Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 4** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.78528

Constrained Rational Cubic Fractal Interpolation Using Function Values

L. Rajesh and K. Mahipal Reddy*

ABSTRACT: This paper presents the development of a constrained rational cubic fractal interpolation using function values based on iterated function systems. The rational cubic fractal interpolation function is constructed using a cubic polynomial in the numerator and a linear polynomial in the denominator, with convergence properties that offer flexibility for modeling complex datasets exhibiting fractal-like behavior. We show that the proposed approximation converges to the original function as the discretization parameter trends to zero. The framework incorporates a single shape-control parameter and enforces constraints through (i) piecewise linear functions, (ii) linear functions, and (iii) rectangular bounds that confine the interpolated curve within a specified range. A detailed performance analysis is provided, along with a systematic approach for selecting scaling factors and shape parameters. The effectiveness of the proposed method is validated through extensive numerical experiments.

Key Words: Fractals, Iterated Function System, Fractal Interpolation Function, Convergence, Constrained Interpolation.

Contents

1. Introduction

In recent decades, interpolation and fractal models have evolved due to iterated function systems (IFS), which have revolutionized data analysis and approximation. Fractal interpolation provides greater flexibility in capturing the nuances of irregular and highly variable datasets than conventional interpolation methods like polynomials and splines. It is perfect for simulating real-world events like financial trends, physiological signals, and geological fluctuations because of its recursive and self-similar character, which preserves fine structural details and effectively handles non-smooth data. M.F. Barnsley [5,7] pioneered the concept of iterated function systems (IFS) to formalize fractal interpolation functions (FIFs). Barnsley and A. N. Harrington [6] introduced smooth FIFs, or C^r-FIFs, where the construction begins with the availability of all r^{th}-order derivatives of the function at the initial endpoint of the domain. Abdulla Sana and K. Mahipal Reddy [1] applied IFS techniques alongside K-means clustering and self-organizing maps for segmentation and design in the fashion industry, showcasing the integration of mathematical principles with digital design methodologies. A.K.B. Chand et al. [8] constructed of

* Corresponding author.
2020 *Mathematics Subject Classification:* 41A30.
Submitted August 21, 2025. Published October 09, 2025

a C^1-rational cubic fractal interpolation function effectively models constrained data while preserving smoothness and shape. The derived error estimates demonstrate the accuracy of rational cubic fractal interpolation function (RCFIF) within $C^k[x_1,\ x_n]$ for $k = 1, 3$. Numerical results confirm that the proposed RCFIF remains within prescribed bounds under suitable IFS conditions. Drakopoulos V. et al. [9] introduced affine fractal interpolation functions using a suitable IFS framework, presenting a brief theoretical foundation for affine FIF models in two dimensions. They derived constraints on contractivity factors to maintain positivity and monotonicity in bivariate affine FIFs, with numerical experiments validating their shape-preserving properties. Katiyar, S.K. et al. [10] investigated how to improve methods for fractal interpolation by creating a sophisticated class of rational cubic spline FIFs with a fixed quadratic denominator and several shape-controlling factors. Greater flexibility is provided by an enhanced method with three form parameters that incorporates a novel tension parameter. Numerical experiments that the suggested method is more accurate, efficient, and superior to conventional interpolation techniques. Gowrisankar A. et al. [2] provided IFS parameter constraints for affine recurrent fractal interpolation function (RFIF), ensuring their placement between piecewise linear segments, and developed an affine recurrent fractal interpolation function using a recurrent iterated function system. Balasubramani N. et al. [3] developed a novel class of α-fractal rational cubic splines based on iterated function systems. This approach utilized a cubic polynomial in the numerator and a quadratic polynomial in the denominator, ensuring boundedness under specific parameter constraints. K. Mahipal Reddy et al. [11] The restricted nature of a novel family of rational cubic fractal interpolation functions in both univariate and bivariate contexts is examined. We present the convergence findings of the RCFIF in relation to an original function. In particular, we establish adequate conditions based on the limitations of iterated function system parameters at fewer discretized values, ensuring that the associated RCFIF retains its inherent properties related to restricted data when the data lies (i) between two piecewise specified lines and (ii) within a rectangle. K. Mahipal Reddy et al. [12] the proposed rational fractal interpolation function (FIF) offers a flexible and effective framework for modeling data using only function values. By incorporating three families of shape parameters, it ensures improved control over the interpolant's shape while maintaining convergence and providing a clear error bound. Numerical examples confirm the method's capability to handle constrained interpolation problems accurately and efficiently. K. Mahipal Reddy et al. [13] the Overveld scheme to third-degree curves, determining curvature continuity and convex hull property. It also identifies conditions for positive-preserving fractal-like Bézier curves in proposed subdivision matrices. The resulting 2D/3D curves resemble fractal images and demonstrate the shape dependence on subdivision matrices' elements. Vijay et al. [17] explored various methods for creating fractal-like Bézier curves in both 2D and 3D environments, including subdivision schemes, Iterated Function System (IFS) theory, and the perturbation of both Bézier curves and Bézier basis functions. It outlines convergence conditions, one-sided approximation conditions, and considerations for perturbed Bézier basis functions. Vijay et al. [15] provided new classes of zipper fractal interpolation curves and surfaces that are continuously differentiable and preserve convexity. Over a rectangular grid, it creates surfaces and builds curves using univariate Hermite interpolation data. When applied to a bivariate data-generating function, these surface interpolants converge uniformly. You can vary the signature vectors in both directions to generate a large number of zipper fractal surfaces. Vijay et al. [16] introduced novel classes of continuously differentiable convexity-preserving zipper fractal surfaces. It constructs curves for univariate Hermite interpolation data and generates surfaces over a rectangular grid. These surface interpolants converge uniformly to a bivariate data-generating function. You can obtain a wide variety of zipper fractal surfaces by varying signature vectors in both directions. K. Mahipal Reddy et al. [14] presented a rational cubic spline function for fractal interpolation developed using a rational iterated function system. The function's graph is positioned within a prescribed rectangle, and a partially blended surface is created. Stability analysis is conducted, and sufficient conditions are investigated for the surface to fit a stipulated cuboid. K. Mahipal Reddy et al. [4] showcased the iterative beauty of fractals through MATLAB-based graphical representations, enhancing the visual understanding of fractal structures.

By the motivation of prior work [11] on the constrained of rational cubic fractal interpolation using function values. This paper introduces a novel approach to Constrained of rational Cubic fractal interpolation using function values based on IFS, focusing on flexibility, efficiency, and convergence. It explores constrained aspects and convergence properties of a new family of rational Cubic fractal interpolation,

ensuring C^1 continuity in univariate cases. We propose a new interpolation method that enhances scientific and engineering applications.
In this paper, In Section 2, we present relevant definitions and essential results concerning fractal interpolation. In section 3, we provide the construction of rational cubic fractal interpolation (RCFI) using function values. In section 4, we derive the convergence analysis. In section 5, we present the constraints on the vertical scaling factors and shape parameters of the constrained rational cubic fractal interpolation using function values. In section 6, we illustrate numerical examples to demonstrate the effectiveness of the proposed approach.

2. Basics of Fractal Interpolation Function (FIF)

Let the interpolation data be given by $\{(x_i,\ t_i) \in \mathbb{R} \times \mathbb{R} : i \in \mathbb{N}_{N+1} = \{1, 2, \dots, N+1\}\}$, where the nodes are ordered such that $x_1 < x_2 < \cdots < x_{N+1}$. Define the domain $\mathbb{I} = [x_1,\ x_{N+1}]$, $\mathbb{J} = [x_1,\ x_N]$ and the subintervals $\mathbb{I}_i^* = [x_i,\ x_{i+1}]$ for $i = 1, 2, \dots, N$. Assume that each subinterval satisfies the condition $x_N - x_1 > x_{i+1} - x_i$, ensuring proper contraction.

