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Analytic Mean Labeling in Certain Classes of Graphs

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ABSTRACT: A graph labeling is a mapping, when the map assigns integers to vertices/edges, it is called vertex/edge labeling and when it assigns to both, it is called total labeling. The idea of mean labeling was proposed by Somasundaram and Ponraj. An injective function $f : V \rightarrow \{t : 0 \leq t \leq q\}$ is called a mean labeling of a graph $G(V, E)$ if for every $u, v \in V$, the edge labels are $\frac{f(u)+f(v)}{2}$ whenever $f(u) + f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ whenever $f(u) + f(v)$ is odd such that edge labels are distinct. The idea of analytic mean labeling was coined by Tharmaraj and Sarasija and they have proved the existence in the following graphs: path,cycle, star, Ladder, bistar, fan, comb and the joint sum of two copies of certain graph [7]. The concept of Analytic even mean labeling is introduced by Sajitha Kumari et., al, and they proved Jewel graph, Jelly Fish graph, Triangular Book graph, Triangular Book with Book Mark admits analytic even mean labeling [4]. Motivated by these studies, here we examine the existence of Analytic Mean labeling and Analytic Even Mean Labeling in the duplicate graphs of ladder, triangular ladder, slant ladder and Zig-Zag ladder graphs.

Key Words: Graph labeling, duplicate graph, path, ladder, slant ladder, triangular ladder, zig zag ladder, analytic mean labeling and analytic even mean labeling .

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1. Introduction

In 1967, Rosa introduced graph labeling [3]. A graph labeling is a mapping: when the map assigns integers to vertices/edges, it is called vertex/edge labeling, and when it assigns to both, it is called total labeling.

Sampath Kumar E. in 1973 introduced duplicate graphs and proved certain characteristics of the same. The duplicate graph of a simple graph $G(V, E)$ with p vertices and q edges is denoted by G'. The graph G' contains $2p$ vertices, and for each $xy \in G$, G' will have two edges xy' and yx' [5].

The mean labeling was introduced by Somasundaram and Ponraj, who proved its existence in certain graphs [6]. Vijayakumar P. et al. proved the existence of mean, odd mean, and even mean labeling in the duplicate graphs of ladder, star, bistar, and double star graphs [13].

The concept of analytic mean labeling was introduced by Tharmaraj and Sarasija, who proved its existence in certain graphs [7]. The concept of analytic even mean labeling was introduced by Sajitha Kumari et al., and they proved the same in some standard graphs [4].

Motivated by these studies, here we prove the existence of analytic mean labeling and analytic even mean labeling in the duplicate graphs of ladder, triangular ladder, slant ladder, and zigzag ladder graphs.

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2. Preliminaries

Definition 2.1 *Mean labeling of a graph $G(V, E)$ is a 1-1 function $f : V \to \{t : 0 \leq t \leq q\}$, such that distinct edge labels from $\{1, 2, ..., q\}$ are induced by the injective function f^* given by $f^*(uv) = \frac{f(u)+f(v)}{2}$. A graph G admitting mean labeling is called a mean graph [6].*

Definition 2.2 *Even mean labeling of a graph $G(V, E)$ is a 1-1 function $f : V \to \{2t : 1 \leq t \leq q\}$ such that distinct edge labels are induced by $f^*(uv) = \frac{f(u)+f(v)}{2}$. A graph admitting even mean labeling is called an even mean graph [13].*

Definition 2.3 *Analytic mean labeling of a graph $G(V, E)$ is a 1-1 map $f : V \to \{t : 0 \leq t \leq p-1\}$ such that distinct edge labels are induced by the function f^* defined by*

$$f^*(uv) = \begin{cases} \frac{[f(v)]^2-[f(u)]^2}{2} & \textit{when } [f(v)]^2 - [f(u)]^2 \textit{ is even,} \\ \frac{[f(v)]^2-[f(u)]^2+1}{2} & \textit{when } [f(v)]^2 - [f(u)]^2 \textit{ is odd.} \end{cases} \tag{2.1}$$

. A graph admitting analytic mean labeling is called analytic mean graph [7].

