evaluated by using boundary conditions (2.9) - (2.12) and (3.8). Here we obtain the following system of equations as
$B_1 + C_1' + \;D_1' = -1,$
$B_2 + C_2' = 0,$
$(6+s)\,B_1 + C_1'\left(\alpha_e^2 + 4 + (2+s)\,\Delta_1(\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + 2sB_2 - sC_2'\Delta_2(\alpha_i) +$
$D_1'\left({\beta_e}^2 + 4 + (2+s)\,\Delta_1(\;\beta_e) - \frac{{\beta_e}^4}{\lambda^2}\right) = 2s,$
$6B_1 + C_1'\left(\alpha_e^2 + 4 + 2\Delta_1(\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + D_1'\left({\beta_e}^2 + 4 + 2\Delta_1(\beta_e) - \frac{{\beta_e}^4}{\lambda^2}\right) - \mu C_2'\left(\alpha_i^2 + 4 + 2\Delta_2(\alpha_i) - \frac{{\alpha_i}^4}{\lambda^2}\right) = 0,$
$C_1'\alpha_e^2\left\{\Delta_1(\alpha_e) + (1+e)\right\} + D_1'\beta_e^2\left\{\Delta_1(\beta_e) + (1+e)\right\} = 0,$
here, $D_1' = D_1\;K_{\frac{3}{2}}(\beta_e),$ Solving the above equations analytically, results to,

$$B_1 = -1 + (\xi_1' - 1\;)\,D_1',\quad C_1' = \;-\xi_1'D_1',\quad \xi_1' = \frac{\beta_e^2[\Delta_1(\beta_e) + (1+e)]}{\alpha_e^2[\Delta_1(\alpha_e) + (1+e)]},$$

$$D_1' = \;\frac{-\,(3s+6)\,g_2' + 6k_2'}{\pi_2'},\quad B_2 = -C_2',\quad \text{and}\quad C_2' = \;\frac{(3s+6)\,g_4' - 6k_4'}{\pi_2'}.$$

Here, $\pi_2' = k_4'g_2' - k_2'g_4'$ where
$k_2' = s\;\left[2 + \Delta_2(\alpha_i)\right],$
$g_4' = \left[\xi_1'\left(\alpha_e^2 - 2 + 2\Delta_1(\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + 2 - {\beta_e}^2 - 2\Delta_1(\beta_e) + \frac{{\beta_e}^4}{\lambda^2}\right],$
$g_2' = \;\mu\left[\alpha_i^2 + 4 + 2\Delta_2(\alpha_i)\right],$
$k_4' = \left[\xi_1'\left(-2 - s + \alpha_e^2 + (2+s)\,\Delta_1(\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + s + 2 - {\beta_e}^2 - (2+s)\,\Delta_1(\beta_e) + \frac{{\beta_e}^4}{\lambda^2}\right].$
Thus, exterior and interior flow stream functions Eqn. (4.1) and Eqn. (4.2) are derived.

## 5. Oscillatory flow of CSF flow beyond a CSF sphere

### 5.1. Formulating the problem:

Consider an oscillatory flow of non-Newtonian fluid over a CSF. The non-Newtonian fluid is a CSF here. The flow is axisymmetric and incompressible. The geometry of the problem is shown in Figure 2.