For each i, define a transformation $W_i : \mathbb{J} \times \mathbb{R} \to \mathbb{I}_i^* \times \mathbb{R}$ by:

$$W_i(x, t) = (V_i(x), \mathbb{L}_i(x, t)),$$

where $V_i : \mathbb{J} \to \mathbb{I}_i^*$ is a continuous affine transformation given by $V_i(x) = a_i x + b_i$, and $\mathbb{L}_i : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that is contractive in the second argument. The pair $(V_i,\ \mathbb{L}_i)$ must satisfy the following conditions:

$$|V_i(x) - V_i(x^*)| \le S_i|x - x^*|, \quad |\mathbb{L}_i(x, t) - \mathbb{L}_i(x, t^*)| \le M_i|t - t^*|,$$

for all $x, x^* \in \mathbb{J}$, $t, t^* \in \mathbb{R}$, with $|S_i| < a_i$, and $|M_i| < a_i$. Moreover, the boundary conditions ensure interpolation:

$$V_i(x_1) = x_i, \quad V_i(x_N) = x_{i+1}, \quad \mathbb{L}_i(x_1, t_1) = t_i, \quad \mathbb{L}_i(x_N, t_N) = t_{i+1}.$$

The set $\{\mathbb{J} \times \mathbb{R}, W_i : i = 1, 2, \dots, N\}$ constitutes an Iterated Function System whose unique attractor $G \subseteq \mathbb{J} \times \mathbb{R}$ corresponds to the graph of a continuous function $g : \mathbb{J} \to \mathbb{R}$. This function precisely interpolates the data points, i.e., $g(x_i) = t_i$ for all $i = 1, 2, \dots, N$, and satisfies a functional equation of self-referential identity:

$$g(V_i(x)) = \mathbb{L}_i(x,\ g(x)), \quad \text{for all } x \in \mathbb{J} \text{ and } i = 1, 2, \dots, N.$$

Proposition 1: [6] Let $\{(x_i, t_i) : i = 1, 2, \dots, N\}$ be a given set of interpolation data. Let each $V_i(x)$ be an affine map, and let $L_i(x, t) = \alpha_i t + q_i(x)$ be a continuous function defined on each subinterval $J = [x_1, x_N] \subseteq [x_1, x_{N+1}]$. Suppose that for some integer $j > 0$, the contraction condition $|\alpha_i| \le s a_i^j$ with $0 < s < 1$ holds, and $q_i \in C^j[x_1, x_N]$ for each $i \in J$. Define the functions

$$F_{i,\kappa}(x, t) = \frac{\alpha_i t + q_i^{(\kappa)}(x)}{a_i^\kappa}, \quad t_{1,\kappa} = \frac{q_1^{(\kappa)}(x_1)}{a_1^\kappa - \alpha_1}, \quad t_{2,\kappa} = \frac{q_N^{(\kappa)}(x_N)}{a_N^\kappa - \alpha_N},$$

for $\kappa = 1, 2, \dots, j$. If the boundary condition

$$F_{i-1,\kappa}(x_N, t_{N,\kappa}) = F_{i,\kappa}(x_1, t_{1,\kappa})$$

is satisfied for all $i = 1, 2, \dots, N$ and $\kappa = 1, 2, \dots, j$,then the system of functions $\gamma_i(x, t) = \{(V_i(x), L_i(x, t)) : i = 1, 2, \dots, N\}$ uniquely defines a **fractal interpolation function** (FIF) $f \in C^j[x_1, x_N]$, where $f^{(\kappa)}$ denotes the κ-th order differentiable FIF constructed by the corresponding maps.

3. Construction of Rational Cubic Fractal Interpolation using function values

Given an interpolation dataset $\{(x_i, t_i) : i \in N_{N+1}\}$, we define for each, $i \in N_N$, a function $\mathbb{L}_i : \mathbb{I}_i \times \mathrm{R} \to \mathrm{R}$ satisfying the conditions $\mathbb{L}_i(x_1,\ t_1) = t_i, \mathbb{L}_i(x_N,\ t_N) = t_{i+1}$. The corresponding map $W_i(x,t) = (V_i(x),\ \mathbb{L}_i(x,t))$, then forms part of an iterated function system, with α_i, serving as the vertical scaling factor for each $i \in N_N$.

The affine map is defined as $V_i(x) = a_i x + b_i$, while the rational vertical map takes the form

$$\mathbb{L}_i(x,t) = \alpha_i t + \frac{P_i(x)}{Q_i(x)}.$$

Consequently, the rational fractal interpolation function satisfies the functional equation Φ.

$$\Phi(V_i(x)) = \alpha_i \Phi(x) + \frac{P_i(x)}{Q_i(x)},\ x \in \mathbb{J}, \tag{3.1}$$

$$\text{where}\ \ P_i(x) = \mathbb{M}_1(1-\theta)^3 + \mathbb{M}_2\theta(1-\theta)^2 + \mathbb{M}_3\theta^2(1-\theta) + \mathbb{M}_4\theta^3,$$
$$Q_i(x) = 1 + (r_i - 2)(1-\theta).$$

$\mathbb{M}_1, \mathbb{M}_2, \mathbb{M}_3, \mathbb{M}_4$, are the coefficient of $P_i(x)$, r_i is the shape parameter

$$\theta = \frac{x - x_1}{x_N - x_1},\ x \in \mathbb{J}.$$

Put $x = x_1$, $\theta = 0$ and $x = x_N$, $\theta = 1$ in (3.1), we get

$$P_i(x_1) = \mathbb{M}_1, \quad Q_i(x_1) = 1 \text{ and } P_i(x_N) = \mathbb{M}_4, \quad Q_i(x_N) = 1.$$

Substitute $x = x_1$ and $x = x_N$ in (3.1), we obtain $\mathbb{M}_1 = (r_i - 1)(t_i - \alpha_i t_1)$ and $\mathbb{M}_4 = t_{i+1} - \alpha_i t_N$. Derivative of Φ in (3.1) with respect to x is

$$\Phi'(V_i(x))a_i = \alpha_i \Phi'(x) + \frac{Q_i(x)P_i'(x) - P_i(x)Q_i'(x)}{(Q_i(x))^2}. \tag{3.2}$$

Substitute, $V_i(x_1) = x_i,\ V_i(x_N) = x_{i+1}$ and $\mathbb{M}_1, \mathbb{M}_4$ expression in (3.2), we get

$$\mathbb{M}_2 = (2r_i - 1)(t_i - \alpha_i t_1) + (r_i - 1)(\Delta_i h_i - \alpha_i \Delta_1(x_N - x_1)),$$
$$\mathbb{M}_3 = (r_i + 1)(t_{i+1} - \alpha_i t_N) - \Delta_{i+1} h_i + \alpha_i \Delta_N(x_N - x_1).$$

The symbol Δ_i indicates the slope between two adjacent points (x_i, t_i) and (x_{i+1}, t_{i+1}) in a set of ordered data points $\{(x_i, t_i) : i = 1, 2, \ldots, N_N\}$. For $i = 1, 2, \ldots, N$, it is expressed as follows:

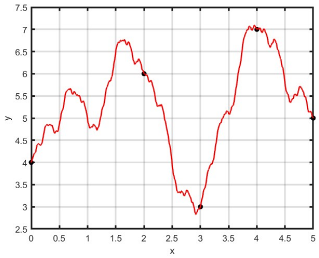
$$\Delta_i = \frac{t_{i+1} - t_i}{x_{i+1} - x_i}.$$

This figure, which is used to generate the rational cubic polynomial $P_i(x)$ in the vertical map of the rational cubic fractal interpolation function, represents the average rate of change of the function values over the subinterval $[x_i, x_{i+1}]$. It is also essential for figuring out the local behavior of the interpolating curve between successive nodes. We obtained a rational cubic fractal interpolation as

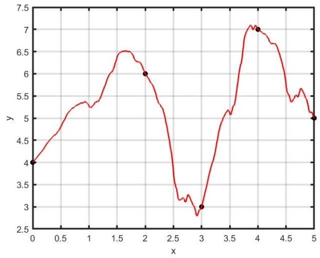
$$\Phi(V_i(x)) = \alpha_i \Phi(x) + \frac{P_i(x)}{Q_i(x)},$$
$$P_i(x) = (r_i - 1)(t_i - \alpha_i t_1)(1-\theta)^3 + \{(2r_i - 1)(t_i - \alpha_i t_1) + (r_i - 1)(\Delta_i h_i - \alpha_i \Delta_1$$
$$(x_N - x_1))\}\theta(1-\theta)^2 + \{(r_i + 1)(t_{i+1} - \alpha_i t_N) - \Delta_{i+1} h_i + \alpha_i \Delta_N (x_N - x_1)\}$$
$$\theta^2(1-\theta) + (t_{i+1} - \alpha_i t_N)\theta^3,$$
$$Q_i(x) = 1 + (r_i - 2)(1-\theta),\ \Delta_i = \frac{t_{i+1} - t_i}{x_{i+1} - x_i} \quad \theta = \frac{x - x_1}{x_N - x_1}, \quad x \in \mathbb{J}.$$

3.1. Numerical Example

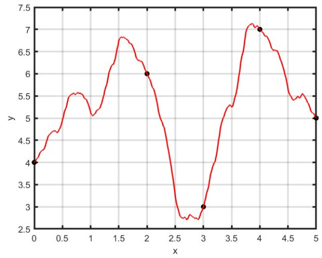
Consider the given data set for interpolation: $\{(x_i,\ t_i)\}_{i=0}^{6} = \{(0,4),(2,6),$ $(3,3),(4,7),(5,5),(6,8)\}$. Figure 1 illustrates the construction of the RCFI. The new scaling factors and shape parameter $\alpha =$ [0.38 0.18 0.17 0.19] and $r_i = [5]_{1\times 5}$ are used to construct Figure 1(a). Similarly, the scaling factors and shape parameter $\alpha = [0.15\ 0.19\ 0.18\ 0.17]$ and $r_i = [4]_{1\times 5}$ are applied in Figure 1(b). For Figure 1(c), we use $\alpha = [0.28\ 0.12\ 0.16\ 0.11]$ and $r_i = [2]_{1\times 5}$. The RCFI shown in Figure 1(d) is obtained using the scaling factors and shape parameter $\alpha = [0\ 0\ 0\ 0]$ and $r_i = [8]_{1\times 5}$.