Definition 2.4 *An analytic even mean labeling of a graph $G(V, E)$ is a 1-1 map $f : V \to \{2t : 0 \leq t \leq q\}$ such that, when $f(u) < f(v)$ distinct edge labels are induced by f^* defined by*

$$f^*(uv) = \begin{cases} \lceil \frac{[f(v)]^2-[f(u)+1]^2}{2} \rceil & when f(u) \neq 0 \\ \lceil \frac{[f(v)]^2}{2} \rceil & when f(u) = 0 \end{cases} \tag{2.2}$$

and are even. A graph admitting an analytic even mean labeling is called analytic even mean graph [4].

We refer [9] for construction of the duplicate graph of ladder graph and [11] for construction of duplicate graph of triangular ladder graph.

2.1. Construction of duplicate graphs of slant ladder graph ($\mathcal{SL}_m$)

.

Let v_t, v'_t $(1 \leq t \leq 2m + 2)$ be the vertices and e_t, e'_t $(1 \leq t \leq 3m)$ be the edges of $\mathcal{SL}_m$.
The edge labels are given as below.
For $1 \leq k \leq m$:
$e_{3k-2} \leftarrow v_{2k-1}v'_{2k+1}$, $e_{3k-1} \leftarrow v_{2k-1}v'_{2k+2}$, $e_{3k} \leftarrow v_{2k}v'_{2k+2}$;
$e'_{3k-2} \leftarrow v_{2k+1}v'_{2k-1}$, $e'_{3k-1} \leftarrow v_{2k+2}v'_{2k-1}$, $e'_{3k} \leftarrow v_{2k+2}v'_{2k}$.

2.2. Construction of duplicate graphs of ZigZag ladder graph ($\mathcal{ZZL}_m$)

Let v_t, v'_t for $1 \leq t \leq 2m$ be the vertices, and e_t, e'_t for $1 \leq t \leq 4m - 3$ be the edges of $\mathcal{ZZL}_m$.
Now, for $1 \leq k \leq 2m - 1$: $e_{2k-1} \leftarrow v_k v'_{k+2}$, $e'_{2k} \leftarrow v_{k+2}v'_k$;
and for $1 \leq k \leq 2m - 2$: $e_{2k} \leftarrow v_k v'_{k+1}$, $e'_{2k-1} \leftarrow v_{k+1}v'_k$.

2.3. Notations

- $\mathcal{L}_m$ – Duplicate graph of ladder graph
- $\mathcal{SL}_m$ – Duplicate graph of slant ladder graph
- $\mathcal{ZZL}_m$ – Duplicate graph of zigzag ladder graph
- $\mathcal{TL}_m$ – Duplicate graph of triangular ladder graph

3. Main Results

Theorem 3.1 *$\mathcal{L}_m$, $m \geq 2$ is analytic mean graph.*

Proof: Let v_t, v'_t $(1 \leq t \leq 2m)$ be the vertices and e_t, e'_t $(1 \leq t \leq 3m-2)$ be the edges of $\mathcal{L}_m$.
The $4m$ vertices of $\mathcal{L}_m$ are labeled as follows. Fix $v'_1 \leftarrow 4m-1$.
When $1 \leq k \leq m$:

$$v_{2k-1} \leftarrow 2k-2, \quad v_{2k} \leftarrow 2k+2m-2, \quad v'_{2k} \leftarrow 2k-1.$$

When $1 \leq k \leq m-1$:

$$v'_{2k+1} \leftarrow 2k+2m-1.$$

By the induced function f^* defined in (2.1), the $6m-4$ edges are labeled as below:
- When $1 \leq k \leq m$, the m edges

$$e_{3k-2}(= v_{2k-1}v'_{2k})$$

receive labels $2k-1$.
- When $1 \leq k \leq m-1$, the $m-1$ edges

$$e_{3k-1}(= v_{2k-1}v'_{2k+1})$$

receive labels $2m^2-2m-1+(4m+2)k$,
the $m-1$ edges

$$e_{3k}(= v_{2k}v'_{2k+2})$$

receive labels $2m^2-4m+2+(4m-6)k$,
the $m-1$ edges

$$e'_{3k}(= v_{2k+2}v'_{2k})$$

receive labels $2m^2+(4m+2)k$,
the $m-1$ edges

$$e'_{3k+1}(= v_{2k+2}v'_{2k+1})$$

receive labels $2m+2k$.
- When $1 \leq k \leq m-2$, the $m-2$ edges

$$e'_{3k+2}(= v_{2k+3}v'_{2k+1})$$

receive labels $2m^2-2m-1+(4m-6)k$.
Finally, the two edges e'_1, e'_2 receive labels $6m^2-4m+1$ and $8m^2-4m-1$.
Thus, all edges are distinctly labeled.
Hence $\mathcal{L}_m$, $m \geq 2$ is analytic mean graph.
An example is given in Figure.1 □