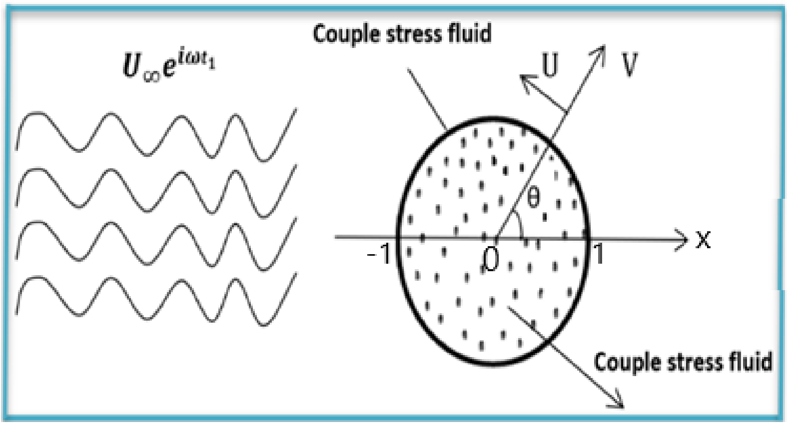


Figure 2: Geometry of oscillatory flow of CSF over a CSF sphere

### 5.2. Solution of the problem:

The solution of Eqn. (3.5) which are systematic for external flow of couple stress fluid ($\psi_e$) and for internal flow couple stress fluid ($\psi_i$) regions by super position process are given by (Aparna et al. [17]):

$$\psi_e = \left[r^2 + \frac{B_1}{r} + C_1\sqrt{r}\ \ K_{\frac{3}{2}}(\alpha_e r) + D_1\sqrt{r}\ \ K_{\frac{3}{2}}(\beta_e r)\right] G_2(x), \text{ and} \tag{5.1}$$

$$\psi_i = \left[B_2\ r^2 + C_2\sqrt{r}\, I_{\frac{3}{2}}(\alpha_i r) + D_2\sqrt{r}\, I_{\frac{3}{2}}(\beta_i r)\right]\ G_2(x). \tag{5.2}$$

The Parameters $B_1, C_1, D_1, B_2, C_2, D_2$ in Eqn. (5.1) - Eqn. (5.2) are evaluated by using boundary conditions (2.9) - (2.12) and (3.8). Here we get the following system of equations as,
$B_1 + C_1' +\ D_1' = -1,$
$B_2 + C_2' +\ D_2' = 0,$
$(6+s)\, B_1 + C_1' \left(\alpha_e^2 + 4 + (2+s)\, \Delta_1\, (\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) 2sB_2 - s\ C_2'\Delta_2\, (\alpha_i) - D_2's\ \Delta_2\, (\beta_i)$
$+\, D_1' \left({\beta_e}^2 + 4 + (2+s)\, \Delta_1\, (\ \beta_e) - \frac{{\beta_e}^4}{\lambda^2}\right) = 2s,$
$6B_1 + C_1' \left(\alpha_e^2 + 4 + 2\Delta_1\, (\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + D_1' \left({\beta_e}^2 + 4 + 2\Delta_1\, (\beta_e) - \frac{{\beta_e}^4}{\lambda^2}\right) - \mu C_2' \left(\alpha_i^2 + 4 + 2\Delta_2\, (\alpha_i) - \frac{{\alpha_i}^4}{\lambda^2}\right) +$
$D_2' \left({\beta_i}^2 + 4 + 2\Delta_2\, (\beta_i) - \frac{{\beta_i}^4}{\lambda^2}\right) = 0,$
$C_1'{\alpha_e}^2\, (\Delta_1(\alpha_e) + (1+e)) + D_1'{\beta_e}^2\, (\Delta_1(\beta_e) + (1+e)) = 0,$
$C_2'{\alpha_i}^2\, (\Delta_2(\alpha_i) + (1+e)) + D_2'{\beta_i}^2\, (\Delta_2(\beta_i) + (1+e)) = 0.$
Solving the above equations analytically, resulted,

$$\begin{aligned} B_1 = -1 + (\xi_1' - 1)\, D_1', \quad C_1' = -\xi_1' D_1' \quad \text{and} \quad D_1' &= \frac{-\,(3s+6)\, g_3' + 6k_3'}{\pi_3'}, \\ B_2 = (\xi_2' - 1\ )\, D_2', \ \ C_2' = -\xi_2' D_2' \quad \text{and} \quad D_2' &= \frac{(3s+6)\, g_4' - 6k_4'}{\pi_3'}. \end{aligned} \tag{5.3}$$