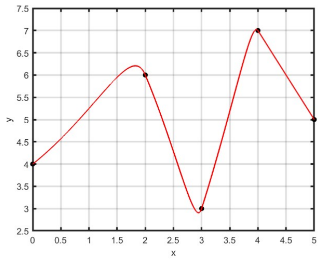
(a) $\alpha = [0.38\ \ 0.18\ \ 0.17\ \ 0.19]$



(b) $\alpha = [0.15\ 0.19\ 0.18\ 0.17]$



(c) $\alpha = [0.28\ 0.12\ 0.16\ 0.11]$



(d) $\alpha = [0\ 0\ 0\ 0]$

Figure 1: Construction of Rational Cubic Fractal Interpolation

4. Convergence Analysis

In this section, we develop the convergence analysis. The dataset $\{(x_i,\ t_i) : i \in N_{N+1}\}$ is assumed to be derived from a generation function $\psi \in \mathcal{C}^1(\mathbb{J})$. Let's C represent the classical interpolation function corresponding to the rational cubic fractal model. To examine the convergence of the fractal interpolant Φ to the true function ψ, we leverage the known convergence behavior of C and the bounded deviation between Φ and C. This analysis is facilitated using the triangle inequality: $\|\Phi - \psi\|_\infty \le \|\Phi - C\|_\infty + \|C - \psi\|_\infty$. We have $\Phi(V_i(x)) = \alpha_i \Phi(x) + \frac{P_i(x)}{Q_i(x)}$. If $\alpha_i = 0$ then, we get

$$
\begin{aligned}
C(x) &= \frac{P_i(x)}{Q_i(x)} = \frac{\mathbb{M}_1(1-\vartheta)^3 + \mathbb{M}_2\vartheta(1-\vartheta)^2 + \mathbb{M}_3\vartheta^2(1-\vartheta) + \mathbb{M}_4\vartheta^3}{1+(r_i-2)(1-\vartheta)},\\
\vartheta &= \frac{x-x_i}{x_{i+1}-x_i},\ x \in \mathbb{I}_i^*.\\
\text{Where}\quad \mathbb{M}_1 &= (r_i-1)(t_i-\alpha_i t_1),\\
\mathbb{M}_2 &= (2r_i-1)(t_i-\alpha_i t_1) + (r_i-1)(\Delta_i h_i - \alpha_i\Delta_1(x_N-x_1)),\\
\mathbb{M}_3 &= (r_i+1)(t_{i+1}-\alpha_i t_N) - \Delta_{i+1}h_i + \alpha_i\Delta_N(x_N-x_1)\\
\text{and } \mathbb{M}_4 &= t_{i+1}-\alpha_i t_N.
\end{aligned}
$$

Now, we express the difference between $\Phi(x)$ and $C(x)$ as:

$$\Phi(x) - C(x) = \frac{-\alpha_i}{Q_i(\vartheta)}\Bigg[(r_i-1)t_1(1-\vartheta)^3 + \{(2r_i-1)t_1 + (r_i-1)\Delta_1(x_N-x_1)\}$$
$$\vartheta(1-\vartheta)^2 + \{(r_i+1)t_N - \Delta_N(x_N-x_1)\}\vartheta^2(1-\vartheta) + t_N\vartheta^3\Bigg].$$

It can also be stated as:

$$\Phi(x) - C(x) = \frac{-\alpha_i}{Q_i(\vartheta)}\left\{[A_0t_1 + A_1t_N + A_2\Delta_1 + A_3\Delta_N]\right\},$$

where $A_0 = (r_i-1)(1-\vartheta)^3 + (2r_i-1)\vartheta(1-\vartheta)^2, A_1 = (r_i+1)\vartheta^2(1-\vartheta) + \vartheta^3$
$A_2 = (r_i-1)h_i\vartheta(1-\vartheta)^2, A_3 = -h_i\vartheta^2(1-\vartheta)$.

$$\frac{A_0 + A_1}{Q_i(\vartheta)} = 1, \quad \frac{A_2 + A_3}{Q_i(\vartheta)} = \frac{h_i\vartheta(1-\vartheta)[r_i(1-\vartheta)-1]}{1+(r_i-2)(1-\vartheta)},$$

$$|\Phi(x) - C(x)| \le |\alpha_i|\left[\max\{t_1, t_N\} + E(\vartheta, h)\max\{\Delta_1, \Delta_N\}\right],$$
$$\text{where} \quad E(\vartheta, h) = \frac{h_i\vartheta(1-\vartheta)[r_i(1-\vartheta)-1]}{1+(r_i-2)(1-\vartheta)}.$$

$$\text{Now, } \psi(x) - C(x) = \psi(x) - \frac{P_i(x)}{Q_i(x)}$$

The maximum approximation error can be evaluated using the infinity norm:

$$|\psi(x) - C(x)| \le \big\{|\psi(x) - t_i|[(r_i-1)(1-\vartheta)^3 + |2r_i-1|\vartheta(1-\vartheta)^2] + |\psi(x) - t_{i+1}|$$
$$[|r_i+1|\vartheta^2(1-\vartheta) + \vartheta^3] + h_i[|\Delta_i|\vartheta(1-\vartheta)^2 + |\Delta_{i+1}|\vartheta^2(1-\vartheta)]\big\}.$$

$$|\psi(x) - C(x)| \le \frac{1}{Q_i(\vartheta_j)}\left[\gamma(\psi; h) + h_i\max\{|\Delta_i|, |\Delta_{i+1}|\}\right],$$

$$\text{where} \quad \gamma(\psi; h) = |r_i-1|(1-\vartheta)^3 + |2r_i-1|\vartheta(1-\vartheta)^2 + |r_i+1|\vartheta^2(1-\vartheta) + \vartheta^3.$$

The expression can be written as,

$$|t|_\infty = \max\{|t_1|, |t_N| : i = 1, 2, \ldots, N\} \le \max\{|t_i| : i = 1, 2, \ldots, N\},$$
$$|\Delta|_\infty = \max\{|\Delta_1|, |\Delta_N| : i = 1, 2, \ldots, N\} \le \max\{|\Delta_i| : i = 1, 2, \ldots, N\},$$
$$|h|_\infty = \max\{|h_i| : i = 1, 2, \ldots, N-1\}, \quad |r|_\infty = \max\{|r_i| : i = 1, 2, \ldots, N\}.$$

$$\text{Thus, } |\Phi(x) - \psi(x)| \le |\alpha_i|\left[\max\{t_1, t_N\} + E(\vartheta, h)\max\{\Delta_1, \Delta_N\}\right]$$
$$+ \frac{1}{Q_i(\vartheta_j)}\left[\gamma(\psi; h) + h_i\max\{|\Delta_i|, |\Delta_{i+1}|\}\right].$$

Convergence Result: Let Φ be a uniformly continuous function defined on ψ, and suppose that $\gamma(\psi; h) \to 0$, as $h \to 0$. Then , by utilizing the classical interpolation C corresponding to ψ we can estimate the error using the inequality: $\|\Phi - \psi\|_\infty \le \|\Phi - C\|_\infty + \|C - \psi\|_\infty$.

5. Constrained Rational Cubic Fractal Interpolation Function Using Function Values

In this section, the focus is on identifying the vector scaling factors and shape parameters that satisfy the imposed constraints for the RCFI. We aim to formulate a constrained RCFI such that the interpolation data $\{(x_i, t_i) : i \in N_{N+1}\}$, lies entirely within two piecewise linear bounding functions, defined as $\mathbb{T}_i^l = \xi_i x + m_i$ and $\mathbb{T}_i^u = \xi_i^* x + m_i^*$. Given the implicit structure of RCFI, we examine two distinct cases for the vertical scaling factor: $\alpha_i \geq 0$ and $\alpha_i < 0$, for all $i \in N_N$.
To ensure that all points generated through iteration respect these bounds when the dataset is enclosed by the piecewise linear functions, the model must satisfy a specific constraint condition.

$$\mathbb{T}_i^l(V_i(x)) \leq F(V_i(x)) \leq \mathbb{T}_i^u(V_i(x)), \quad V_i(x) = a_i x + b_i, \ i \in N_N.$$

We can rewrite it for a non-negative scaling factor as follows:

$$\xi_i(V_i(x)) + m_i \leq \alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(V_i(x)) + m_i^*, \ i \in N_N,$$

which further simplifies to:

$$\xi_i(a_i x + b_i) + m_i \leq \alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(a_i x + b_i) + m_i^*,$$

$$\text{where} \ \ \theta_j = \frac{x_j - x_1}{x_N - x_1}.$$

We aim to derive conditions on the IFS parameters that guarantee the subsequent inequalities hold:

$$\xi_i(a_i x + b_i) + m_i \leq \alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \quad \text{and} \quad \alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(a_i x + b_i) + m_i^*. \tag{5.1}$$

We now present a theorem that establishes the conditions for the constrained RCFI.