Theorem 3.2 *theorem $\mathcal{SL}_m$, $m \geq 2$ is analytic mean graph.*

Proof: Let v_t, v'_t $(1 \leq t \leq 2m+2)$ be the vertices and e_t, e'_t $(1 \leq t \leq 3m)$ be the edges of $\mathcal{SL}_m$.
The $4m+4$ vertices of $\mathcal{SL}_m$ are labeled as below:
Fix $v'_1 \leftarrow 2m+1$, $\quad v'_2 \leftarrow 4m+3$.
When $1 \leq k \leq m+1$:

$$v_{2k-1} \leftarrow 2k-2, \quad v_{2k} \leftarrow 2m+2k.$$

When $1 \leq k \leq m$:

$$v'_{2k} \leftarrow 2m+2k+1, \quad v'_{2k+1} \leftarrow 2k-1.$$

By the induced function defined in (2.1), all $6m$ edges are labeled as below:

- For $1 \le k \le m$:

$$\begin{aligned} e_{3k-2}\,(= v_{2k-1}v'_{2k+1}) &\text{ receives label } 2k-1,\\ e_{3k-1}\,(= v_{2k-1}v'_{2k+2}) &\text{ receives label } 2m^2+2m-1+(4m+6)k,\\ e_{3k}\,(= v_{2k}v'_{2k+2}) &\text{ receives label } 2m+2k+1, \end{aligned}$$

For $1 \le k \le m-1$:

$$\begin{aligned} e'_{3k+1}\,(= v_{2k+3}v'_{2k+1}) &\text{ receives label } 2+6k,\\ e'_{3k+2}\,(= v_{2k+4}v'_{2k+1}) &\text{ receives label } 2m^2+8m+8+(4m+10)k,\\ e'_{3k+3}\,(= v_{2k+4}v'_{2k+2}) &\text{ receives label } 6m+6k+8. \end{aligned}$$

- The remaining 3 edges e'_1, e'_2, e'_3 are labeled respectively with

$$2m^2+2m-1, \quad 6m+8, \quad 6m^2+4m-3.$$

Thus, all edges are labeled distinctly.
Hence $\mathcal{SL}_m$, $m \ge 2$ is analytic mean graph.

Example 3.1 *We illustrate the analytic mean labeling of $\mathcal{L}_4$ and $\mathcal{SL}_4$ in the following figure.*

