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On the Stability of Circular Rayleigh Problem of Hydrodynamic Stability

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ABSTRACT: This study presents a detailed analytical investigation into the stability of incompressible swirling flows. A necessary condition is derived for the emergence of both neutral and unstable disturbance modes within subcritical region. To determine a sufficient condition for the existence of unstable modes, the variational structure of the inviscid stability problem is examined under the assumption of a monotonic axial velocity profile. A Sturm-Liouville framework is then employed to establish the existence of a regular neutral mode. Moreover, it is demonstrated that the total number of linearly unstable modes does not exceed the number of neutral modes admitted by the system.

Keywords: Hydrodynamic stability, axisymmetric disturbances, axial flows, Sturm-Liouville problem.

Contents

1. Introduction

Here we examine the linear stability of axial flows of an inviscid, incompressible, and homogeneous fluid when subjected to axisymmetric disturbances. The analysis is carried out in cylindrical polar coordinates (r, θ, z), where the fluid occupies an annular domain defined by $r_1 < r < r_2$, with r_1 and r_2 representing the radii of the inner and outer cylinders, respectively. The undisturbed flow is purely axial, characterized by a velocity profile $(0, 0, W(r))$ and uniform pressure p_0. Upon introducing a perturbation, the flow acquires additional components, with the perturbed velocity given by $(u'_r, 0, W + u'_z)$ and pressure becoming $p_0 + p'$. The perturbations are assumed to have a normal mode form, expressed as a product of a radial-dependent function and the exponential $\exp(ik(z - ct))$, where $k > 0$ denotes the axial wave number and $c = c_r + ic_i$ is the complex phase velocity of the disturbance. Specifically, the radial velocity perturbation is written as $u'_r = \tilde{u}(r) \exp(ik(z - ct))$, with $\tilde{u}(r)$ representing its amplitude. This type of stability analysis is referred to as the circular Rayleigh problem, which arises as the inviscid counterpart of the circular Orr-Sommerfeld problem. The latter addresses the stability of viscous axial flows confined between two coaxial cylinders under axisymmetric perturbations. This problem has recently gained attention due to its relevance in biomedical engineering, particularly in applications such as those described by Walton [1]. One notable example is the modeling of thread annular flow, which is employed to represent the flow within certain medical implants. In this context, the flow occurs between two concentric cylinders, with the inner cylinder moving at a constant, non-zero speed.

The circular Rayleigh problem has been extensively investigated over the years, with numerous analytical insights compiled, particularly in the work of Chandrasekhar [2] (see reference 78(a)). When the eigenvalue $c = c_r + ic_i$ has a positive imaginary part $c_i > 0$, the corresponding eigenmode (c, u)

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represents an unstable disturbance growing at the rate kc_i. A fundamental requirement for such instability is that the function $\Psi(r) = rD(DW/r)$, where $D = d/dr$, must undergo at least one sign change within the interval (r_1, r_2). This necessary condition was originally derived by Rayleigh, as noted in Chandrasekhar's text. Batchelor and Gill [3] later refined this condition. They introduced a criterion involving a point $r = r_s \in (r_1, r_2)$ where $\Psi(r_s) = 0$ and defined $W_s = W(r_s)$. For instability to occur, it is necessary that $\Psi(W - W_s) < 0$ at least once in the domain. Chandrasekhar [2] also introduced the function $K(r) = -\frac{\Psi(r)}{r(W-W_s)}$ to analyze the instability characteristics of specific axial flow profiles, showing that a positive value of $K(r)$ is a necessary condition for instability. Batchelor and Gill [3] confirmed that $K(r) > 0$ must hold for an instability to arise. One of their key results is that the phase velocity $c = c_r + ic_i$ of any unstable mode (i.e., where $c_i > 0$) must lie within a semicircle in the complex c_r-c_i plane. This semicircle has a radius of $(b - a)/2$ and is centered at $((a + b)/2, 0)$, where a and b are the minimum and maximum values of the axial velocity profile W, respectively. This geometric constraint was also derived from the findings of Howard and Gupta [4]. Further analysis by Iype and Subbiah [5] demonstrated that as the wave number k becomes large, the growth rate kc_i approaches zero. They also established bounds on the real parts of the eigenvalues and provided numerical evidence of instability for certain axial velocity profiles. In their 2010 study, Iype and Subbiah [5] examined a series of basic axial velocity profiles by plotting the growth rate kc_i against the wave number k. Their results indicated that some velocity profiles exhibited stability across all considered wave numbers, as the growth rate remained zero. In contrast, other profiles demonstrated instability, showing a positive growth rate over a certain range of k. Notably, instability was only observed for small values of k, implying that only long-wavelength disturbances are susceptible to growth.

