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RT-conjugate codes in the Rosenbloom-Tsfasman metric

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ABSTRACT: Linear Complementary Dual (LCD) codes are a special class of linear error-correcting code used in data transmission and storage. These codes possess specific algebraic properties that make them useful in applications, such as communication systems, cryptography, and data storage devices. These are particularly valuable in scenarios that require a high degree of error detection and correction. This study explores the characteristics of RT-conjugate codes within the Rosenbloom-Tsfasman metric (RT-metric). In this study, we focus on a specific subclass of LCD codes characterized by conjugate conditions. In particular, we establish sufficient conditions under which a linear code in the RT metric qualifies as an LCD code through its conjugate structure. We also analyzed the weight distribution of the dual of these codes in terms of their type and proposed several construction methods for RT-conjugate codes.

Key Words: linear code, LCD code, RT-conjugate code, RT-metric, dual code.

Contents

1. Introduction

Linear Complementary Dual (LCD) code has a wide range of applications in cryptography, communication systems, and data storage. Massey first introduced these codes [1] and demonstrated that they offer an optimal linear coding method for a two-user binary adder channel. Subsequently, Yang and Massey [2] identified a necessary and sufficient condition for classifying cyclic codes as LCD codes. Carlet et al. [3] explored the existence of q-ary LCD MDS codes. They successfully addressed this issue in the Euclidean case and introduced certain classes of Hermitian LCD MDS codes. Research on LCD codes, including Euclidean, Galois, Hermitian, and σ-LCD codes, is theoretically and practically important. Several studies have been conducted in this area (see [4,5,6,7,8,9,10]).

The Rosenbloom-Tsfasman metric (RT-metric) was first introduced by Rosenbloom and Tsfasman [11] in coding theory. Similarly, this metric was independently introduced by Martin and Stinson [13] and Skriganov [14] in the context of uniform distribution theory. As a generalization of the classical Hamming metric, the RT-metric has rapidly attracted the interest of coding theorists, resulting in a continuous stream of research on the codes defined by this metric. Much of this research has focused on various aspects, including bounds on codes [15], weight distributions and MacWilliams identities [17,16,18,19], linearity properties [20,21,22], maximum distance separability [14,23], automorphism groups [24], burst error enumeration [26,25,27], covering properties [28], normality [29], construction of self-dual codes [30], the existence of LCD codes [31], and properties of reversible codes [32] across different algebraic structures.

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The inner product used in the RT-metric (see [14]) differs from the conventional inner product, which defines duality in the Hamming metric. Consequently, many LCD codes in the Hamming metric do not retain this property in the RT-metric. Therefore, it is essential to investigate the existence of LCD codes within this different inner product framework and study the characteristics of such codes if they are found. The aim of this investigation was to address this issue by outlining the necessary criteria for classifying linear code in the RT-metric as an LCD code. Additionally, we explore the potential weight distribution of their dual codes, which we describe in terms of the "type" of the code. These codes are effective in handling asymmetric and burst errors in delay-sensitive communication systems under the RT-metric [11]. The algebraic structure of LCD codes enables efficient decoding, enhances error detection and correction, and provides cryptographic advantages such as resistance to side-channel attacks [12]. These properties make them suitable for applications in communication, cryptography, and data storage.

The organization of the paper is as follows. In Section 2, we present the basic definitions and concepts that are essential for the results discussed in the subsequent sections. A linear code of dimension k over $\mathbb{F}_q$ in the RT metric has exactly k distinct non-zero weights. Based on this observation, Section 3 introduces and formally defines RT-conjugate codes, and establishes sufficient conditions under which such codes are LCD under the RT metric. The relationship between the Hamming and RT metrics is also discussed in this section. In Section 4, we investigate the duals of these codes, their weight distributions, and their covering radii. Section 5 presents some construction methods along with illustrative examples. Finally, Section 6 concludes the paper.

2. Preliminaries

The RT-distance between two vectors $x = (x_1, x_2, \ldots, x_\eta)$ and $y = (y_1, y_2, \ldots, y_\eta)$ in space $\mathbb{F}_q^\eta$ is determined by the maximum index i where the corresponding components of x and y differ provided that $1 \leq i \leq \eta$. This can be expressed as $d_\rho(x, y) = \max\{i \mid x_i \neq y_i\}$. The subsets of $\mathbb{F}_q^\eta$ equipped with this metric are called q-ary RT-metric codes or simply q-ary codes in the RT-metric. If these subsets form vector spaces, they are referred to as linear RT-metric codes.