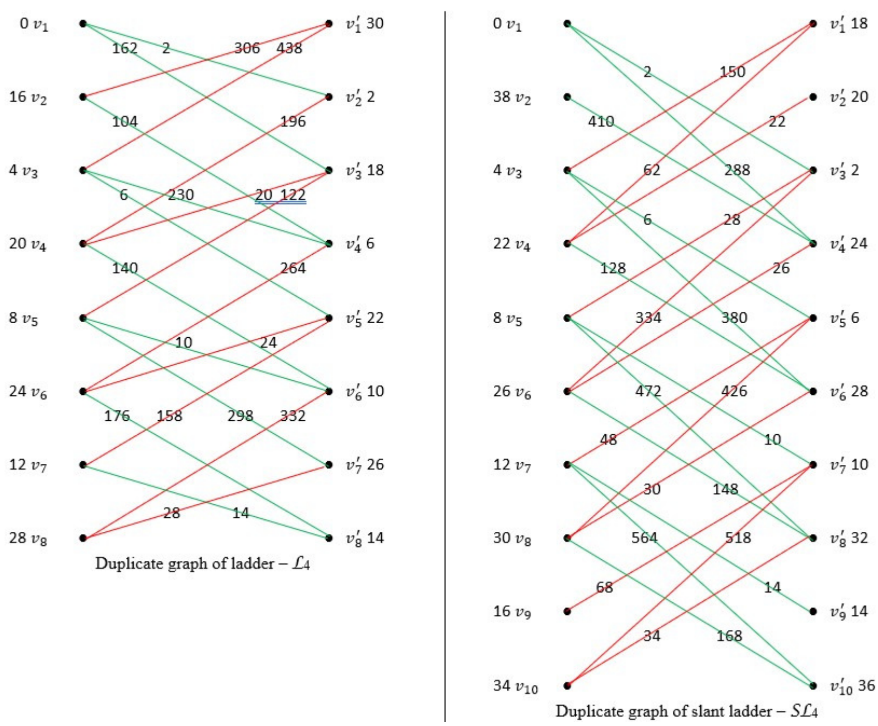

Figure 1: AML in $\mathcal{L}_4$ and $\mathcal{SL}_4$

□

Theorem 3.3 *$\mathcal{ZZL}_m$, $m \geq 2$ is analytic mean graph.*

Proof: Let v_t, v'_t $(1 \leq t \leq 2m)$ be the vertices and e_t, e'_t $(1 \leq t \leq 4m-3)$ be the edges of $\mathcal{ZZL}_m$.
The $4m$ vertices of $\mathcal{ZZL}_m$ are labeled as follows:
Fix $v'_1 \leftarrow 4m-1$.
When $1 \leq k \leq m$:

$$v_{2k-1} \leftarrow 2k-2, \quad v_{2k} \leftarrow 2m+2k-2, \quad v'_{2k} \leftarrow 2k-1.$$

When $1 \leq k \leq m-1$:

$$v'_{2k+1} \leftarrow 2m+2k-1.$$

By the induced function defined in (2.1), the $8m-6$ edges are labeled as below:
- For $1 \leq k \leq m$:

$$e_{4k-3}(= v_{2k-1}v'_{2k}) \text{ receives label } 2k-1.$$

- For $1 \leq k \leq m-1$:

$$\begin{aligned}
e_{4k-1}(= v_{2k}v'_{2k+1}) &\text{ receives label } 2m+2k-1,\\
e_{4k-2}(= v_{2k-1}v'_{2k+1}) &\text{ receives label } 2m^2-2m-1+(4m+2)k,\\
e_{4k}(= v_{2k}v'_{2k+2}) &\text{ receives label } 2m^2-4m+2+(4m-6)k,\\
e'_{4k+1}(= v_{2k+2}v'_{2k+1}) &\text{ receives label } 2m+2k,\\
e'_{4k-1}(= v_{2k+1}v'_{2k}) &\text{ receives label } 2k,\\
e'_{4k}(= v_{2k+2}v'_{2k}) &\text{ receives label } 2m^2+(4m+2)k.
\end{aligned}$$

- For $1 \leq k \leq m-2$:

$$e'_{4k+2}(= v_{2k+3}v'_{2k+1}) \text{ receives label } 2m^2-2m-1+(4m-6)k.$$

Hence, the two edges e'_1, e'_2 receive the labels $6m^2-4m+1$ and $8m^2-4m-1$, respectively. So all edges are labeled distinctly. Hence, $\mathcal{ZZL}_m$, $m \geq 2$, is an analytic mean graph.
An example is given in Figure **??**. □

Theorem 3.4 *$\mathcal{TL}_m$, $m \geq 2$, is an analytic mean graph.*

Proof: Let v_t, v'_t $(1 \leq t \leq 2m)$ be the vertices and e_t, e'_t $(1 \leq t \leq 4m-3)$ be the edges of $\mathcal{TL}_m$.
The $4m$ vertices of $\mathcal{TL}_m$ are labeled as follows:
Fix $v'_2 \leftarrow 4m-1$.
When $1 \leq k \leq m$:

$$v_{2k-1} \leftarrow 2m+2k-2, \quad v_{2k} \leftarrow 2k-2, \quad v'_{2k-1} \leftarrow 2k-1.$$

When $1 \leq k \leq m-1$:

$$v'_{2k+2} \leftarrow 2m+2k-1.$$

By the induced function f^* defined in (2.1), the $8m-6$ edges are labeled as below:
- For $1 \leq k \leq m$:

$$e'_{4k-3}(= v_{2k}v'_{2k-1}) \text{ receives label } 2k-1.$$

- For $1 \leq k \leq m-1$:

$$\begin{aligned}
e_{4k+1}(= v_{2k+1}v'_{2k+2}) &\text{ receives label } 4m+4k,\\
e_{4k-2}(= v_{2k-1}v'_{2k+1}) &\text{ receives label } 2m^2-2m+2+(4m-4)k,\\
e_{4k-1}(= v_{2k-1}v'_{2k+2}) &\text{ receives label } 2m+2k-1,\\
e_{4k}(= v_{2k}v'_{2k+2}) &\text{ receives label } 2m^2-2m-1+(4m+2)k,\\
e'_{4k-2}(= v_{2k+1}v'_{2k-1}) &\text{ receives label } 4m^2-1+(8m+4)k,\\
e'_{4k-1}(= v_{2k+2}v'_{2k-1}) &\text{ receives label } 2k.
\end{aligned}$$