The energy equation, when averaged along the z-direction, highlights the importance of the Reynolds stress, given by

$$\tau = -\frac{k}{2\pi} \int_{z_0}^{z_0+\pi/k} uw \, dr,$$

in mediating energy exchange between the primary flow and perturbations. Batchelor and Gill [3] demonstrated that, for neutral disturbances bordering unstable modes, the total change in Reynolds stress across all critical layers $r = r_c$, where the basic flow velocity W equals the phase speed c, must sum to zero.

In studying the circular Rayleigh problem, it is essential to consider its classical counterpart Rayleigh's stability problem for plane parallel flows of inviscid, incompressible, homogeneous fluids. However, the circular case presents added complexity due to the cylindrical coordinate system.

For unstable modes where $c_i > 0$, the semicircle theorem implies that $W(r)$ must lie strictly between its minimum and maximum values, $a < W(r) < b$. For neutral modes adjacent to these unstable ones, the condition relaxes to $a \leq W(r) \leq b$. A specific location in the flow where $W = c$ is referred to as the critical layer, denoted by $r = r_c$.

Mott and Joseph [6] showed that as $c_i \to 0^+$, the derivative $D(r\tau) \to 0$. Consequently, for neutral modes close to instability, the Reynolds stress behaves as $\tau = \frac{\tau_0}{r}$ across regions where $W \neq c$, with τ_0 being a constant. At the critical layer, a discontinuity in $r\tau$ may arise, and this jump was computed explicitly by Mott and Joseph. This behavior differs from that in plane parallel shear flows, where the Reynolds stress remains constant except for possible discontinuities at the critical layer.

As discussed earlier, the primary condition for instability of axisymmetric disturbances in axial flows is the sign change of the quantity Ψ across the flow domain. Remarkably, this condition also extends to non-axisymmetric disturbances. In such cases, a necessary condition for instability is that the derivative DQ changes sign within the domain, where

$$Q = r\left(\frac{DW}{m^2 + k^2r^2}\right),$$

and m is the azimuthal wave number (refer to Batchelor and Gill [3]). Although stability analyses of inviscid, constant-density axial flows may have limited direct applications, these theoretical results offer valuable insight into the instability mechanisms of more complex and realistic flows. One notable example is the inviscid jet with the velocity profile

$$W(r) = \frac{1}{(1+r^2)^2},$$

studied by Batchelor and Gill [3] in the absence of confining cylinders. This flow was shown to be stable to axisymmetric disturbances due to the monotonic nature of Ψ, which does not change sign. Another illustrative case is that of Walton [1], who analyzed axial flow between two concentric cylinders, with $r = 1$ as the outer boundary and $r = \delta$ representing the inner cylinder, which moves with a dimensionless speed V_0. The corresponding steady-state velocity profile, derived from the Navier-Stokes equations and considered for stability, is given by:

$$W(r) = 1 - r^2 + \frac{V_0 - 1 + \delta^2}{\ln \delta} \ln r, \quad \text{for } \delta \le r \le 1.$$

This particular flow also satisfies the Euler equations and, according to Rayleigh's general stability criterion, is stable with respect to axisymmetric disturbances. However, Walton [1] demonstrated that the inclusion of viscosity leads to instability, indicating that the destabilizing mechanism in this case is due to the fluid's viscous properties.

These two examples underscore the value of general theoretical results related to the circular Rayleigh problem in identifying the sources of instability in more physically relevant scenarios. Specifically, the instability in the inviscid jet considered by Batchelor and Gill [3] arises due to the presence of non-axisymmetric modes, while in Walton's case, the instability stems from viscous effects in an otherwise stable Euler flow.

As discussed earlier, the semicircle theorem developed by Batchelor and Gill [3] outlines a region in the complex plane where instability can occur, but it does not explicitly involve the function Ψ, which plays a crucial role in determining the stability characteristics of inviscid axial flows. Therefore, there is significant interest in identifying instability criteria that directly incorporate Ψ. The function Ψ plays a significant role in determining the stability of axial flows. Insights into this aspect can be drawn from the work of Joseph [8], particularly in the context of the circular Orr-Sommerfeld problem. In his analysis, Joseph [8] derived an improved estimate for the growth rate of arbitrary unstable modes, which enhances the inviscid stability bounds. While no well-defined instability regions have been identified for viscous flows, Joseph [8] established lower and upper limits on the wave speed c_r for unstable modes in the circular Orr-Sommerfeld framework.