Theorem 5.1 *Consider the interpolation dataset $\{(x_i, t_i) : i \in N_{N+1}\}$ positioned between a pair of piecewise linear functions. If the bounds are given by $\mathbb{T}_i^l = \xi_i x + m_i$ and $\mathbb{T}_i^u = \xi_i^* x + m_i^*$, $i \in N_N$, then the rational cubic fractal interpolation will remain between these functions, provided that the IFS parameters are selected appropriately.*

$$(i) \ 0 \leq \alpha_i < min\left[a_i, \frac{t_i - \mathbb{T}_i^l(x_i)}{t_1 - \mathbb{T}_i^l(x_1)}, \frac{t_{i+1} - \mathbb{T}_i^l(x_{i+1})}{t_N - \mathbb{T}_i^l(x_N)}, \frac{\mathbb{T}_i^u(x_i) - t_i}{\mathbb{T}_i^u(x_1) - t_1}, \frac{\mathbb{T}_i^u(x_{i+1}) - t_{i+1}}{\mathbb{T}_i^u(x_N) - t_N}\right],$$

$$\begin{aligned}(ii) r_i > max\Bigg[0, &\frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^l(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*| + \xi_i(\alpha_i - a_i)|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^l(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*| + \xi_i(\alpha_i - a_i)|J^*|}, \\ &\frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^l(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1}))) - \alpha_i(t_N - \mathbb{T}_i^l(x_N)) + \xi_i(\alpha_i - a_i)|J^*|}, \\ &\frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*| + \xi_i^*(a_i - \alpha_i)|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*| + \xi_i^*(a_i - \alpha_i)|J^*|}, \\ &\frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N)] - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N) + \xi_i^*(a_i - \alpha_i)|J^*|}\Bigg],\end{aligned}$$

and $|J^*| = |x_N - x_1|$.

Proof: We can represent the left-hand side equation (5.1) in the form,

$$\alpha_i f(x) Q_i(\theta_j) + P_i(\theta_j) - \{[\xi_i(a_i x + b_i) + m_i] Q_i(\theta_j)\} \geq 0.$$

Assume that $\alpha_i \geq 0$. Thus, $f(x) \geq \xi_i x + m_i \Rightarrow \alpha_i f(x) Q_i(\theta_j) \geq \alpha_i(\xi_i x + m_i) Q_i(\theta_j)$.

As a result, the established conditions naturally guarantee the fulfillment of the subsequent inequalities.

$$\begin{aligned}
&\alpha_i(\xi_i x + m_i)Q_i(\theta_j) + P_i(\theta_j) - (\xi_i(a_i x + b_i) + m_i)Q_i(\theta_j) \geq 0,\\
&\Rightarrow \alpha_i[\xi_i(x_1 + \theta_j(x_N - x_1) + m_i]Q_i(\theta_j) + P_i(\theta_j) - [\xi_i(a_i[x_1 + \theta_j(x_N - x_1)\\
&\quad + b_i] + m_i]Q_i(\theta_j) \geq 0,\\
&\Rightarrow \alpha_i[\xi_i x_1 + m_i]Q_i(\theta_j) + \alpha_i\xi_i(x_N - x_1)\theta_j Q_i(\theta_j) + P_i(\theta_j) - [\xi_i(a_i x_1 + b_i)\\
&\quad + m_i]Q_i(\theta_j) - \xi_i a_i(x_N - x_1)\theta_j Q_i(\theta_j) \geq 0.
\end{aligned}$$

Which can be written as,

$$\begin{aligned}
&\alpha_i[\xi_i x_1 + m_i]Q_i(\theta_j) + \alpha_i\theta_j(x_N - x_1)\theta_j Q_i(\theta_j) + P_i(\theta_j)\\
&- [\xi_i(a_i x_1 + b_i) + m_i]Q_i(\theta_j) - \xi_i a_i(x_N - x_1)\theta_j Q_i(\theta_j) \geq 0.
\end{aligned} \tag{5.2}$$

Using degree elevation technique on $Q_i(\theta_j)$ and $\theta_j Q_i(\theta_j)$ we obtain,

$$\begin{aligned}
&Q_i(\theta_j) = (r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta + (r_i + 1)\theta^2(1-\theta) + \theta^3,\\
&\theta_j(Q_i(\theta_j)) = (r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3.
\end{aligned} \tag{5.3}$$

Using (5.2) and (5.3), we get

$$\begin{aligned}
&\{\alpha_i(\xi_i x_1 + m_i) - [\xi_i(a_i x_1 + b_i) + m_i]\}\{(r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta\\
&+ (r_i + 1)\theta^2(1-\theta) + \theta^3\} + [\xi_i(\alpha_i - a_i)|J^*|]\{(r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3\}\\
&+ \{[(r_1 - 1)(t_i - \alpha_i t_1)](1-\theta)^3 + [(2r_i - 1)(t_i - \alpha_i t_1 + (r_i - 1)\{h_i\Delta_i - \alpha_i\Delta_1|J^*|\}]\\
&(1-\theta)^2\theta + [(r_i + 1)(t_{i+1} - \alpha_i t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|](1-\theta)\theta^2 + [(t_{i+1} - \alpha_i t_N)]\\
&\theta^3\} \geq 0,
\end{aligned}$$

By rearranging and simplifying the given inequality, we obtain

$$\begin{aligned}
&\Rightarrow (r_i - 1)[(t_i - (\xi_i(a_i x_1 + b_i) + m_i)) - \alpha_i(t_1 - (\xi_i x_1 + m_i))](1-\theta)^3\\
&\quad + \{(2r_i - 1)[(t_i - (\xi_i(a_i x_1 + b_i) + m_i)) - \alpha_i(t_1 - (\xi_i x_1 + m_i))]\\
&\quad + (r_i - 1)[h_i\Delta_i - \alpha_i\Delta_1|J^*| + \xi_i(\alpha_i - a_i)|J^*|]\}(1-\theta)^2\theta + \{(r_i + 1)\\
&\quad [(t_{i+1} - (\xi_i(a_i x_1 + b_i) + m_i)) - \alpha_i(t_N - (\xi_i x_1 + m_i))] - h_i\Delta_{i+1}\\
&\quad + \alpha_i\Delta_N|J^*| + r_i(\xi_i(\alpha_i - a_i)|J^*|)\}\theta^2(1-\theta) + \{[(t_{i+1} - (\xi_i(a_i x_1 + b_i)\\
&\quad + m_i)) - \alpha_i(t_N - (\xi_i x_1 + m_i)) + (\xi_i(\alpha_i - a_i)|J^*|)\}\theta^3 \geq 0
\end{aligned}$$

$$\text{Hence, } 0 \leq \alpha_i < min\left[a_i,\quad \frac{t_i - \mathbb{T}_i^l(x_i)}{t_1 - \mathbb{T}_i^l(x_1)},\quad \frac{t_{i+1} - \mathbb{T}_i^l(x_{i+1})}{t_N - \mathbb{T}_i^l(x_N)}\right]$$

$$\begin{aligned}
r_i > max\Bigg[0, &\frac{(t_i - \mathbb{T}_i^l) - \alpha_i(t_1 - \mathbb{T}_i^l) + h_i\Delta_i - \alpha_i\Delta_1|J^*| + \xi_i(\alpha_i - a_i)|J^*|}{2[(t_i - \mathbb{T}_i^l) - \alpha_i(t_1 - \mathbb{T}_i^l)] + h_i\Delta_i - \alpha_i\Delta_1|J^*| + \xi_i(\alpha_i - a_i)|J^*|},\\
&\frac{-[(t_{i+1} - \mathbb{T}_i^l) - \alpha_i(t_N - \mathbb{T}_i^l) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l) - \alpha_i(t_N - \mathbb{T}_i^l) + \xi_i(\alpha_i - a_i)|J^*|}\Bigg]
\end{aligned}$$

The right-hand side inequality of (5.1) can be expressed as,

$$\mathbb{T}_i^l(V_i(x)) \leq F(V_i(x)) \leq \mathbb{T}_i^u(V_i(x)),\ V_i(x) = a_i x + b_i,\ j \in N_N.$$

$$\xi_i(V_i(x)) + m_i \leq \alpha_i f(x_i) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(V_i(x)) + m_i^*, \quad \theta_j = \frac{x - x_1}{x_N - x_1},\ j \in N_N.$$

Here $f(x) \geq \xi_i^* x + m_i^* \Rightarrow \alpha_i f(x) \geq \alpha_i(\xi_i^* x + m_i^*)$

$$\Rightarrow \alpha_i f(x) Q_(\theta_j) \geq \alpha_i(\xi_i^* x + t_i^*) Q_(\theta_j) \Rightarrow \alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(a_i x + b_i) + m_i^*$$
$$\Rightarrow \alpha_i f(x) Q_i(\theta_j) + P_i(\theta_j) - Q_i(\theta_j) - [\xi_i^*(a_i x + b_i) + m_i^*] \leq 0$$
$$\Rightarrow \alpha_i(\xi_i^* x + m_i^*) Q_i(\theta_j) + P_i(\theta_j) - (Q_i(\theta_j))[\xi_i^*(a_i x + b_i) + m_i^*] \leq 0$$
$$\Rightarrow [\alpha_i(\xi_i^*(x_1 + m_i^*) - \xi_i^*(a_i x_1 + b_i) + t_i^*)] Q_i(\theta_j) - [\alpha_i \xi_i^* |J^*| - \xi_i^* a_i |J^*|] \theta_j Q_i(\theta_j) + P_i(\theta_j) \leq 0$$