For k-dimensional linear code $\mathscr{C} \subseteq \mathbb{F}_q^\eta$, the generator matrix G, $k \times \eta$, contains rows that form the basis of code $\mathscr{C}$. A set of k linearly independent columns from G is referred to as the information set for $\mathscr{C}$.

RT-ball $B_\rho(x; r)$, or ρ-ball, is defined as the set $\{y \in \mathbb{F}_q^\eta \mid d_\rho(x, y) \leq r\}$, where $x \in \mathbb{F}_q^\eta$ is the center and r is the radius. The packing radius of a code is the maximum radius r for which ρ-balls centered at distinct codewords do not intersect. Conversely, the covering radius is the minimum radius R such that ρ-balls centered at codewords cover the entire space $\mathbb{F}_q^\eta$. The code is considered perfect if its packing radius is equal to its covering radius. As introduced in [29], the concept of partition number provides a simplified approach for determining the covering radius for codes in the RT-metric.

For an RT-metric code $\mathscr{C}$ of minimum ρ-distance d_ρ and length η, the Singleton bound is given by $|\mathscr{C}| \leq q^{\eta - d_\rho + 1}$. Linear codes with dimension k were translated into $k \leq \eta - d_\rho + 1$. Codes that meet this bound are referred to as the Maximum Distance Separable (MDS). Unless otherwise specified, all the codes discussed in this paper are RT-metric codes over $\mathbb{F}_q$.

RT-metric code $\mathscr{C}$ is classified based on its specific structural properties. It is called self-orthogonal if it is entirely contained within its dual code $\mathscr{C}^\perp$. A code is considered self-dual when it satisfies equality $\mathscr{C} = \mathscr{C}^\perp$. By contrast, an LCD code is characterized by having no nonzero codewords in common with its dual. Additionally, code $\mathscr{C}$ is termed reversible if, for every vector $(v_1, v_2, \ldots, v_\eta)$ in $\mathscr{C}$, its reversed form $(v_\eta, v_{n-1}, \ldots, v_2, v_1)$ also belongs to $\mathscr{C}$.

We use the notations $[\eta, k, \tau_\rho]_q$ to denote a q-ary linear code with minimum ρ-distance d_ρ, dimensions k, and length η. In addition, the notation $[\eta]$ represents the set $\{1, 2, \ldots, \eta\}$.

3. RT-conjugate codes in the RT-metric

Definition 3.1 *(see [30]): Consider the set* $[\eta] = \{1, 2, \ldots, \eta\}$. *Two elements* $\mu, \nu \in [\eta]$ *are said to be conjugate if they satisfy the condition* $\mu + \nu = \eta + 1$.

Definition 3.2 *(see [30]): Let* $A = (a_{ij})$ *be a* $p \times r$ *matrix. The flip of the matrix* A, *denoted by* $Flip(A)$, *is defined by* $Flip(A) = (a_{ik})$ *where* $k = r - j + 1$ *for* $1 \leq i \leq p$ *and* $1 \leq j \leq r$.

Definition 3.3 *(see [30]): Any generator matrix G of the linear code $\mathscr{C}$ is equivalent to G'. A linear code having G' as its generator matrix in standard form is said to be of type $(\tau_1, \tau_2, \ldots, \tau_k)$.*

Definition 3.4 *If $\mathscr{C}$ is $[\eta, k, \tau]$ RT-metric code of type $(\tau_1, \tau_2, \ldots, \tau_k)$ is said to be RT-conjugate if τ_i's are pairwise conjugates.*

Theorem 3.1 *Let $\mathscr{C}$ be any $[\eta, k, \tau]$ RT-metric code of type $(\tau_1, \tau_2, \ldots, \tau_k)$ over $\mathbb{F}_q$. Let the generator matrix of $\mathscr{C}$ in the standard form (see [30]) be*

$$G = \begin{bmatrix} a_{1,1} & \ldots & a_{1,\tau_1-1} & a_{1,\tau_1} & 0 & \ldots & 0 & \ldots & 0 & 0 \\ a_{2,1} & \ldots & a_{2,\tau_1-1} & 0 & a_{2,\tau_1+1} & \ldots & a_{2,\tau_2} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1,1} & \ldots & a_{k-1,\tau_1-1} & 0 & a_{k-1,\tau_1+1} & \ldots & 0 & \ldots & a_{k-1,\tau_{k-1}} & 0 \\ a_{k,1} & \ldots & a_{k,\tau_1-1} & 0 & a_{k,\tau_1+1} & \ldots & 0 & \ldots & 0 & a_{k,\tau_k} \end{bmatrix}$$