Pavithra and Subbiah [7] established the existence of a critical axial wave number k_c, beyond which disturbances are guaranteed to be stable. This implies that short axisymmetric disturbances (with $k > k_c$) cannot grow, and thus, any potential instability must be associated with long-wave disturbances. The value of k_c is explicitly calculated for a specific flow configuration.

As previously noted, the function $K(r)$ is pivotal in assessing the potential for instability in axial flows. It is a necessary condition for instability that $K(r) > 0$ somewhere in the flow domain. In this work, we prove that if the function $K(r)$ satisfies

$$0 < K(r) < \frac{r_1^2 \pi^2}{r_2^3 (r_2 - r_1)^2},$$

throughout the domain, then the flow is stable. This result provides a sufficient condition for stability in terms of $K(r)$. In this work explores the stability characteristics of incompressible swirling flows. The primary focus is on examining the behavior of subsonic disturbances within such flow configurations. A necessary criterion is established for the existence of both neutral and unstable subsonic modes. To determine a sufficient condition for instability, following Shivamoggi [9] a variational technique is employed under the assumption that the axial velocity profile varies smoothly either increasing or decreasing. A Sturm-Liouville framework is then used to rigorously show the existence of at least one well-defined neutral mode. Moreover, it is proven that the number of subsonic unstable modes cannot exceed the number of neutral modes.

2. Eigenvalue Problem

The stability of inviscid incompressible homogeneous fluid to axisymmetric disturbances is formulated by Chandrasekhar [2] is given by

$$D(D_* u) - k^2 u - \left[\frac{rD(\frac{DW}{r})}{W - c} \right] u = 0 \tag{2.1}$$

with the boundary conditions

$$u(r) = 0; r = r_1, r_2, \tag{2.2}$$

where D is the derivative $D = \frac{d}{dr}$, and $D_* = D + \frac{1}{r}$.

3. Mathematical Analysis

Theorem 3.1 *The neutral mode exists only if* $[\mathcal{W}]_c = 0$ *or several jumps* $[\mathcal{W}]_c$ *cancel out, then* $\left[D\left(\frac{DW}{r}\right)\right]$ *changes sign at various points where* $W = c$ *(i.e.,* $c = c_r$*), where*

$$\mathcal{W} = \frac{c\left[r(uDu^* - u^*Du)\right]}{2}.$$

Proof: Multiplying (2.1) by ru^* and integrate from r_1 and r_2 then using (2.2), we get

$$-\int_{r_1}^{r_2} r\left[|D_* u|^2 + k^2|u|^2\right] dr = \int_{r_!}^{r_2} \left[\frac{r^2 D(\frac{DW}{r})}{W - c} \right] |u|^2 dr. \tag{3.1}$$

The imaginary part of equation (3.1) gives

$$c_i \int_{r_1}^{r_2} \left[\frac{r^2 D(\frac{DW}{r})}{|W - c|^2} \right] |u|^2 dr = 0. \tag{3.2}$$

If $c_i \neq 0$ then from the above expression $\left[r^2 D\left(\frac{DW}{r}\right)\right]$ must change sign atleast once in the flow domain $[r_1, r_2]$. $\left[r^2 D\left(\frac{DW}{r}\right)\right]$ is continuous

$$\left[r^2 D\left(\frac{DW}{r}\right)\right]_{r=r_s} = 0. \tag{3.3}$$

To show that in case of a neutral mode $W(r_s) = c$, $(c = c_r)$, rewrite the equation (2.1) as

$$D^2 u + \frac{Du}{r} - \frac{u}{r^2} - k^2 u - \left[\frac{D^2 W - \frac{DW}{r}}{W - c} \right] u = 0,$$

multiply the above equation by ru^*, and subtract from the resulting equation by its complex conjugate, we obtain

$$D\left[r(u^* Du - uDu^*)\right] - (c^* - c)r \left[\frac{D^2 W - \frac{DW}{r}}{|W - c|^2} \right] |u|^2 = 0. \tag{3.4}$$

Introducing the "Wronskian"

$$\mathcal{W} = \frac{c\left[r(uDu^* - u^*Du)\right]}{2}, \tag{3.5}$$

equation (3.4) becomes

$$D\mathcal{W} - c_i \left[\frac{r^2 D(\frac{DW}{r})}{|W - c|^2} \right] |u|^2 = 0. \tag{3.6}$$

as $c_i \to 0$ in equation (3.6), we obtain

$$\mathcal{W} = const., \tag{3.7}$$

integrating equation (3.6) from r_c^- to r_c^+ then

$$\mathcal{W}(r_c^+) - \mathcal{W}(r_c^-) = \int_{r_c-}^{r_c+} \frac{r^2 D\left(\frac{DW}{r}\right)|u|^2 c_i}{W'[(W-c_r)^2 + c_i^2]} dW, \tag{3.8}$$