When $1 \le k \le m-2$, the $m-2$ edges

$$e'_{4k+4}(=v_{2k+4}v'_{2k+2})$$

receive labels

$$2m^2 - 2m - 1 + (4m-6)k.$$

The two edges e_1 and e'_4 receive labels

$$6m^2 - 4m + 1 \quad \text{and} \quad 8m^2 - 4m - 1,$$

respectively.

Thus, all edges are labeled distinctly. Hence, $\mathcal{TL}_m$, $m \ge 2$, is an analytic mean graph.

Example 3.2 *We illustrate the analytic mean labeling of $\mathcal{TL}_2$ and $\mathcal{ZZL}_3$ in the following figure.*

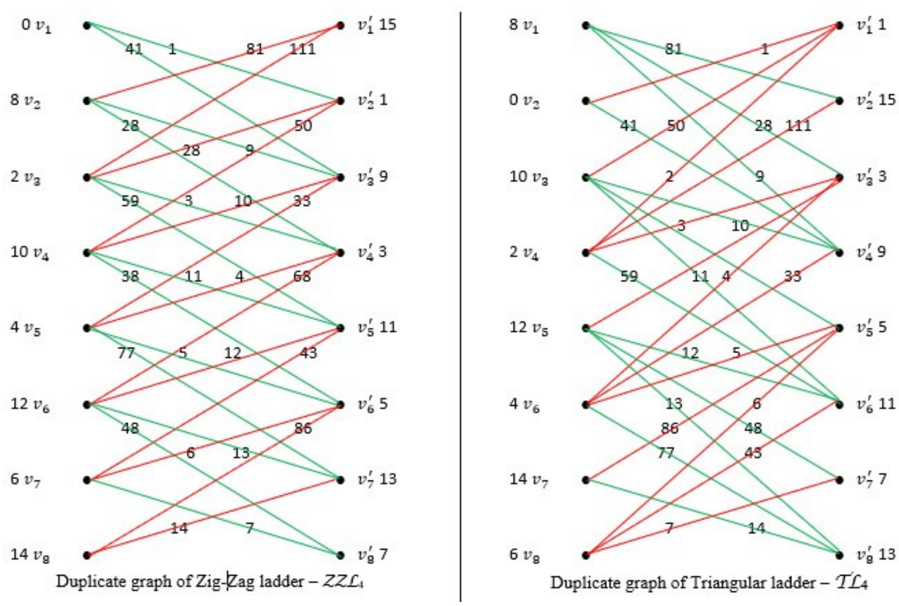

Figure 2: AML in $\mathcal{TL}_2$ and $\mathcal{ZZL}_3$

□

Theorem 3.5 *$(\mathcal{L}_m)$, for $m \ge 2$ is analytic even mean graph.*

Proof: Let v_t, v'_t $(1 \le t \le 2m)$ be the vertices and e_t, e'_t $(1 \le t \le 3m-2)$ be the edges of $\mathcal{L}_m$.

The $4m$ vertices of $DG(\mathcal{L}_m)$ are labeled as follows:

Fix $v'_1 \leftarrow 8m-2$.

When $1 \le k \le m$:

$$v_{2k-1} \leftarrow 4k-4, \quad v_{2k} \leftarrow 4m+4k-4, \quad v'_{2k} \leftarrow 4k-2.$$