where $a_{i,\tau_i} \neq 0$, $a_{i,\tau_j} = 0$ for $j \neq i$ and $\tau_1, \tau_2, ..., \tau_k$ is the set of k possible RT-weights such that $1 \leq \tau = \tau_1 < \tau_2 < ... < \tau_k \leq \eta$ and $\tau_i = \eta - \tau_{k-i+1} + 1$, for every $i = 1, 2, \ldots, k$ (all τ_i are pairwise conjugates). Then, $\mathscr{C}$ is LCD in the RT-metric.

Proof. Given $\mathscr{C}$ is any $[\eta, k, \tau]$ RT-metric code of type $(\tau_1, \tau_2, \ldots, \tau_k)$ over $\mathbb{F}_q$, Standard form of generator matrix of $\mathscr{C}$:

$$G = \begin{bmatrix} a_{1,1} & \ldots & a_{1,\tau_1-1} & a_{1,\tau_1} & 0 & \ldots & 0 & \ldots & 0 & 0 \\ a_{2,1} & \ldots & a_{2,\tau_1-1} & 0 & a_{2,\tau_1+1} & \ldots & a_{2,\tau_2} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1,1} & \ldots & a_{k-1,\tau_1-1} & 0 & a_{k-1,\tau_1+1} & \ldots & 0 & \ldots & a_{k-1,\tau_{k-1}} & 0 \\ a_{k,1} & \ldots & a_{k,\tau_1-1} & 0 & a_{k,\tau_1+1} & \ldots & 0 & \ldots & 0 & a_{k,\tau_k} \end{bmatrix}$$

where $a_{i,\tau_i} \neq 0$, $a_{j,\tau_j} = 0$ for $j \neq i$ and $\tau_1, \tau_2, \ldots, \tau_k$ is the set of k possible RT-weights such that $1 \leq \tau = \tau_1 < \tau_2 < \cdots < \tau_k \leq \eta$ and satisfies the condition $\tau_i = \eta - \tau_{k-i+1} + 1$, where $i = 1, 2, \ldots, k$. From Theorem 3.4 in [31]," G is a generator matrix for the $[\eta, k, \tau]$ linear code in the RT-metric over $\mathbb{F}_q$. Then, $\mathscr{C}$ is the LCD code if and only if the $k \times k$ matrix $GG^{\diamond}$ is non-singular, where $G^{\diamond} = [Flip(G)]^T$", which implies that

$$GG^{\diamond} = \begin{bmatrix} a_{1,1} & \ldots & a_{1,\tau_1-1} & a_{1,\tau_1} & 0 & \ldots & 0 & \ldots & 0 & 0 \\ a_{2,1} & \ldots & a_{2,\tau_1-1} & 0 & a_{2,\tau_1+1} & \ldots & a_{2,\tau_2} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1,1} & \ldots & a_{k-1,\tau_1-1} & 0 & a_{k-1,\tau_1+1} & \ldots & 0 & \ldots & a_{k-1,\tau_{k-1}} & 0 \\ a_{k,1} & \ldots & a_{k,\tau_1-1} & 0 & a_{k,\tau_1+1} & \ldots & 0 & \ldots & 0 & a_{k,\tau_k} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \ldots & 0 & a_{k,\tau_k} \\ 0 & 0 & \ldots & a_{k-1,\tau_{k-1}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{2,\tau_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{2,\tau_1+1} & \ldots & a_{k-1,\tau_1+1} & a_{k,\tau_1+1} \\ a_{1,\tau_1} & 0 & \ldots & 0 & 0 \\ a_{1,\tau_1-1} & a_{2,\tau_1-1} & \ldots & a_{k-1,\tau_1-1} & a_{k,\tau_1-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1,1} & a_{2,1} & \ldots & a_{k-1,1} & a_{k,1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 & a_{1,,\tau_1}a_{k,\tau_k} \\ 0 & 0 & \dots & a_{2,\tau_2}a_{k-1,\tau_{k-1}} & a'_{2,k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{k-1,\tau_{k-1}}a_{2,\tau_2} & \dots & a'_{k-1,k-1} & a'_{k-1,k} \\ a_{k,\tau_k}a_{1,\tau_1} & a'_{k,2} & \dots & a'_{k,k-1} & a'_{k,k} \end{bmatrix}$$

because $a_{i,\tau_i} \neq 0$ and $a'_{i,j} \in \{0, 1, \dots, q-1\}$ are the elements below the anti-diagonal elements. These anti-diagonal in $k \times k$ matrix are nonzero. This implies that, $GG^{\diamond}$ is non-singular and hence, $\mathscr{C}$ over $\mathbb{F}_q$ is the LCD in the RT-metric.