$$\mathcal{W}(r_c^+) - \mathcal{W}(r_c^-) = -\lim_{\epsilon\to 0}\lim_{c_i\to 0^+} \int_{r_c-\epsilon}^{r_c+\epsilon} \frac{r^2\left[D\left(\frac{DW}{r}\right)\right]}{W'}|u|^2 \left[\frac{1}{1+\left(\frac{W-c_i}{c_i}\right)^2}\frac{dW}{c_i}\right],$$

$$\mathcal{W}(r_c^+) - \mathcal{W}(r_c^-) = [\mathcal{W}]_c = -\frac{r^2\left[D\left(\frac{DW}{r}\right)\right]_{r=r_c}}{W'_c}|u_c|^2\pi. \tag{3.9}$$

The above equation shows that for a neutral mode to exists, there are two possibilities either all the jumps $[\mathcal{W}]_c = 0$, $\left[D\left(\frac{DW}{r}\right)\right]_{r=r_c} = 0$ at all points where $W = c$, $(c = c_r)$; or some jumps $[\mathcal{W}]_c = 0$ cancel out, then $\left[D\left(\frac{DW}{r}\right)\right]$ changes sign at various points where $W = c$, $(c = c_r)$.

□

Theorem 3.2 *For a monotonic axial velocity profiles $W(r)$, the global necessary condition for stability is that*

$$\left[r^2 D\left(\frac{DW}{r}\right)\right](W - W_s) \le 0$$

throughout the flow domain $r_1 \le r \le r_2$.

Proof: The real part of equation (3.1) gives

$$\int_{r_1}^{r_2} (W - c_r)\frac{r^2 D\left(\frac{DW}{r}\right)}{|W-c|^2}|u|^2 dr = -\int_{r_1}^{r_2}\left[\frac{1}{r}|D(ru)|^2 + k^2 r|u|^2\right] dr. \tag{3.10}$$

Adding,

$$(c_r - W_s)\int_{r_1}^{r_2} \frac{r^2 D\left(\frac{DW}{r}\right)}{|W-c|^2}|u|^2 dr = 0,$$

to left hand side of equation (3.10),

$$\int_{r_1}^{r_2} (W - W_s)\frac{r^2 D\left(\frac{DW}{r}\right)}{|W-c|^2}|u|^2 dr = -\int_{r_1}^{r_2}\left[\frac{1}{r}|D(ru)|^2 + k^2 r|u|^2\right] dr, \tag{3.11}$$

From which one can obtain a global necessary condition for the existence of a non-neutral or neutral subsonic mode

$$r^2 D\left(\frac{DW}{r}\right)(W - W_s) < 0,$$

where,for $r_1 \le r \le r_2$:

$$\left[r^2 D\left(\frac{DW}{r}\right)\right]_{W=W_s} = 0,$$

and:

$$W_s = W(r_s).$$

In particular, if $W(r)$ is monotonic function and:

$$\left[r^2 D\left(\frac{DW}{r}\right)\right](W - W_s) \le 0, \tag{3.12}$$

throughout the flow domain $r_1 \le r \le r_2$.

□

Theorem 3.3 *If c is real (i.e., $c = c_r$) and falls within the range of W_s and $\left[rD\left(\frac{DW}{r}\right)\right]_{W=c} \neq 0$, then a singular neutral subsonic mode is possible , i.e., $u(r_c) = u_c \neq 0$.*

Proof: If $D^2W \neq 0$ in $[r_1, r_2]$ where $W = c$, the neutral mode is given by

$$u(r) = Au_1(r) + Bu_2(r),$$

where A and B are constants and

$$u_1(r) = (r - r_c) + \left(\frac{r_c W_c'' - 2W_c'}{2r_c W_c'}\right)(r - r_c)^2 + \frac{1}{6}\left(\frac{6}{r_c^2} + k^2 - \frac{3W_c''}{r_c W_c'} + \frac{W_c'''}{W_c'}\right)(r - r_c)^3 + ...,$$

and

$$u_2(r) = (r - r_c) + \left[\frac{W_c'''}{2W_c'} - \frac{W_c''}{2W_c' r_c} + \frac{1}{r_c^2} + \frac{k^2}{2} - \frac{(rW'' - W')^2 (r_c)}{r_c^3 (W_c')^2}\right](r - r_c)^2 + ...$$