Here, $\pi_3' \ = \ k_4' g_3' - k_3' g_4'$, where
$k_4' = \left[\xi_1' \left(-2 - s + \alpha_e^2 + (2+s)\Delta_1(\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + s + 2 - {\beta_e}^2 - (2+s)\Delta_1(\beta_e) + \frac{{\beta_e}^4}{\lambda^2}\right],$
$k_3' = s\ \ [\xi_2'\, (-2 - \Delta_2\, (\alpha_i)) + \Delta_2\, (\beta_i) + 2]\, ,$
$g_4' = \left[\xi_1' \left(-2 + \alpha_e^2 + 2\Delta_1\, (\alpha_e) - \frac{{\alpha_e}^4}{\lambda^2}\right) + 2 - {\beta_e}^2 - 2\Delta_1\, (\beta_e) + \frac{{\beta_e}^4}{\lambda^2}\right],$
$g_3' = \ \mu \left[\xi_2' \left(-\alpha_i^2 - 4 - 2\Delta_2\, (\alpha_i) + \frac{{\alpha_i}^4}{\lambda^2}\right) + {\beta_i}^2 + 4 + 2\Delta_2\, (\beta_i) - \frac{{\beta_i}^4}{\lambda^2}\right].$
Thus, exterior and interior flow stream functions Eqn. (5.1) and Eqn. (5.2) are derived.

### 5.3. Drag force on a sphere:

The limit form of the drag force on a body which is placed in an oscillatory flow given by Srinivasa charya and Iyengar [9] is

$$D_f = i\rho\omega UV_0 + 4\pi i\rho\omega \lim_{r\to\infty}\left[\frac{r\left(\psi_e^* - \psi_\infty^*\right)}{sin^{\,2}\,\theta}\right], \tag{5.4}$$

where $\psi_\infty^*$ denotes the stream function correlates with the fluid velocity at infinity, $v_0$ is the volume of spherical body, the stream function $(\psi_e^* - \psi_\infty^*)$ gives a state of rest at infinity by Happel and Brenner [25].

Substituting Eqn. (2.9), (5.1), and $V_0 = \frac{4}{3}\pi r^3$ and simplifying, we get

$$D_f = \frac{4}{3}\pi i\rho\omega e^{i\omega t_1}\left(1 + B_1 + C_1' + D_1'\right) + 2\pi i\rho\omega e^{i\omega t_1}B_1, \tag{5.5}$$

$$D_f = 2\pi i\rho\omega e^{i\omega t_1}B_1 \quad (\text{with } B_1 + C_1' + D_1' = -1), \tag{5.6}$$

$$D_f = (T + iT1) \ \ (\text{using Eqn. } (5.3)), \tag{5.7}$$

where $T$ (Real drag) = -$2\pi\rho\omega B_1 sin\omega t_1$, $T_1$ (Imaginary drag) = $2\pi\rho\omega B_1 cos\omega t_1$.

The real drag $T$ and imaginary drag $T1$ are calculated for distinct values and slip parameter ($s$).

As $\alpha_e \to \infty, \mu \to \infty,\ s \to \infty$, then it changes to oscillatory viscous flow on the solid sphere by using no-slip condition, which matches to the drag force evaluated by (Lakshmana Rao and Bhujanga Rao [8]).

## 6. Results and discussion

The outer and inner stream functions of Eqn. (5.1) and Eqn. (5.2) are calculated using the B. Cs from Eqn. (2.9) to Eqn. (2.12) and Eqn. (3.8). The drag force is determined and is given in Eqn. (5.6). Real and imaginary drag values are analyzed, and their variations related to varied slip parameters and couple stress parameter values are originated in the following graphs.

i). Real drag ($T$) vs slip parameter ($s$) for dissimilar couple stress parameter ($e$) are plotted in Fig. 3. It was identified that with a increase in slip parameter ($s$) values there is reduce in real drag ($T$) values. Also identified that with a rise in couple stress parameter ($e$) values there is reduce in real drag ($T$) values. The numerical outcomes are presented in Table 1.