By performing degree elevation on $Q_i(\theta_j)$ and $\theta_j Q_i(\theta_j)$ we derive,

$$[(\xi_i^*(a_i x_1 + b_i) + m_i^*) - \alpha_i(\xi_i^* x_1 + m_i^*)]\{(r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta + (r_i + 1)\theta^2(1-\theta) + \theta^3\} + [\xi_i^*(a_i - \alpha_i)|J^*|]\{(r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3\} - \{[(r_1 - 1)(t_i - \alpha_i t_1)](1-\theta)^3 + [(2r_i - 1)(t_i - \alpha_i t_1 + (r_i - 1)\{h_i\Delta_i - \alpha_i\Delta_1|J^*|\}] (1-\theta)^2\theta + [(r_i + 1)(t_{i+1} - \alpha_i t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|](1-\theta)\theta^2 + [(t_{i+1} - \alpha_i t_N)] \theta^3\} \geq 0.$$

Rewriting and simplifying the inequality, we derive

$$\Rightarrow (r_i - 1)[((\xi_i^*(a_i x_1 + b_i) + m_i^*) - t_i) - \alpha_i((\xi_i^* x_1 + m_i^*) - t_1](1-\theta)^3 + \{(2r_i - 1) [((\xi_i^*(a_i x_1 + b_i) + m_i^*) - t_i) - \alpha_i((\xi_i^* x_1 + m_i^*) - t_1] + (r_i - 1)[\xi_i^*(a_i - \alpha_i)|J^*| - h_i\Delta_i + \alpha_i\Delta_1|J^*|]\}(1-\theta)^2\theta + \{(r_i + 1)[((\xi_i^*(a_i x_1 + b_i) + m_i^*) - t_{i+1}) - \alpha_i((\xi_i^* x_1 + m_i^*) - t_N)] + r_i[\xi_i^*(a_i - \alpha_i)|J^*|] + h_i\Delta_{i+1} - \alpha_i\Delta_N|J^*|\}\theta^2(1-\theta) + [((\xi_i^*(a_i x_1 + b_i) + m_i^*) - t_{i+1}) - \alpha_i((\xi_i^* x_1 + m_i^*) - t_N) + \xi_i^*(a_i - \alpha_i)|J^*|]\theta^3 \geq 0.$$

$$\text{Hence,}\ \ 0 \leq \alpha_i < min\Big[a_i,\ \frac{\mathbb{T}_i^u(x_i) - t_i}{\mathbb{T}_i^u(x_1) - t_1},\ \frac{\mathbb{T}_i^u(x_{i+1}) - t_{i+1}}{\mathbb{T}_i^u(x_N) - t_N}\Big],$$

$$r_i > max\Bigg[0,\ \frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*| + \xi_i^*(a_i - \alpha_i)|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*| + \xi_i^*(a_i - \alpha_i)|J^*|},\ \frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N)] - \Delta_{i+1} + \alpha_i\Delta_N|J^*|}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N) + \xi_i^*(a_i - \alpha_i)|J^*|}\Bigg]$$

□

Next, we consider the constrained RCFI when the scaling factors are negative.

Theorem 5.2 *Let* $\{(x_i, t_i) : i \in \mathbb{N}_{N+1}\}$ *be a given interpolation dataset such that the inequality* $\mathbb{T}_i^l(x) \leq t_i \leq \mathbb{T}_i^u(x)$ *holds for all* $i \in \mathbb{N}_N$. *In order to ensure that the rational cubic fractal interpolation with negative scaling remains strictly confined within the bounding functions* $\mathbb{T}_i^l(x) = \xi_i x + m_i$ *and* $\mathbb{T}_i^u(x) = \xi_i^* x + m_i^*$, *certain conditions on the Iterated Function System (IFS) parameters must be imposed.*

$$(i)\ 0 \geq \alpha_i > max\Bigg[-a_i,\ \frac{t_i - \mathbb{T}_i^l(x_i)}{t_1 - \mathbb{T}_i^u(x_1)},\ \frac{t_{i+1} - \mathbb{T}_i^l(x_{i+1})}{t_N - \mathbb{T}_i^u(x_N)},\ \frac{\mathbb{T}_i^u(x_i) - t_i}{\mathbb{T}_i^l(x_1) - t_1},\ \frac{\mathbb{T}_i^u(x_{i+1}) - t_{i+1}}{\mathbb{T}_i^l(x_N) - t_N}\Bigg],$$

$$(ii)\ r_i > max\Bigg[0,\ \frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|},\ \frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|},\ \frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1)) - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|},\ \frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N)|J^*|]}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|)}\Bigg].$$

Proof: In this case, the scaling factor α_i is required to satisfy the constraint $-a_i < \alpha_i < 0$. The left-hand side of an inequality (5.1) can be expressed as follows,

$$\xi_i(a_i x + b_i) + m_i \leq \alpha_i f(x) + \frac{P_i(\theta)}{Q_i(\theta)}$$

$$\begin{aligned}
\text{Here,}\;\; & f(x) \geq \xi_i^* x + m_i^* \Rightarrow \alpha_i f(x) \geq \alpha_i(\xi_i^* x + m_i^*) \\
& \Rightarrow \alpha_i f(x) Q_i(\theta) \geq \alpha_i(\xi_i^* x + m_i^*) Q_i(\theta) \Rightarrow \alpha_i f(x) + \frac{P_i(\theta)}{Q_i(\theta)} - [\xi_i(a_i x + b_i) + m_i] \geq 0 \\
& \Rightarrow \alpha_i f(x) Q_i(\theta) + P_i(\theta)[\xi_i(a_i x + b_i) + m_i] Q_i(\theta) \geq 0 \\
& \Rightarrow \{\alpha_i(\xi_i^* x + m_i^*) - [\xi_i(a_i x + b_i) + m_i]\} Q_i(\theta) + P_i(\theta) \geq 0 \\
& \Rightarrow \{\alpha_i(\xi_i^*(x_1 + \theta|J^*|) + m_i^*) - [\xi_i(a_i(x_1 + \theta|J^*|) + b_i) + m_i]\} Q_i(\theta) + P_i(\theta) \geq 0 \\
& \Rightarrow [\alpha_i(\xi_i^* x_1 + m_i^*) - (\xi(a_i x_1 + b_i) + m_i)] Q_i(\theta) + \{[\xi_i^* \alpha_i - \xi_i a_i]|J^*|\} \theta Q_i(\theta) \\
& \quad + P_i(\theta) \geq 0.
\end{aligned}$$

Utilizing the degree of elevation, we obtain

$$\begin{aligned}
Q_i(\theta_j) &= (r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta + (r_i + 1)\theta^2(1-\theta) + \theta^3, \\
\theta_j(Q_i(\theta_j)) &= (r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow & [\alpha_i(\xi_i^* x_1 + m_i^*) - (\xi(a_i x_1 + b_i) + m_i)][(r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta \\
& + (r_i + 1)\theta^2(1-\theta) + \theta^3] + \{[\xi_i^* \alpha_i - \xi_i a_i]|J^*|\}[(r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3] \\
& + \{[(r_1 - 1)(t_i - \alpha_i t_1)](1-\theta)^3 + [(2r_i - 1)(t_i - \alpha_i t_1 + (r_i - 1)\{h_i\Delta_i - \alpha_i\Delta_1|J^*|\}] \\
& (1-\theta)^2\theta + [(r_i + 1)(t_{i+1} - \alpha_i t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|](1-\theta)\theta^2 + [(t_{i+1} - \alpha_i t_N)] \\
& \theta^3\} \geq 0.
\end{aligned}$$

Which is simplified as,

$$\begin{aligned}
& \Rightarrow (r_i - 1)[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)](1-\theta)^3 + \{[2[(t_i - \mathbb{T}_i^l(x_i)) \\
& - \alpha_i(t_1 - \mathbb{T}_i^u(x_1))] + h_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|]r_i - [(t_i - \mathbb{T}_i^l(x_i)) \\
& - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)) + h_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|]\}(1-\theta)^2\theta \\
& + \{r_i[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|)] + [(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) \\
& - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]\}\theta^2(1-\theta) + [(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N \\
& - \mathbb{T}_i^u(x_N)) + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|)]\theta^3 \geq 0,
\end{aligned}$$

$$\text{Hence,}\;\; 0 \geq \alpha_i > max\left[-a_i, \frac{t_i - \mathbb{T}_i^l(x_i)}{t_1 - \mathbb{T}_i^u(x_1)}, \frac{t_{i+1} - \mathbb{T}_i^l(x_{i+1})}{t_N - \mathbb{T}_i^u(x_N)}\right],$$

$$\begin{aligned}
r_i > max\Bigg[0, & \frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}, \\
& \frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}\Bigg].
\end{aligned}$$