Assume that $u_c = 0$. Then $B = 0$, so that $u(r) = Au_1(r)$, and u' exists at r_c. Integration of equation (2.1) between r to r_c gives:

$$(W - c)\left[\frac{D(ru)}{r}\right] - (ru)\left(\frac{DW}{r}\right) = \int_{r_c}^{r} k^2 (W - c) u dr, \tag{3.13}$$

rearranging above equation one obtains:

$$-(ru^2)\left[\frac{W - c}{ru}\right]' = \int_{r_c}^{r} k^2 (W - c) u dr. \tag{3.14}$$

Since a change of sign for ru is always allowable, Assume for $r > r_c$,

$$(W - c)(u) > 0. \tag{3.15}$$

Then, equation (3.14) gives for $r > r_c$,

$$-\left[\frac{W - c}{ru}\right]' > 0. \tag{3.16}$$

By repeating this argument, it follows that

$$\left[\frac{W - c}{ru}\right] > 0, \tag{3.17}$$

i.e., $(W - c)$ and u are of same sign, and that $\frac{(W-c)}{u}$ decreases as r increases, so that u cannot vanish at $r = r_2$ as it should. Therefore the contradiction, and so:

$$u_c \neq 0. \tag{3.18}$$

□

Theorem 3.4 *Regular neutral subsonic modes, i.e.,*

$$u = u_s, \quad k = k_s, \quad c = c_s = W_s, \tag{3.19}$$

are possible if the mean flow profiles are monotonic, and if there is satisfied in the flow-field

$$\left[rD\left(\frac{DW}{r}\right)\right]_{r=r_s} = 0.$$

Proof: Equation (2.1) is written as

$$[N(r)D_*(u)]' + K(r)u + \lambda u = 0, \tag{3.20}$$

where:

$$N(r) = 1,$$
$$K(r) = -\frac{rD(\frac{DW}{r})}{W - c_s},$$
$$\lambda = -k^2.$$

with boundary conditions (2.2).
Let there be a point $r = r_s$ such that:

$$\left[rD\left(\frac{DW}{r}\right)\right]_{r=r_s} = 0,$$

and

$$\left[\frac{DW}{r}\right]_{r=r_s} \neq 0.$$

Multiplying equation (3.20) with ru^*, and integrating from r_1 to r_2, and using the boundary condition (2.2) one can obtain:

$$\lambda = \frac{\int_{r_1}^{r_2} r\left[N|D_*(u)|^2 - K(r)|u|^2\right]dr}{\int_{r_1}^{r_2} r|u|^2 dr}, \tag{3.21}$$

$$\delta \int_{r_1}^{r_2} r\left[|D_*(u)|^2 + \left(\frac{rD\left(\frac{DW}{r}\right)}{W - c_s}\right)|u|^2\right]dr = 0,$$

subject to:

$$\int_{r_1}^{r_2} r|u|^2 dr = const. \tag{3.22}$$

$$\delta\lambda = \frac{2I\int_{r_1}^{r_2}[r(ND_*u)\delta(D_*u) - K(ru)\delta(ru)]dr - 2J\int_{r_1}^{r_2}(ru)(\delta u)dr}{I^2}, \tag{3.23}$$

where:

$$I = \int_{r_1}^{r_2} r|u|^2 dr; \quad J = \int_{r_1}^{r_2} r\left[N|D_*u|^2 - K(r)|u|^2\right]dr.$$

Noting:

$$\int_{r_1}^{r_2} rN(D_*u)\delta(D_*u)dr = -\int_{r_1}^{r_2} r(ND_*u)'\delta(u)dr,$$

equation (3.23) becomes:

$$-\frac{I}{2}\delta\lambda = -\int_{r_1}^{r_2} r\left[(ND_*u)' + Ku + \lambda u\right]\delta u dr. \tag{3.24}$$

From (3.20) one finds:

$$\delta\lambda = 0. \tag{3.25}$$

On the other hand, if $\delta\lambda$ is zero for all admissible variations δu those that meet the boundary conditions and have square integrable derivatives like u then equation (3.24) implies that the corresponding function u must satisfy equation (3.20). While equation (3.25) indicates only that λ is a stationary value, to establish that λ represents a minimum, we observe from equation (3.20) that when $c_i \neq 0$, the condition is satisfied.