Example 3.1 Let $\mathscr{C}$ be a $[6,4,1]$ RT-conjugate code in the RT-metric over $GF(2)$, whose generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 2, \tau_3 = 5 \textit{ and } \tau_4 = 6$. Hence, $\mathscr{C}$ is an LCD code. This is an example to supports theorem 3.1.

Example 3.2 Let $\mathscr{C}$ be a $[7,5,1]$ RT-conjugate code in the RT-metric over $GF(5)$, whose generator matrix is:

$$G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 3, \tau_3 = 4, \tau_4 = 5 \textit{ and } \tau_5 = 7$. Therefore, $\mathscr{C}$ is an LCD code. This is an example to support Theorem 3.1.

Remark 3.1 *As illustrated in examples 3.3, 3.4, and 3.5, although RT-conjugate code $\mathscr{C}$ is an LCD code in the RT-metric, it need not be an LCD code in the Hamming metric. However, in certain cases, these are LCD codes in the Hamming metric, as shown in Theorem 3.2.*

Example 3.3 Let $\mathscr{C}$ be a $[7,4,2]$ RT-conjugate code in the RT-metric over $GF(2)$, whose generator matrix is:

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Here $\tau_1 = 2, \tau_2 = 3, \tau_3 = 5 \textit{ and } \tau_4 = 6$. Therefore, $\mathscr{C}$ is an LCD in Hamming metric.

Example 3.4 Let $\mathscr{C}$ be a $[7,3,1]$ RT-conjugate code in the RT-metric over $GF(2)$, whose generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 4, \textit{ and } \tau_3 = 7$. Therefore, $\mathscr{C}$ was not an LCD code in Hamming metric.

Example 3.5 Let $\mathscr{C}$ be a $[6,2,1]$ RT-conjugate code in the RT-metric over $GF(5)$, whose generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 3, \tau_3 = 4, \tau_4 = 5$ *and* $\tau_5 = 7$. Therefore, $\mathscr{C}$ was not an LCD in Hamming metric.

Theorem 3.2 *If $\mathscr{C}$ is $[\eta, k, \tau]$ RT-conjugate reversible code over $\mathbb{F}_q$, then $\mathscr{C}$ is an LCD code in Hamming metric.*

Proof.

$$\begin{aligned} \text{Consider } \mathscr{C} \text{ as an LCD code in the RT-metric} &\Leftrightarrow GG^{\diamond} \text{ is non-singular} \\ &\Leftrightarrow G(Flip(G))^T \text{ is non-singular} \\ &\Leftrightarrow G(PG)^T \text{ is non-singular} \\ &\qquad (\because\ \mathscr{C} \text{ is RT-conjugate reversible}) \\ &\Leftrightarrow GG^T P^T \text{ is non-singular} \\ &\qquad (\because \text{P is non-singular}) \\ &\Leftrightarrow GG^T \text{ is non-singular} \\ &\Leftrightarrow \mathscr{C} \text{ is LCD in the Hamming metric} \end{aligned}$$

Example 3.6 Let $\mathscr{C}$ be a $[6,2,1]$ RT-conjugate code in the RT-metric over $GF(5)$, whose generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 3, \tau_3 = 4, \tau_4 = 5$ *and* $\tau_5 = 7$. Hence, $\mathscr{C}$ is an LCD code in the Hamming metric. This is an example to illustrates Theorem 3.2.

Theorem 3.3 *If $\mathscr{C}$ is $[\eta, k, \tau]$ RT-conjugate code with minimum distance $\tau > 1$, then the covering radius of $\mathscr{C}$ is η.*

Proof. Based on this hypothesis, the minimum distance is $\tau = \tau_1 > 1$. It follows that $1 < \tau_1 = \eta - \tau_k + 1$, which implies $1 < \tau_1 < \eta$ and $1 < \tau_1 = \eta - \tau_k + 1 < \eta$. Consequently, $0 < \eta - \tau_k$; therefore, at least one coordinate exists in all codewords that are zero, and the maximum distance τ_k of $\mathscr{C}$ is less than η. Thus, the last $\eta - \tau_k$ coordinates of all codewords in $\mathscr{C}$ are zero, which implies that the partition number of $\mathscr{C}$ is zero. Hence, the covering radius of $\mathscr{C}$ is η.