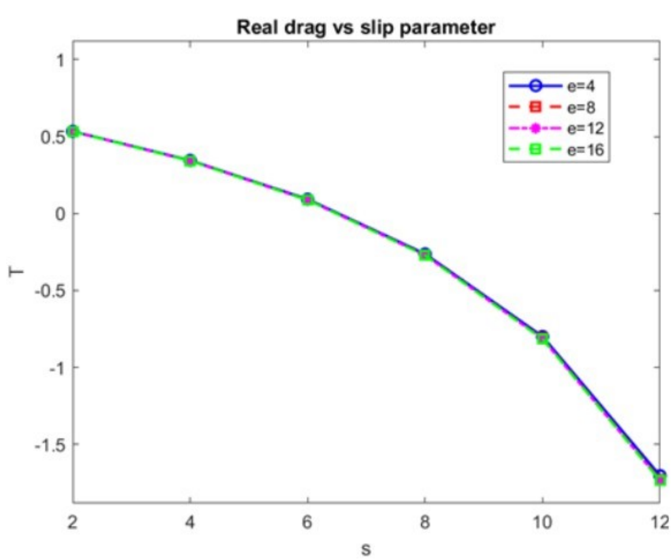


Figure 3: Real drag ($T$) vs slip parameter ($s$) for dissimilar couple stress parameter ($e$) assuming fixed constant frequencies for the parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

Table 1: Real drag ($T$) vs slip parameter ($s$) for dissimilar couple stress parameter ($e$) assuming fixed constant frequencies for the parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

| $T \diagdown s$ | $e = 4$ | $e = 8$ | $e = 12$ | $e = 16$ |
|---|---|---|---|---|
| 2 | 0.5334 | 0.5321 | 0.5318 | 0.5317 |
| 4 | 0.3446 | 0.3423 | 0.3418 | 0.3415 |
| 6 | 0.0921 | 0.0881 | 0.0872 | 0.0866 |
| 8 | -0.2631 | -0.2699 | -0.2715 | -0.2724 |
| 10 | -0.7996 | -0.8116 | -0.8156 | - -0.8161 |
| 12 | -1.7038 | -1.7273 | -1.7351 | -1.7362 |

ii). Imaginary drag ($T1$) vs slip parameter ($s$) for dissimilar couple stress parameter ($e$) are plotted in Fig. 4. It was observed that with increase in slip parameter ($s$) values, there is a rise in imaginary drag ($T1$) and there is a very small variation in the values of imaginary drag ($T1$) with a variation in values of couple stress parameter ($e$). The numerical results are obtained in Table 2.

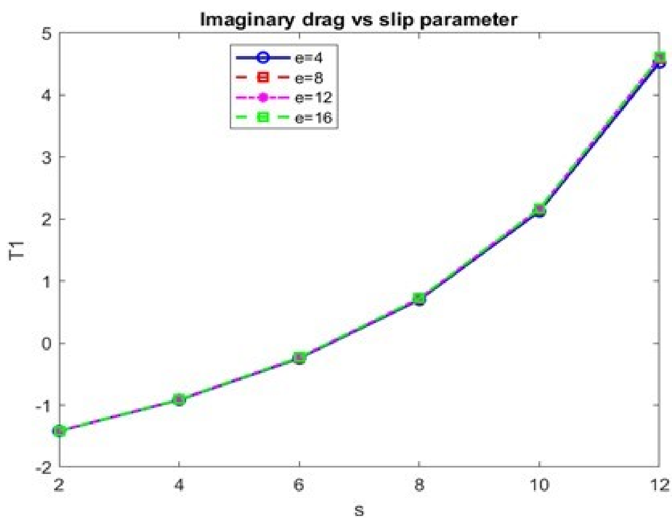


Figure 4: Imaginary drag ($T1$) veruss slip parameter ($s$) for different couple stress parameter ($e$) assuming fixed constant frequencies for the parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