$$-\int_{r_1}^{r_2}\left[r|D_*u|^2 - \lambda_1 r|u|^2\right]dr = \int_{r_1}^{r_2}\frac{rD\left(\frac{DW}{r}\right)}{W - c}r|u|^2$$

$$= \int_{r_1}^{r_2} \frac{(W-c_s)}{|W-c|^2} rD\left(\frac{DW}{r}\right) r|u|^2 dr,$$

$$-\int_{r_1}^{r_2} \left[|D_* u|^2 - \lambda_1 r|u|^2\right] dr = -\int_{r_1}^{r_2} \frac{(W-c_s)^2}{|W-c|^2} Kr|u|^2$$

$$< -\int_{r_1}^{r_2} Kr|u|^2 dr.$$

From which,

$$-\lambda_1 < -\frac{\int_{r_1}^{r_2} r\left[|D_* u|^2 - K|u|^2\right]}{\int_{r_1}^{r_2} r|u|^2} dr,$$

$$\lambda < \lambda_1,$$

Thus:

$$\lambda = min(\lambda_1),$$

where:

$$\lambda_1 = -\frac{\int_{r_1}^{r_2} r\left[|D_*(\chi)|^2 - K|\chi|^2\right]}{\int_{r_1}^{r_2} r|\chi|^2} dr, \tag{3.26}$$

here χ is a square integrable function satisfying the condition (2.2). Replacing χ by the test function W, we get

$$I_1\lambda = \int_{r_1}^{r_2} r\frac{\left[rD\left(\frac{DW}{r}\right)\right] WW_s}{W - W_s} dr,$$

where:

$$I_1 = \int_{r_1}^{r_2} rW^2 dr \neq I,$$

$$I_1\lambda = -\int_{r_1}^{r_2} rK(r)WW_s dr, \tag{3.27}$$

provided $W > 0$. Therefore, $\lambda < 0$, k is real and positive, for which there is a neutrally stable mode. □

4. Existence of Self-Excited Neutral Mode

Next we deduce the existence of unstable mode whose limit as $c_i \to 0^+$ is the neutral mode. If:

$$D\left[\frac{DW}{r}\right]_{W=c=c_r} = 0,$$

holds somewhere in the flow domain, it will ensure the existence of neutral disturbances. If the conditions at such a point are denoted by a subscript s, one has a solution $c_s = W$ (where c_s is real). Consider the eigen value problem of equation (2.1) with the parameter $-k^2 = \lambda$. We have necessary condition in the form:

$$-\frac{r}{W-c} D\left[\frac{DW}{r}\right] > 0, \tag{4.1}$$

and let $DW > 0$. Under these assumptions, it follows from equation (3.23) that the eigenvalue λ_s is both real and negative, implying that k_s must be real and positive. Additionally, the associated eigenfunction u_s is also real. To demonstrate that a neutral solution is accompanied by a nearby self-excited mode, we will analyze how the parameter c varies with $\lambda = -k^2$ in the context of the eigen value problem of (2.1). Note that the eigenfunction u_s corresponds to the eigenvalue $c = c_s$, with $\lambda_s = -k_s^2$, allowing us to express:

$$D[D_* u_s] - \frac{ru_s}{W-c_s} D\left[\frac{DW}{r}\right]_{c=c_s} + \lambda_s u_s = 0. \tag{4.2}$$

Multiplying equation (2.1) by ru_s and (4.2) by ru and subtracting we get

$$r[D(D_*u)u_S - D(D_*u_s)u] + r^2 D\left(\frac{DW}{r}\right)\left[\frac{1}{W-c_s} - \frac{1}{W-c}\right]uu_s + r(\lambda-\lambda_s)uu_s = 0. \tag{4.3}$$

Integrating over (r_1, r_2)

$$(\lambda-\lambda_s)\int_{r_1}^{r_2} ruu_s\,dr = \int_{r_1}^{r_2} r^2 D\left(\frac{DW}{r}\right)\left[\frac{1}{W-c} - \frac{1}{W-c_s}\right]uu_s\,dr$$

as $\lambda \to \lambda_s$, $c \to c_s$,$u \to u_s$ gives

$$-\left(\frac{d\lambda}{dc}\right)_{k=k_s}\int_{r_1}^{r_2} r|u_s|^2 dr = \lim_{c\to c_s}\int_{r_1}^{r_2} r\left[\frac{rD\left(\frac{DW}{r}\right)}{W-c_s}\right]\frac{|u_s|^2}{W-c}dr. \tag{4.4}$$

The right hand side of above equality is

$$\int_{r_1}^{r_2}\left[\frac{rD\left(\frac{\rho_\circ DW}{r}\right)_{c=c_s}}{W-c_s}\right]\frac{r\,|u_s|^2}{W-c}dr = \int_{r_1}^{r_2}\left[\frac{rD\left(\frac{\rho_\circ DW}{r}\right)_{c=c_s}}{W-c_s}\right]\frac{r\,|u_s|^2\left[(W-c_r)+ic_i\right]}{|W-c|^2}dr.$$