When $1 \le k \le m-1$:

$$v'_{2k+1} \leftarrow 4m+4k-2.$$

The induced function f^* with $f(u) < f(v)$ defined in (2.2) allots labels to the $6m-4$ edges as follows:
- For $1 \le k \le m$:

$$e_{3k-2}(=v_{2k-1}v'_{2k}) \text{ receives label } 4k-2.$$

- For $1 \le k \le m-1$:

$$\begin{aligned} e_{3k-1}(=v_{2k-1}v'_{2k+1}) &\text{ receives label } 8m^2-8m-2,\\ e_{3k}(=v_{2k}v'_{2k+2}) &\text{ receives label } 8m^2-16m+4+(16m-28)k,\\ e'_{3k}(=v_{2k+2}v'_{2k}) &\text{ receives label } 8m^2+(16m+4)k,\\ e'_{3k+1}(=v_{2k+2}v'_{2k+1}) &\text{ receives label } 4m+4k. \end{aligned}$$

- For $1 \le k \le m-2$:

$$e'_{3k+2}(=v_{2k+3}v'_{2k+1}) \text{ receives label } 8m^2-8m-10+(16m-28)k.$$

The two edges

$$e'_1(=v_2v'_1), \quad e'_2(=v_3v'_1)$$

receive labels

$$24m^2-20m+2, \quad 32m^2-16m-10,$$

respectively.

Thus, all edge labels are distinct. Hence, $\mathcal{L}_m$, for $m \ge 2$, is an analytic even mean graph.
An example is given in Figure.3 □

Theorem 3.6 *($\mathcal{SL}_m$), $m \ge 3$ is analytic even mean graph.*

Proof: Let v_t, v'_t $(1 \le t \le 2m+2)$ be the vertices and e_t, e'_t $(1 \le t \le 3m)$ be the edges of $\mathcal{SL}_m$.
The $4m+4$ vertices of $\mathcal{SL}_m$ are labeled as follows:

$$v'_1 \leftarrow 4m+2, \quad v_2 \leftarrow 8m+6.$$

For $1 \le k \le m$:

$$v_{2k+2} \leftarrow 4m+4k+2, \quad v'_{2k+1} \leftarrow 4k-2.$$

For $1 \le k \le m+1$:

$$v_{2k-1} \leftarrow 4k-4, \quad v'_{2k} \leftarrow 4m+4k.$$

By the induced function f^* defined in (2.2), the $6m$ edges are labeled as follows:
- For $1 \le k \le m$:

$$\begin{aligned} e_{3k-2}(=v_{2k-1}v'_{2k+1}) &\text{ receives label } 4k-2,\\ e_{3k-1}(=v_{2k-1}v'_{2k+2}) &\text{ receives label } 8m^2+32m+32+(16m+28)k,\\ e'_{3k}(=v_{2k+2}v'_{2k}) &\text{ receives label } 4m+4k+2. \end{aligned}$$

- For $1 \le k \le m-1$:

$$\begin{aligned} e_{3k+3}(=v_{2k+2}v'_{2k+4}) &\text{ receives label } 20m+20k+28,\\ e'_{3k+1}(=v_{2k+3}v'_{2k+1}) &\text{ receives label } 20k+8,\\ e'_{3k+2}(=v_{2k+4}v'_{2k+1}) &\text{ receives label } 8m^2+24m+18+(16m+28)k. \end{aligned}$$

The three edges

$$e'_1(=v_3v'_1), \quad e'_2(=v_4v'_1), \quad e_3(=v_2v'_4)$$

receive labels

$$8m^2 + 8m - 10, \quad 12m + 14, \quad 24m^2 + 12m - 22,$$

respectively.

Thus, all edges are labeled distinctly. Hence, $\mathcal{SL}_m$, $m \geq 2$, is an analytic even mean graph. $\square$

Example 3.3 *We illustrate the analytic mean labeling of $\mathcal{L}_4$ and $\mathcal{SL}_4$ in the following figure.*