Table 2: Imaginary drag ($T1$) veruss slip parameter ($s$) for different couple stress parameter ($e$) assuming fixed constant frequencies for the parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

| $T1 \diagdown s$ | $e = 4$ | $e = 8$ | $e = 12$ | $e = 16$ |
|---|---|---|---|---|
| 2 | -1.4170 | -1.4137 | -1.4129 | -1.4126 |
| 4 | -0.9155 | -0.9094 | -0.9080 | -0.9074 |
| 6 | -0.2446 | -0.2341 | -0.2316 | -0.2307 |
| 8 | 0.6991 | 0.7170 | 0.7212 | 0.7229 |
| 10 | 2.1244 | 2.1562 | 2.1638 | 2.1667 |
| 12 | 4.5265 | 4.5890 | 4.6040 | 4.6098 |

iii). Real drag ($T$) versus couple stress parameter ($e$) for dissimilar slip parameter ($s$) values are plotted in Fig 5. It was observed that at lesser values of couple stress parameter ($e$) i.e., less than 4 there is fall in real drag ($T$) values after that there is a slight increase in real drag value up to 6, thevalues after are stable. The impact of the couple stress parameter ($e$) is in the range of $2 < e < 6$. Also identified that with a rise in slip parameter $(s)$ there is fall in real drag $(T)$ values.

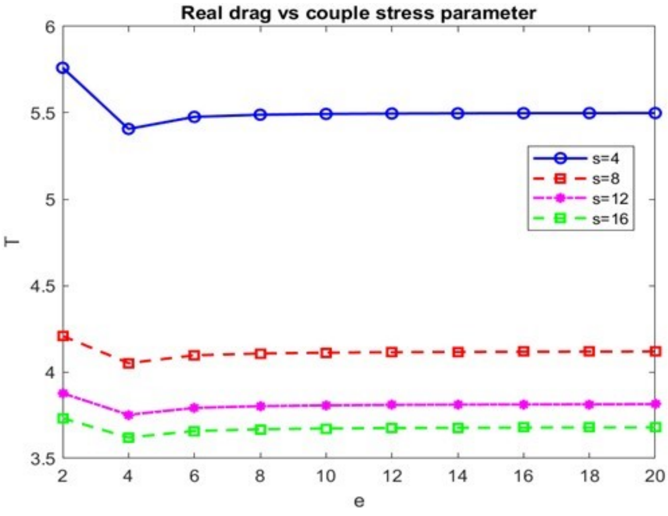


Figure 5: Real drag $(T)$ versus couple stress parameter $(e)$ for different slip parameter $(s)$ at fixed values to frequency parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

Table 3: Real drag $(T)$ versus couple stress parameter $(e)$ for different slip parameter $(s)$ at fixed values to frequency parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

| $T \diagdown e$ | $s = 4$ | $s = 8$ | $s = 12$ | $s = 16$ |
|---|---|---|---|---|
| 2 | 5.7606 | 4.2082 | 3.8766 | 3.6513 |
| 4 | 5.4067 | 4.0512 | 3.7520 | 3.5467 |
| 6 | 5.4762 | 4.0960 | 3.7919 | 3.5833 |
| 8 | 5.4888 | 4.1068 | 3.8020 | 3.5930 |
| 10 | 5.4934 | 4.1116 | 3.8067 | 3.5975 |
| 12 | 5.4956 | 4.1142 | 3.8093 | 3.6001 |
| 14 | 5.4968 | 4.1158 | 3.8109 | 3.6017 |
| 16 | 5.4976 | 4.1169 | 3.8120 | 3.6028 |
| 18 | 5.4981 | 4.1177 | 3.8128 | 3.6035 |
| 20 | 5.4984 | 4.1182 | 3.8133 | 3.6041 |

iv). Imaginary drag $(T1)$ versus couple stress parameter $(e)$ for dissimilar slip parameter $(s)$ values are plotted in Fig. 6. It was observed that at lower values of couple stress parameter $(e)$ i.e., less than 4 there is an increase in imaginary drag $(T1)$ values. After that there is a slight decrease in imaginary drag values and there after the values are stable. Also observed that with a rise in slip parameter $(s)$ there is rise in imaginary drag $(T1)$ values.