We can represent the right-hand side of an inequality (5.1) can be expressed as follows,

$$\alpha_i f(x) + \frac{P_i(\theta_j)}{Q_i(\theta_j)} \leq \xi_i^*(a_i x + b_i) + m_i^*$$

Here, $f(x) \geq \xi_i x + m_i \Rightarrow \alpha_i f(x) \geq \alpha_i(\xi_i x + m_i)$

$$\Rightarrow \alpha_i f(x) Q_i(\theta) \geq \alpha_i(\xi_i x + m_i) Q_i(\theta)$$

$$\Rightarrow \alpha_i f(x) + \frac{P_i(\theta)}{Q_i(\theta)} - [\xi_i^*(a_i x + b_i) + m_i^*] \le 0$$

$$\Rightarrow [\xi_i^*(a_i x + b_i) + m_i^*]Q_i(\theta) - \alpha_i f(x)Q_i(\theta) + P_i(\theta) \ge 0$$
$$\Rightarrow [\xi_i^*(a_i x + b_i) + m_i^*]Q_i(\theta) - (\alpha_i(\xi_i x + m_i)Q_i(\theta) + P_i(\theta)) \ge 0$$
$$\Rightarrow [\xi_i^*(a_i(x_1 + \theta|J^*|) + b_i) + m_i^*]Q_i(\theta) - \alpha_i(\xi_i(x_1 + (\theta)|J^*|) + m_i)Q_i(\theta) - P_i(\theta) \ge 0.$$

Applying the degree of elevation, we get

$$[(\xi_i^*(a_i(x_1 + \theta|J^*|) + b_i) + m_i^*) - \alpha_i(\xi_i(x_1 + \theta|J^*|) + m_i)][(r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta + (r_i + 1)\theta^2(1-\theta) + \theta^3] + [(\xi_i^* a_i - \xi_i\alpha_i)|J^*|][(r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3] - P_i(\theta) \ge 0.$$

By extending the above approach, the following inequality is derived:

$$[(\xi_i^*(a_i(x_1 + \theta|J^*|) + b_i) + m_i^*) - \alpha_i(\xi_i(x_1 + \theta|J^*|) + m_i)][(r_i - 1)(1-\theta)^3 + (2r_i - 1)(1-\theta)^2\theta + (r_i + 1)\theta^2(1-\theta) + \theta^3] + [(\xi_i^* a_i - \xi_i\alpha_i)|J^*|][(r_i - 1)(1-\theta)^2\theta + r_i\theta^2(1-\theta) + \theta^3] - \{[(r_1 - 1)(t_i - \alpha_i t_1)](1-\theta)^3 + [(2r_i - 1)(t_i - \alpha_i t_1 + (r_i - 1)\{h_i\Delta_i - \alpha_i\Delta_1|J^*|\}](1-\theta)^2\theta + [(r_i + 1)(t_{i+1} - \alpha_i t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|](1-\theta)\theta^2 + [(t_{i+1} - \alpha_i t_N)]\theta^3\} \ge 0.$$

Hence, $0 \ge \alpha_i > max\left[-a_i, \dfrac{\mathbb{T}_i^u(x_i) - t_i}{\mathbb{T}_i^l(x_1) - t_1}, \dfrac{\mathbb{T}_i^u(x_{i+1}) - t_{i+1}}{\mathbb{T}_i^l(x_N) - t_N}\right]$,

$$r_i > max\left[0, \frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|}, \frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N)|J^*|]}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|)}\right].$$

□

Remark 5.1 *Let the interpolation data set* $\{(x_i, t_i) : i \in \mathbb{N}_{N+1}\}$ *be such that each data point satisfies the condition* $\mathbb{T}^l(x) \le t_i \le \mathbb{T}^u(x)$, *for all* $i \in \mathbb{N}_{N+1}$, $j \in \mathbb{N}_N$ *. Assume that the bounding functions* $\mathbb{T}^l(x) = \xi x + m$ *and* $\mathbb{T}^u(x) = \xi^* x + m^*$ *are distinct linear expressions defined at the knots* x_j*, with* $i \in \mathbb{N}_{N-1}$*. The rational cubic fractal interpolation function exhibits a controlled negative scaling trend, strictly constrained within the region bounded by* $\mathbb{T}^l$ *and* $\mathbb{T}^u$*, which is achieved through a suitable choice of IFS parameters that satisfy the prescribed criteria.*

$$(i)\ 0 \ge \alpha_i > max\left[-a_i, \frac{t_i - \mathbb{T}_i^l(x_i)}{t_1 - \mathbb{T}_i^u(x_1)}, \frac{t_{i+1} - \mathbb{T}_i^l(x_{i+1})}{t_N - \mathbb{T}_i^u(x_N)}, \frac{\mathbb{T}_i^u(x_i) - t_i}{\mathbb{T}_i^l(x_1) - t_1}, \frac{\mathbb{T}_i^u(x_{i+1}) - t_{i+1}}{\mathbb{T}_i^l(x_N) - t_N}\right],$$

$$(ii)\ r_i > max\left[0, \frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*| + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}, \frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) + (\xi_i^*\alpha_i - \xi_i a_i)|J^*|}, \frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*| + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|}, \frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N)|J^*|]}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) + (\xi_i^* a_i - \xi_i\alpha_i)|J^*|)}\right].$$

Remark 5.2 *If $\xi_i = 0$ when $m_i = \mathrm{s}_1$ and $\xi_i = 0$ when $m_i^= \mathrm{s}_2$, then the resulting RCFI will definitely stay within the limits $\mathrm{s}_1 \leq f(x) \leq \mathrm{s}_2$ according to Theorems 5.1 and 5.2. A certain condition must be met by the parameters defining the iterated function system to guarantee that the graph of the RCFI stays entirely contained within the rectangle $[x_1,\ x_N] \times [\mathrm{s}_1,\ \mathrm{s}_2]$.*
(i) To ensure the intended interpolation characteristics, we choose each vertical scaling factor α_i from the interval (α_i^L, α_i^R) for $i = 1, 2, \ldots, N.$,

$$\text{where}\quad \alpha_i^R = min\left[a_i, \frac{t_i - \mathrm{s}_1}{t_1 - \mathrm{s}_1}, \frac{t_{i+1} - \mathrm{s}_1}{t_N - \mathrm{s}_1}, \frac{\mathrm{s}_2 - t_i}{\mathrm{s}_2 - t_1}, \frac{\mathrm{s}_2 - \mathrm{y}_{i+1}}{\mathrm{s}_2 - t_N}\right],$$

$$\alpha_i^L = max\left[-a_i, \frac{t_i - \mathrm{s}_1}{t_1 - \mathrm{s}_2}, \frac{t_{i+1} - \mathrm{s}_1}{t_N - \mathrm{s}_2}, \frac{\mathrm{s}_2 - t_i}{\mathrm{s}_1 - t_1}, \frac{\mathrm{s}_2 - t_{i+1}}{\mathrm{s}_1 - t_N}\right],$$

(ii) The shape parameter r_i is selected as

$$r_i > max\Bigg[0, \frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^l(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^l(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*|},$$
$$\frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^l(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1}))) - \alpha_i(t_N - \mathbb{T}_i^l(x_N))},$$
$$\frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^u(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*|},$$
$$\frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N)] - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^u(x_N) - t_N)}\Bigg].$$

$$r_i > max\Bigg[0, \frac{(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1)) + h_i\Delta_i - \alpha_i\Delta_1|J^*|}{2[(t_i - \mathbb{T}_i^l(x_i)) - \alpha_i(t_1 - \mathbb{T}_i^u(x_1))] + h_i\Delta_i - \alpha_i\Delta_1|J^*|},$$
$$\frac{-[(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N)) - h_i\Delta_{i+1} + \alpha_i\Delta_N|J^*|]}{(t_{i+1} - \mathbb{T}_i^l(x_{i+1})) - \alpha_i(t_N - \mathbb{T}_i^u(x_N))},$$
$$\frac{(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1) - h_i\Delta_i + \alpha_i\Delta_1|J^*|}{2[(\mathbb{T}_i^u(x_i) - t_i) - \alpha_i(\mathbb{T}_i^l(x_1) - t_1)] - h_i\Delta_i + \alpha_i\Delta_1|J^*|},$$
$$\frac{-[(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N) - h_i\Delta_{i+1} + \alpha_i\Delta_N)|J^*|]}{(\mathbb{T}_i^u(x_{i+1}) - t_{i+1}) - \alpha_i(\mathbb{T}_i^l(x_N) - t_N)}\Bigg].$$

6. Result and discussion

This section provides various examples of how to create a constrained RCFI, in which the graphs are constrained by two piecewise linear functions and straight lines that remain within a defined rectangular area. This condition happens when $\mathbb{T}_i^u$ and $\mathbb{T}_i^l$ contain the provided interpolation data. Generally, if we construct an RCFI using arbitrary IFS parameters, it might fall outside these ranges. We set specific limits on the scaling factor α_i and the shape parameters r_i based on the data, as explained in section 6, to ensure that the RCFI keeps the structure of the restricted data. To evaluate the accuracy of the proposed RCFI and confirm its efficacy in comparison to conventional methods, we analyze numerous numerical examples. We created an algorithm that employs input data points, scaling factors, and form parameters to repeatedly apply the IFS approach to generate the RCFI graph.