The limit of the real part of (4.4) becomes

$$\int_{r_1}^{r_2}\left[\frac{rD\left[\frac{\rho_\circ DW}{r}\right]_{c=c_s}}{W-c_s}\right]\frac{r\,|u_s|^2}{W-c_s}dr = R(say),$$

and the limit of the imaginary part of (4.4)

$$\int_{r_1}^{r_2}\left[\frac{rD\left[\frac{\rho_\circ DW}{r}\right]_{c=c_s}}{(DW)_s}\right]\frac{r\pi\,|u_s|^2}{W-c_s} = S > 0(say).$$

As $c_i \to 0^+$, equation (4.4) becomes $-\frac{d\lambda}{dc} = R + iS$, $S > 0$.

$$-\frac{dc}{d\lambda} = \frac{dc}{d(k^2)} = \frac{R-iS}{R^2+S^2}. \tag{4.5}$$

If k^2 decreases slightly, c_i becomes positive. Equation (4.5) shows the existence of a self-excited disturbances in the neighborhood of the neutral disturbances.

5. The Number of Unstable Modes

Let the necessary condition for the existence of non-neutral modes in the form

$$K(r) = -\frac{rD(\frac{DW}{r})}{W-c_s} \geq 0,$$

hold, where c_s is real. Assume that $(W-c_s)$ has only one zero in (r_1, r_2), otherwise the flow is stable. Consider the eigenvalue problem:

$$D\left[N(r)D_*\chi\right] + K(r)\chi + \lambda\chi = 0, \tag{5.1}$$

$$\chi(r_1) = 0 = \chi(r_2). \tag{5.2}$$

as in (3.20). For subsonic disturbances this eigenvalue problem is an ordinary Sturm-Liouville problem, and has infinitely many eigenvalues in the increasing order $\lambda_1 > \lambda_2 > \ldots$. The eigenvalue λ_1 is the characterized as the minimum of the functional:

$$\Im(u) = \int_{R_1}^{R_2} \left[N(r) \left| D_* u \right|^2 - K(r) \left| u \right|^2 \right] r dr,$$

where u is continuously differentiable function satisfying the boundary condition (5.2) and normalized by

$$\int_{r_1}^{r_2} r \left| u \right|^2 dr = 1. \tag{5.3}$$

The minimum value is achieved when $u = \chi_1$, the first eigenfunction. In a similar manner, the eigenvalue λ_n corresponds to the minimum of the functional under the constraint that u is orthogonal to the first $n-1$ eigenfunctions. This minimum is attained when $u = \chi_n$. For those eigenvalues λ_k that are negative, we denote them as $\lambda_k = -k_{sk}^2$, and they are associated with neutral eigenfunctions χ_k. Based on the sufficient conditions previously derived, instability is expected to occur within the neutral modes whenever $k^2 < k_{sk}^2$.

Theorem 5.1 *For the mean flow profiles that satisfy the global necessary condition*

$$-rD\left[\frac{DW}{r}\right] \frac{1}{W - c_s} \geq 0, \tag{5.4}$$

holds in the flow domain $[R_1, R_2]$, there can be no more than $(n-1)$ linearly independent unstable eigen function of (5.1), if $k^2 \geq -\lambda_n$.

Proof: Suppose u_1 and u_2 are two unstable eigenfunctions of equation (2.1), corresponding to the same eigenvalue k^2, and associated with eigenvalues c_1 and c_2 respectively, where both have positive imaginary parts ($c_i > 0$). Then, we have the following

$$\int_{r_1}^{r_2} \left[-ru_1^* D\left[\frac{D(ru_2)}{r}\right] + k^2 r u_1^* u_2 \right] dr = -\int_{r_1}^{r_2} rD\left[\frac{DW}{r}\right] \frac{ru_1^* u_2}{W - c_2} dr, \tag{5.5}$$

and

$$\int_{r_1}^{r_2} \left[-ru_2^* D\left[\frac{D(ru_1)}{r}\right] + k^2 r u_1^* u_2 \right] dr = -\int_{r_1}^{r_2} rD\left[\frac{DW}{r}\right] \frac{ru_1^* u_2}{W - c_1} dr, \tag{5.6}$$

setting

$$F_k = \frac{u_k}{W - c_k}, \tag{5.7}$$

we obtain

$$\int_{r_1}^{r_2} K(r) F_1^* F_2 (W - c_s) r dr = 0, \tag{5.8}$$

where

$$K(r) = -rD\left[\frac{DW}{r}\right] \frac{1}{W - c_s} > 0. \tag{5.9}$$

If u_1, u_2,..,u_n are n linearly independent unstable eigenfunctions of (2.1) for the same eigenvalue k^2 and values c_1, c_1,..,c_n of c then one may construct a function ψ