Example 3.7 Let $\mathscr{C}$ be a $[7,4,2]$ RT-conjugate code in the RT-metric over $GF(3)$, whose generator matrix is:

$$G = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Here $\tau_1 = 2, \tau_2 = 3, \tau_3 = 5$ *and* $\tau_4 = 6$. Hence, the covering radius of $\mathscr{C}$ is 7. This is an example to support Theorem 3.3.

Theorem 3.4 *If $\mathscr{C}$ is an $[\eta, k, \tau]$ RT-metric linear code of type $(\tau_1, \tau_2, \ldots, \tau_k)$, and if at least one pair of τ_i's are pairwise conjugates with $\eta > k$, then $\mathscr{C}$ cannot be an MDS code.*

Proof. Theorem 3.4 can be easily proven using the definition of RT-conjugate codes, notations, and simple algebraic manipulations.

Example 3.8 Let $\mathscr{C}$ be a $[5,4,1]$ RT-conjugate code in the RT-metric over $GF(3)$, whose generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 1, \tau_2 = 3, \tau_3 = 4$ *and* $\tau_4 = 5$. Hence, $\mathscr{C}$ is not an MDS code. This is an example to support Theorem 3.4.

4. On duality of RT-conjugate codes in the RT-metric

To establish MacWilliams-type identities for codes in the RT-metric, a specialized inner product in space $Mat_{m\times\eta}(\mathbb{F}_q)$ was introduced in [13]. This inner product is crucial for studying codes in the RT-metric, as it leads to significant results such as the fact that the dual MDS code under this inner product is also an MDS. This inner product plays a central role in shaping the properties and duality of codes within the RT-metric framework.

Theorem 4.1 *Let $\mathscr{C}$ be an $[\eta, k, \tau]$ RT-conjugate code of type $(\tau_1, \tau_2, \dots, \tau_k)$ over $\mathbb{F}_q$: The dual code $\mathscr{C}^\perp$ is also an $[\eta, \eta-k, \tau^\perp]_q$ RT-conjugate code for $\mathbb{F}_q$. Additionally, the relationship between the sets of RT-weights of $\mathscr{C}$ and $\mathscr{C}^\perp$ is such that the RT-weights of $\mathscr{C}^\perp$, $\{\tau_1^\perp, \tau_2^\perp, \dots, \tau_{\eta-k}^\perp\}$, are precisely the complement of the RT-weights of $\mathscr{C}$, i.e., $[\eta] \setminus \{\tau_1, \tau_2, \dots, \tau_k\}$.*

Proof. From Theorem 2 in [30], " the dual $\mathscr{C}^\perp$ of $\mathscr{C}$ is a $[\eta, \eta-k, \tau^\perp]_q$ linear code of type $(\tau_1^\perp, \tau_2^\perp, \dots, \tau_{\eta-k}^\perp)$ such that $\{\tau_1^\perp, \tau_2^\perp, \dots, \tau_{\eta-k}^\perp\} = [\eta] \setminus \{\eta - \tau_1 + 1, \eta - \tau_2 + 1, \dots, \eta - \tau_k + 1\}$". As $\mathscr{C}$ is an RT conjugate, $\{\tau_1^\perp, \tau_2^\perp, \dots, \tau_{\eta-k}^\perp\} = [\eta] \setminus \{\tau_1, \tau_2, \dots, \tau_k\}$.

Example 4.1 Let $\mathscr{C}$ be a $[7,4,2]$ RT-conjugate code in the RT-metric over $GF(3)$, whose generator matrix is:

$$G = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 \end{bmatrix} \text{ and } G^\perp = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 2, \tau_2 = 3, \tau_3 = 5$ *and* $\tau_4 = 6$. Hence, $\{\tau_1^\perp, \tau_2^\perp, \tau_3^\perp\} = [7] \setminus \{2,3,5,6\} = \{1,4,7\}$. This is an example to support Theorem 4.1.