6.1. Example of the constrained RCFI lies between two piecewise lines:

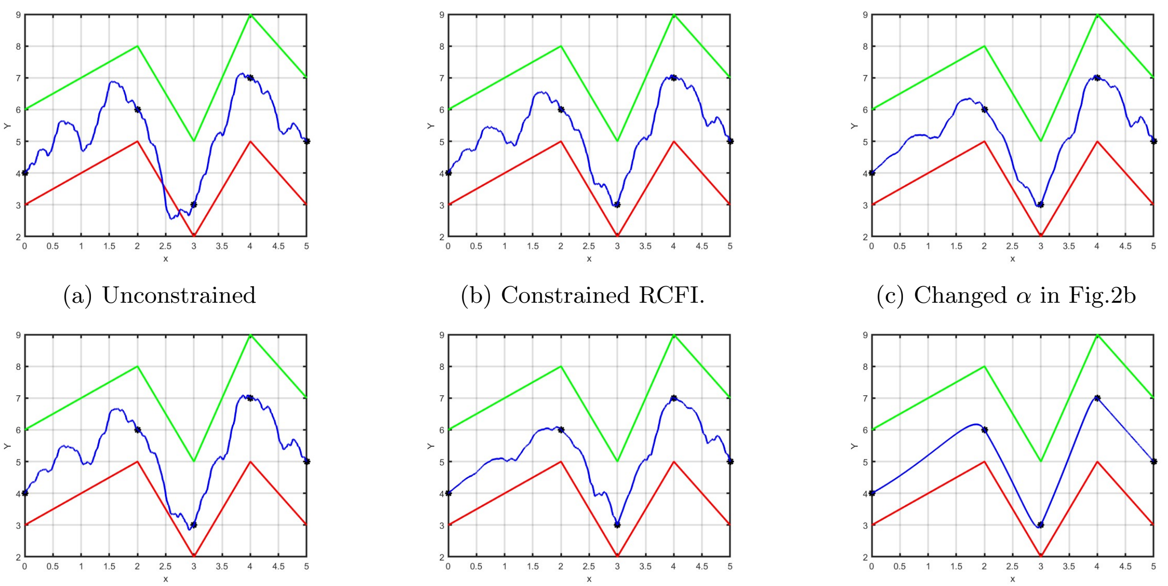
Examine a dataset $\{(x_i, t_i)_{i=0}^{6} = (0,\ 4), (2,\ 6), (3,\ 3), (4,\ 7), (5,\ 5), (6,\ 8)\}$ that is restricted by two piecewise linear functions. An extra point $(6,\ 8)$ is added to make this into an acceptable interpolation dataset with related upper and lower bounds $\mathbb{T}_i^l$ and $\mathbb{T}_i^u$. This feature allows derivatives at the grid points to be approximated by setting $\Delta_i = \frac{t_{i+1}-t_i}{x_{i+1}-x_i}$.

$$\mathbb{T}^l = \begin{cases} x+3 & \text{if} \quad 0 \leq x \leq 2 \\ -3x+11 & \text{if} \quad 2 \leq x \leq 3 \\ 3x-7 & \text{if} \quad 3 \leq x \leq 4 \\ -2x+13 & \text{if} \quad 4 \leq x \leq 5 \end{cases} \quad \text{and} \quad \mathbb{T}^u = \begin{cases} x+6 & \text{if} \quad 0 \leq x \leq 2 \\ -3x+14 & \text{if} \quad 2 \leq x \leq 3 \\ 4x-7 & \text{if} \quad 3 \leq x \leq 4 \\ -2x+17 & \text{if} \quad 4 \leq x \leq 5 \end{cases}$$

To build restricted rational cubic fractal models (RCFIs), Figure 2 shows how the RCFI is structured and emphasizes how shape parameter choices affect the results. According to Table 2, when the IFS parameters change in Figure 2(a), the RCFI is no longer contained inside the two piecewise linear limits. However, Figure 2(b) shows that when the shape parameters and scaling factors meet the criteria in Theorem 2, the RCFI stays within the boundaries. The resultant RCFI is shown in Figure 2(c) when the scaling factor α_1 and the shape parameter r are changed. Likewise, r and α_2 variations are reflected in the RCFI in Figure 2(d). Additionally, beginning with the configuration in Figure 2(b), the limited RCFI in Figure 2(e) is obtained by suitably adjusting the scaling and geometric parameters corresponding to α_3 and r. In Figure 2(f), the restricted RRFIF is the consequence of varying the scaling parameters r and α_4. In summary, Figures 2(a) and (b) depict the unconstrained situation, the restricted configuration is established, and the effects of certain parameter adjustments are examined in Figures 2(c)–2(f).

Table 1: Scale control parameters and tension parameters in constructing the piecewise lines RCFI in Figure

Scale factor (α)	Fig	r	Fig
[0.36 0.18 0.16 0.15]	2a	[2 2 2 2 2]	2a
[0.31 0.17 0.19 0.18]	2b,d	[10 10 10 10 10]	2b,c,f
[0.17 0.11 0.15 0.19]	2c	[5 5 5 5 5]	2d
[0.14 0.19 0.15 0.17]	2e	[100 100 100 100 100]	2e
[0 0 0 0]	2f		



(a) Unconstrained (b) Constrained RCFI. (c) Changed α in Fig.2b

(d) Changed r in Fig.2b. (e) Changed α & r in Fig.2b. (f) Classical of RCFI in Fig.2b.

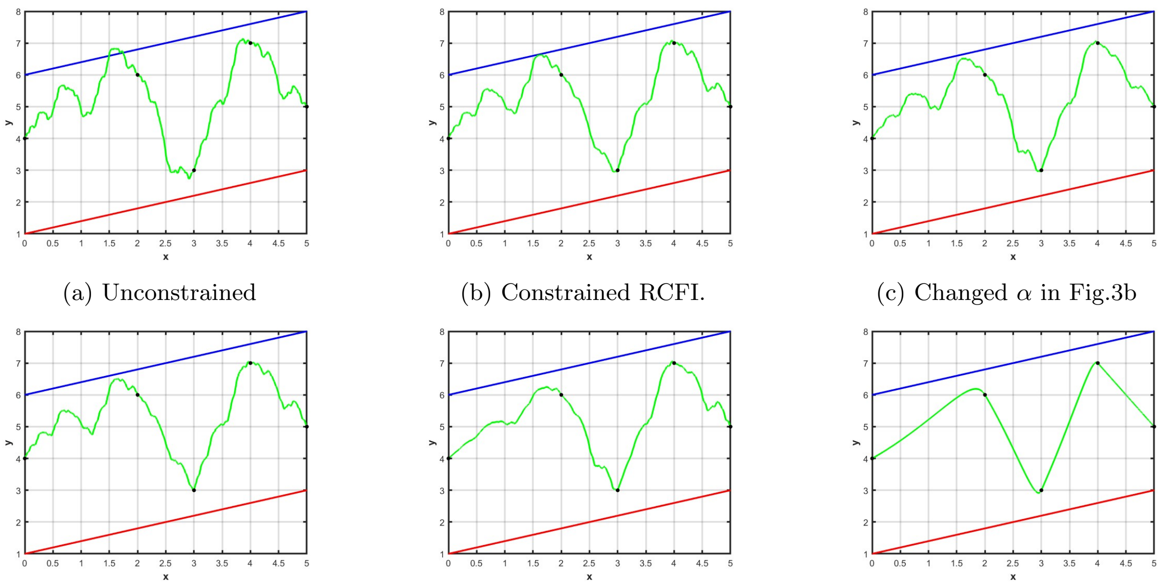
Figure 2: RCFI lies between piecewise lines

6.2. Example of constrained RCFI lies between two straight lines:

The dataset $\{(x_i, t_i)_{i=0}^{6} = (0,\ 4), (2,\ 6), (3,\ 1), (4,\ 7), (5,\ 5), (6,\ 8)\}$, shows that the data points are located both above and below the two boundary lines, $\mathbb{T}^l = \frac{2x}{5} + 1$ and $\mathbb{T}^u = \frac{2x}{5} + 6$. The effects of shape parameter selections on the RCFI structure are shown in Figure 3.The unconstrained Rational Cubic Fractal Interpolation, as depicted in Figure 3(a), exceeds the defined bounding limits. Parameter adjustments based on the provided theorem 5.2 yield the constrained interpolation seen in Figure 3(b). Figures 3(c) to 3(e) systematically examine how tuning scaling and shape parameters modifies the interpolation relative to Figure 3(b). In Figure 3(f), a completely bounded RCFI is obtained by setting all scaling coefficients to zero.