$$\psi = \sum_{l=1}^{n} a_l u_l, \tag{5.10}$$

which is orthogonal to the first $(n-1)$ eigenfunctions ψ_1, ψ_2,..,ψ_{n-1} of the Sturm-Liouville problem (2.1) ; let

$$\int_{r_1}^{r_2} r \left| \psi \right|^2 dr = 1.$$

Now

$$D\left[\frac{D(r\psi)}{r}\right]-k^2u=\sum_{l=1}^{n}a_l rD\left[\frac{DW}{r}\right]\frac{\phi_l}{W-c_l}$$
$$=-\sum_{l=1}^{n}K(r)a_lF_l(W-c_s),$$

$$D\left[\frac{D(r\psi)}{r}\right]+K(r)\psi=k^2u+\sum_{l=1}^{n}a_lF_lK(r)(c_s-c_l). \tag{5.11}$$

Multiplying with $r\psi^*$, and integrating over flow domain,we get

$$\int_{r_1}^{r_2}[-|D_*\psi|^2+K|\psi|^2]rdr=k^2\int_{r_1}^{r_2}r|\psi|^2dr+\int_{r_1}^{r_2}rK\sum_{k=1}^{n}\sum_{l=1}^{n}a_l^*F_l^*(u-c_l)a_kF_k(c_s-c_k)dr,$$

rewrite the above equality as

$$\int_{r_1}^{r_2}[-|D_*\psi|^2+K|\psi|^2]rdr=k^2\int_{r_1}^{r_2}r|\psi|^2dr+\int_{r_1}^{r_2}rK\sum_{k=1}^{n}\sum_{l=1}^{n}a_l^*F_l^*(c_s-c_l)a_kF_k(c_s-c_k)dr$$
$$+\sum_{k=1}^{n}\sum_{l=1}^{n}\int_{r_1}^{r_2}rKa_l^*F_l^*(u-c_s)a_kF_kdr,$$

$$-k^2=\int_{r_1}^{r_2}[|D_*\psi|^2-K|\psi|^2]rdr+\int_{r_1}^{r_2}rK\sum_{k=1}^{n}\sum_{l=1}^{n}a_l^*F_l^*(c_s-c_l)a_kF_k(c_s-c_k)dr,$$
$$-k^2=\int_{r_1}^{r_2}[|D_*\psi|^2-K|\psi|^2]rdr+\int_{r_1}^{r_2}rK\left|\sum_{k=1}^{n}a_kF_k(c_s-c_k)\right|^2dr,$$
$$-k^2\geq\lambda_n. \tag{5.12}$$

Therefore, if there a n linearly independent unstable eigen functions of (2.1) then $k^2\leq-\lambda_n$. Indeed, $k^2<-\lambda_n$, since if the equality is to hold, then

$$\sum_{k=1}^{n}a_k(c_s-c_k)F_k=0,$$

so that

$$\sum_{k=1}^{n}a_k(c_s-c_k)[D(D_*u_k)-k^2u_k]=\sum_{k=1}^{n}a_k(c_s-c_k)\frac{rD\left(\frac{DW}{r}\right)}{W-c_k}u_k$$
$$=rD\left(\frac{DW}{r}\right)\sum_{k=1}^{n}a_k(c_s-c_k)F_k$$
$$=0.$$

Taking

$$g=\sum_{k=1}^{n}a_k(c_s-c_k)u_k \tag{5.13}$$

we get

$$\int_{r_1}^{r_2}r[|D_*g|^2+k^2|g|^2]dr=0, \tag{5.14}$$

$$g = 0. \tag{5.15}$$

Since, $c_s - c_k \neq 0$, it follows that the eigenfunctions $u_1, u_2, \ldots, u_n$ must be linearly dependent, which contradicts our initial assumption. Therefore, the flow remains stable whenever $k^2 > |-\lambda_1|$, where $|-\lambda_1|$ represents the absolute value of the lowest eigenvalue. As λ_n eventually becomes positive, its connection to the stability analysis becomes irrelevant. In fact, any criterion that guarantees all eigenvalues λ of the Sturm-Liouville problem (5.1) are positive serves as a sufficient condition for stability. □

6. Concluding Remarks

This analysis provides a comprehensive theoretical framework for understanding the linear stability of incompressible swirling flows. By deriving both necessary and sufficient conditions for instability, the study offers valuable insight into the behavior of subcritical disturbance modes. The use of variational principles, combined with a Sturm-Liouville approach, not only confirms the existence of regular neutral modes but also constrains the number of possible unstable modes relative to them. These results contribute significantly to the theoretical foundation of flow stability and may serve as a basis for future investigations into more complex or nonlinear flow configurations.

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