Example 4.2 Let $\mathscr{C}$ be a $[6,4,3]$ MDS code in the RT-metric over $GF(2)$, whose generator matrix is:

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } G^\perp = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\tau_1 = 3, \tau_2 = 4, \tau_3 = 5$ *and* $\tau_4 = 6$. Hence, $\{\tau_1^\perp, \tau_2^\perp\} = [7] \setminus \{3,4,5,6\} \neq \{5,6\}$. This is an example to support Theorem 4.1.

Theorem 4.2 *If $\mathscr{C}$ is $[\eta, k, \tau]$ RT-conjugate code of type $(\tau_1, \tau_2, \dots, \tau_k)$ with minimum distance $\tau = 1$, then the covering radius of $\mathscr{C}^\perp$ is $\eta - k$.*

Proof. Theorem 4.2 can be easily proven using notations and simple algebraic techniques.

Example 4.3 Let $\mathscr{C}$ be a $[7, 4, 2]$ code in the RT-metric over $GF(3)$, whose generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \textit{ and } \quad G^{\perp} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Here $\tau_1 = 1,$ *and* $\tau_2 = 5$. Hence, the covering radius of $\mathscr{C}^{\perp}$ is $\eta - k = 2$. This is an example to support Theorem 3.3.

5. Construction of RT-conjugate codes in the RT-metric

Theorem 5.1 *Let $\mathscr{C}$ be an $[\eta, k, \tau]$ RT-conjugate code over $\mathbb{F}_q$. If $\eta' \leq \eta$ and $k' \leq k$, then the restriction of $\pi : \mathbb{F}_q^{\eta} \to \mathbb{F}_q^{\eta'}$ to $\mathscr{C}$ is injective and $\pi(\mathscr{C})$ is an $[\eta', k', \tau']$ RT-conjugate code over $\mathbb{F}_q^{\eta'}$.*

Example 5.1 Here is an example of an RT-conjugate code that was created with this method. Let us assume a $[7, 4, 2]$ binary RT-conjugate code $\mathscr{C}$, whose generator matrix G, given as follows:

$$G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

It is simple to determine that these codes, $\mathscr{C}$ is RT-conjugates. Consider the generator matrix of code $\pi(\mathscr{C})$, which is given by

$$\pi(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

From $\pi(G)$, we observe that code $\pi(\mathscr{C})$ is a $[5, 3, 2]$ RT-conjugate.

Theorem 5.2 *Let G_1 and G_2 be the generator matrices of RT-conjugate codes $\mathscr{C}_1$ and $\mathscr{C}_2$ over $\mathbb{F}_q^n$ respectively. Then, the code $\mathscr{C}$ generated by $G = G_1 \oplus G_2 \oplus G_1$ is also an RT conjugate.*

Example 5.2 Here is an example of an RT-conjugate code that was constructed using this method. Let us assume that a $[4, 2, 1]$ binary RT-conjugate code $\mathscr{C}_1$ and a $[5, 3, 2]$ binary RT-conjugate code $\mathscr{C}_2$, whose generator matrix G_1 and G_2, respectively, are given as follows:

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \textit{ and } \quad G_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It is simple to determine that these two codes, $\mathscr{C}_1$ and $\mathscr{C}_2$, are RT-conjugates. Consider code $\mathscr{C}$'s generator matrix, which is given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

From G, we observe that code $\mathscr{C}$ is an $[13, 7, 1]$ RT-conjugate.

Theorem 5.3 *Let G' be a generator matrix of RT-conjugate code $\mathscr{C}'$. Then, the code $\mathscr{C}$ generated by $G = G' \oplus G' \oplus \cdots \oplus G'$ is also an RT-conjugate code.*

Example 5.3 Here is an example of an RT-conjugate code that was created using this method. Let us assume a $[5,3,1]$ ternary RT-conjugate code $\mathscr{C}'$, whose generator matrix G', given as follows:

$$G' = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

It is simple to determine that this code, $\mathscr{C}'$ is RT-conjugate. Consider a generator matrix of the code $\mathscr{C}$, which is given by

$$G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

From G, we observe that code $\mathscr{C}$ is an $[15,9,1]$ RT-conjugate.

6. Conclusion

In this study, a specific subclass of Linear Complementary Dual (LCD) codes in the Rosenbloom-Tsfasman metric (RT-metric) was developed, focusing on their conjugate structures. Through this analysis, we identified the sufficient conditions for a linear code to be an LCD code in this metric. Further, we analyzed the weight distribution of their duals, and proposed methods to construct RT-conjugate codes of larger length and dimension using those with smaller length and dimension. This results can be extended to NRT codes.

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