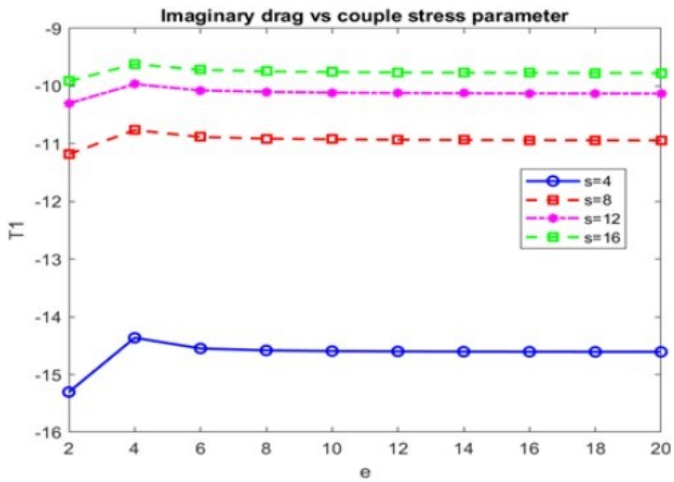


Figure 6: Imaginary drag ($T1$) versus couple stress parameter ($e$) for slip parameter ($s$) at fixed values to frequency parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

Table 4: Imaginary drag ($T1$) versus couple stress parameter ($e$) for slip parameter ($s$) at fixed values to frequency parameters $\mu = 10$, $\omega = 0.6$, $\rho = 0.6$, $\lambda = 0.5$, $t = 0.6$.

| $T1 \diagdown e$ | $s = 4$ | $s = 8$ | $s = 12$ | $s = 16$ |
|---|---|---|---|---|
| 2 | -15.3043 | -11.1800 | -10.2990 | -9.7005 |
| 4 | -14.3640 | -10.7629 | -9.9680 | -9.4226 |
| 6 | -14.5488 | -10.8820 | -10.0739 | -9.5198 |
| 8 | -14.5822 | -10.9107 | -10.1010 | -9.5456 |
| 10 | -14.5945 | -10.9234 | -10.1133 | -9.5576 |
| 12 | -14.6003 | -10.9303 | -10.1202 | -9.5644 |
| 14 | -14.6036 | -10.9346 | -10.1245 | -9.5687 |
| 16 | -14.6056 | -10.9375 | -10.1274 | -9.5715 |
| 18 | -14.6069 | -10.9395 | -10.1295 | -9.5736 |
| 20 | -14.6079 | -10.9410 | -10.1309 | -9.5751 |

## 7. Conclusions

This study aims at obtaining an analytical solution for oscillatory flow over a fluid sphere by using slip condition on its surface. The flow is presumed to be non-compressible and axisymmetric. The internal and external stream functions for oscillatory flow of viscous fluid, flowing beyond a Newtonian fluid sphere, oscillatory flow of Newtonian fluid flow past a CSF sphere, oscillatory flow of CSF past a viscous fluid sphere and oscillatory flow of CSF flow over a CSF sphere with slip condition were obtained. The drag forces are computed, and observations were documented as in comparison with data in the literature. From the obtained results, the real drag ($T$) values a decrease along with increase in slip parameter ($s$) values at a fixed couple stress parameter ($e$) value. Also, it was noticed that the values of an imaginary drag force ($T1$) increase within the rise in slip parameter values at a fixed couple stress parameter ($e$) value.

There is a scope for further extension of this work for other non-Newtonian fluids which arise in chemical engineering problems. The exact solutions obtained above are useful for researchers who extend the work for intermediate Reynolds numbers which are to be solved by using semi analytic methods or numerical methods and can compare with our solutions for special cases.

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