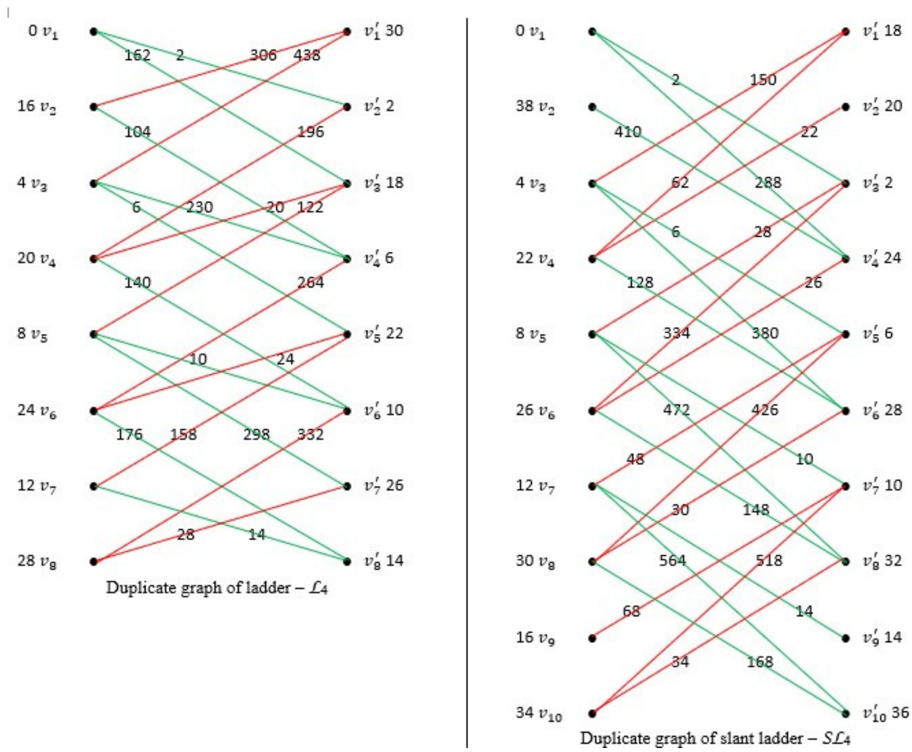

Figure 3: AEML in $\mathcal{L}_4$ and $\mathcal{SL}_4$

Theorem 3.7 *$(\mathcal{ZZL}_m)$, $m \geq 3$ is analytic even mean graph.*

Proof: Let v_t, v_t' $(1 \leq t \leq 2m)$ be the vertices and e_t, e_t' $(1 \leq t \leq 4m-3)$ be the edges of $\mathcal{ZZL}_m$.

The $4m$ vertices of $\mathcal{ZZL}_m$ are labeled as follows:

$$v_1' \leftarrow 8m - 2.$$

For $1 \leq k \leq m$:

$$v_{2k-1} \leftarrow 4k - 4, \quad v_{2k} \leftarrow 4m + 4k - 4, \quad v_{2k}' \leftarrow 4k - 2.$$

For $1 \leq k \leq m - 1$:

$$v_{2k+1}' \leftarrow 4m + 4k - 2.$$

The function defined in (2.2) induces the edge labels as follows:

- For $1 \leq k \leq m$:

$$e_{4k-3} (= v_{2k-1} v_{2k}') \text{ receives label } 4k - 2.$$

- For $1 \le k \le m-1$:

$$\begin{aligned}
e_{4k-1}(= v_{2k-1}v'_{2k}) &\text{ receives label } 4m+4k-2,\\
e_{4k-2}(= v_{2k-1}v'_{2k+1}) &\text{ receives label } 8m^2-8m-2+(16m+4)k,\\
e_{4k}(= v_{2k}v'_{2k+2}) &\text{ receives label } 8m^2-16m+4+(16m-28)k,\\
e'_{4k+1}(= v_{2k+2}v'_{2k+1}) &\text{ receives label } 4m+4k,\\
e'_{4k-1}(= v_{2k}v'_{2k+1}) &\text{ receives label } 4k,\\
e'_{4k}(= v_{2k+2}v'_{2k}) &\text{ receives label } 8m^2+(16m+4)k.
\end{aligned}$$

- For $1 \le k \le m-2$:

$$e'_{4k+2}(= v_{2k+3}v'_{2k+1}) \text{ receives label } 8m^2-8m-10+(16m-28)k.$$

The two edges

$$e'_1 \text{ and } e'_2$$

receive labels

$$24m^2-20m+2 \quad \text{and} \quad 32m^2-16m-10,$$

respectively.

Thus, all $8m-6$ edges are labeled distinctly. Hence, $\mathcal{ZZL}_m$, $m \ge 3$, is an analytic even mean graph. An example is given in Figure.4 □

Theorem 3.8 *$\mathcal{TL}_m$, $m \ge 2$ is analytic even mean graph.*

Proof: Let v_t, v'_t $(1 \le t \le 2m)$ be the vertices and e_t, e'_t $(1 \le t \le 4m-3)$ be the edges of $\mathcal{TL}_m$.