Table 2: Scale control parameters and tension parameters in constructing the piecewise lines RCFI in Figure

Scale factor (α)	Fig	r	Fig
[0.38 0.19 0.19 0.19]	3a	[3 3 3 3 3]	3a
[0.35 0.12 0.18 0.17]	3b,d	[9 9 9 9 9]	3b,c,f
[0.29 0.19 0.14 0.11]	3c	[40 40 40 40 40]	3d
[0.15 0.16 0.13 0.18]	3e	[15 15 15 15 15]	3e
[0 0 0 0]	3f		



(a) Unconstrained (b) Constrained RCFI. (c) Changed α in Fig.3b

(d) Changed r in Fig.3b. (e) Changed α & r in Fig.2b. (f) Classical of RCFI in Fig.3b.

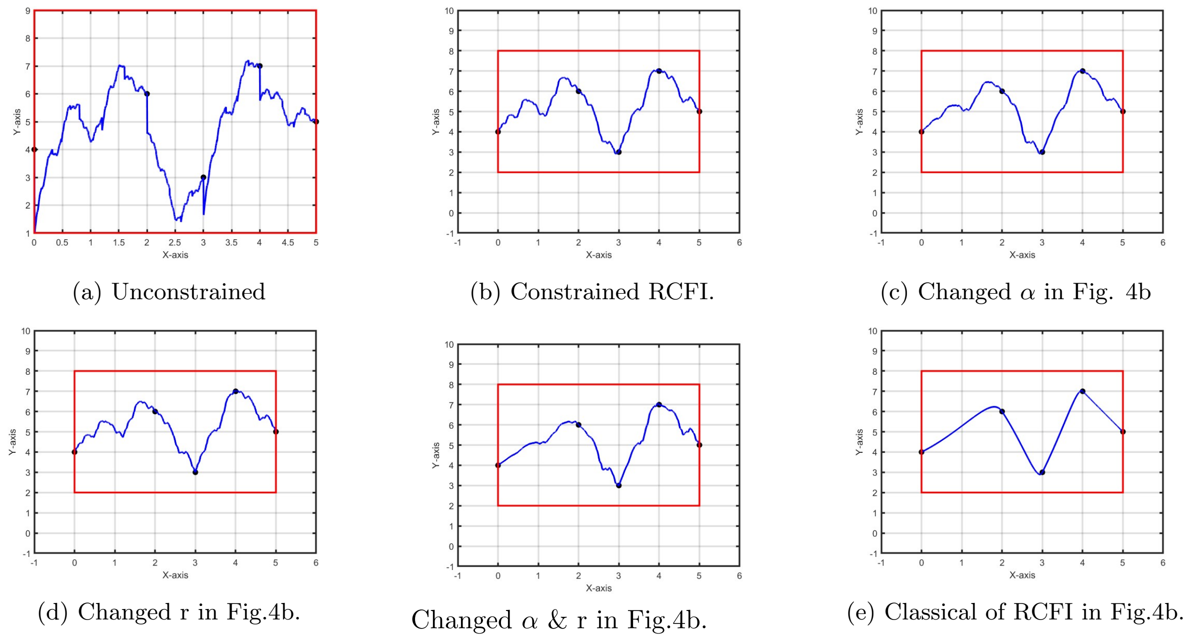
Figure 3: RCFI lies between straight lines.

6.3. Example of the constrained RCFI within the Rectangle:

The dataset $\{(x_i,\ t_i)\}$ demonstrates how varying IFS parameters affect the structure of the RCFI within the domain $[0, 5] \times [2, 8]$, as shown in Figure 4. Figure 4(a) shows the unconstrained case where the RCFI exceeds boundaries. Figure 4(b) presents a constrained RCFI achieved by selecting parameters per Remark 5.2. Figures 4(c)–4(e) show effects of modifying scaling and shape parameters relative to 4(b), while 4(f) displays a fully restricted RCFI with all scaling factors set to zero.

Table 3: Scale control parameters and tension parameters in constructing the rectangle lines RCFI in Figure

Scale factor (α)	Fig	r	Fig
[0.39 0.18 0.18 0.16]	4a	[1 1 1 1 1]	4a
[0.36 0.17 0.18 0.17]	4b,d	[7 7 7 7 7]	4b,c,f
[0.22 0.19 0.11 0.12]	4c	[100 100 100 100 100]	4d
[0.12 0.19 0.17 0.19]	4e	[20 20 20 20 20]	4e
[0 0 0 0]	4f		



(a) Unconstrained

(b) Constrained RCFI.

(c) Changed α in Fig. 4b

(d) Changed r in Fig.4b.

Changed α & r in Fig.4b.

(e) Classical of RCFI in Fig.4b.

Figure 4: RCFI lies within the rectangle bounds.

7. Conclusion

In this work, we developed a constrained rational cubic fractal interpolation using function values that preserves key form properties while modeling complicated, irregular data. The approach provides a linear denominator and a cubic numerator. A single shape-control parameter plus various constraint techniques, such as piecewise linear limits, straight-line conditions, and rectangular enclosures, guarantees that the interpolated curve complies with the intended geometric and functional characteristics. A systematic parameter selection process and theoretical convergence analysis serve to further support the approach's resilience and flexibility. Numerical tests show that the technique may produce fractal interpolants that are both aesthetically pleasing and preserve form on datasets. These findings confirm that the proposed framework is suitable for practical applications in data visualization, geometric modeling, and scientific computing, where dealing with irregularities and enforcing constraints is important.

References

1. Abdulla, Sana, and K. Mahipal Reddy. Machine Learning Clustering Techniques for Segmenting IFS-Generated Garment Patterns. *Proceedings of the 2024 International Conference on Intelligent Computing and Emerging Communication Technologies (ICEC)*, IEEE, 2024, pp. 1–6.

2. Balasubramani, N., and Gowrisankar, A. Affine Recurrent Fractal Interpolation Functions. *The European Physical Journal Special Topics*, vol. 230, no. 21, 2021, pp. 3765–3779.

3. Balasubramani, N., Prasad, M.G., and Natesan, S. Shape-Preserving α-Fractal Rational Cubic Splines. *Calcolo*, vol. 57, no. 3, 2020, pp. 21.

4. Banerjee, S., Gowrisankar, A., and Reddy, K.M. *Fractal Patterns with MATLAB*. Springer, 2023.

5. Barnsley, M.F. Fractal Functions and Interpolation. *Constructive Approximation*, vol. 2, 1986, pp. 303–329.

6. Barnsley, M.F., and Harrington, A.N. The Calculus of Fractal Interpolation Functions. *Journal of Approximation Theory*, vol. 57, no. 1, 1989, pp. 14–34.

7. Barnsley, M.F. *Fractals Everywhere*. Academic Press, 2014.

8. Chand, A.K.B., and Tyada, K.R. Constrained Shape-Preserving Rational Cubic Fractal Interpolation Functions. *The Rocky Mountain Journal of Mathematics*, vol. 48, no. 1, 2018, pp. 75–105.

9. Drakopoulos, V., and Vijender, N. Univariable Affine Fractal Interpolation Functions. *Theoretical and Mathematical Physics*, vol. 207, no. 3, 2021, pp. 689–700.

10. Katiyar, S.K., Chand, A.K.B., and Kumar, G.S. A New Class of Rational Cubic Spline Fractal Interpolation Function and Its Constrained Aspects. *Applied Mathematics and Computation*, vol. 346, 2019, pp. 319–335.

11. K. Mahipal Reddy, and Chand, A.K.B. Constrained Univariate and Bivariate Rational Fractal Interpolation. *International Journal for Computational Methods in Engineering Science and Mechanics*, vol. 20, no. 5, 2019, pp. 404–422.

12. K. Mahipal Reddy, Chand, A.K.B., and Viswanathan, P. Data Visualization by Rational Fractal Function Based on Function Values. *The Journal of Analysis*, vol. 28, 2020, pp. 261–277.

13. K. Mahipal Reddy, Saravana Kumar, G., and Chand, A.K.B. Family of Shape-Preserving Fractal-Like Bézier Curves. *Fractals*, vol. 28, no. 6, 2020, pp. 2050105.

14. K. Mahipal Reddy and Vijender, N. A Fractal Model for Constrained Curve and Surface Interpolation. *The European Physical Journal Special Topics*, vol. 232, no. 7, 2023, pp. 1015–1025.

15. Vijay, and Chand, A.K.B. Convexity-Preserving Rational Cubic Zipper Fractal Interpolation Curves and Surfaces. *Mathematical and Computational Applications*, vol. 28, no. 3, 2023, pp. 74.

16. Vijay, and Chand, A.K.B. Shape-Preserving Aspects of a Novel Class of Bi-Cubic Partially Blended Rational Zipper Fractal Interpolation Surfaces. *Journal of Analysis*, vol. 32, no. 6, 2024, pp. 3475–3505.

17. Vijay, Saravana Kumar, G., and Chand, A.K.B. A Comprehensive Discussion on Various Methods of Generating Fractal-Like Bézier Curves. *Computational and Applied Mathematics*, vol. 43, no. 6, 2024, pp. 368.

L. Rajesh and K. Mahipal Reddy
Department of Mathematics,
VIT-AP UNIVERSITY,
INDIA.
E-mail address: mahipal.reddy@vitap.ac.in