The $4m$ vertices of $\mathcal{TL}_m$ are labeled as follows:

$$v'_2 \leftarrow 8m-2.$$

For $1 \le k \le m$:

$$v_{2k-1} \leftarrow 4m+4k-4, \quad v_{2k} \leftarrow 4k-4, \quad v'_{2k-1} \leftarrow 4k-2.$$

For $1 \le k \le m-1$:

$$v'_{2k+2} \leftarrow 4m+4k-2.$$

By the induced function f^* defined in (2.2), the $8m-6$ edges are labeled as follows:

- For $1 \le k \le m$:

$$e'_{4k-3}(= v_{2k}v'_{2k-1}) \text{ receives label } 4k-2.$$

- For $1 \le k \le m-1$:

$$\begin{aligned}
e_{4k+1}(= v_{2k+1}v'_{2k+2}) &\text{ receives label } 4m+4k,\\
e_{4k-2}(= v_{2k-1}v'_{2k+1}) &\text{ receives label } 8m^2-16m+4+(16m-28)k,\\
e_{4k-1}(= v_{2k-1}v'_{2k+2}) &\text{ receives label } 4m+4k-2,\\
e_{4k}(= v_{2k}v'_{2k+2}) &\text{ receives label } 8m^2-8m-2+(16m+4)k,\\
e'_{4k-2}(= v_{2k+1}v'_{2k-1}) &\text{ receives label } 8m^2+(16m+4)k,\\
e'_{4k-1}(= v_{2k+2}v'_{2k-1}) &\text{ receives label } 4k.
\end{aligned}$$

- For $1 \le k \le m-2$:

$$e'_{4k+4}(= v_{2k+4}v'_{2k+2}) \text{ receives label } 8m^2-8m-10+(16m-28)k.$$

The two edges

$$e_1 \text{ and } e'_4$$

receive labels

$$24m^2 - 20m + 2 \quad \text{and} \quad 32m^2 - 16m - 10,$$

respectively.

Thus all edges are labeled distinctly. Hence, $\mathcal{TL}_m$, $m \geq 2$, is an analytic even mean graph. $\square$

Example 3.4 *We illustrate the analytic even mean labeling of $\mathcal{TL}_4$ and $\mathcal{ZZL}_4$ in the following figure.*

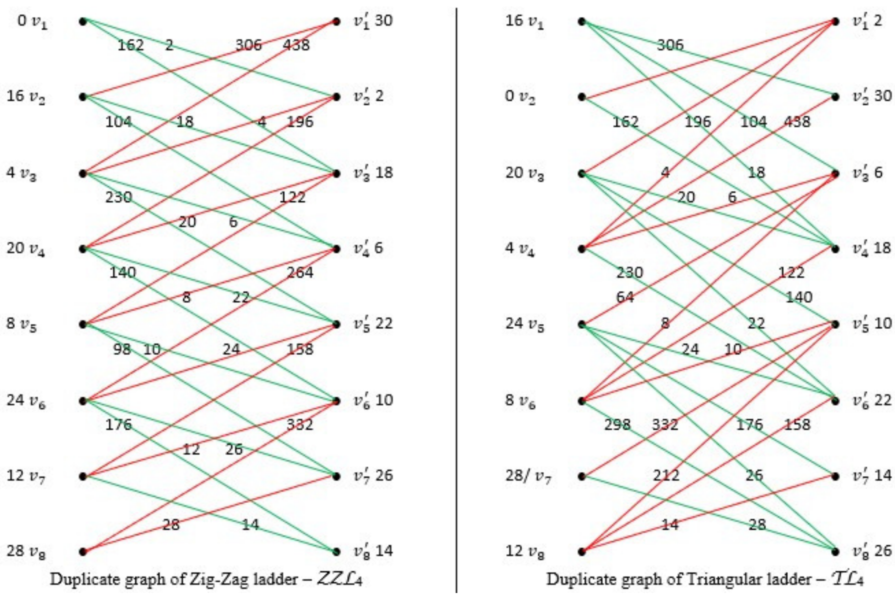

Figure 4: AEML in $\mathcal{TL}_4$ and $\mathcal{ZZL}_4$

4. Conclusion

We have proved that the graphs $\mathcal{L}_m$, $\mathcal{SL}_m$, $\mathcal{ZZL}_m$ and $\mathcal{TL}_m$ are analytic mean graphs and analytic even mean graphs.

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