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Exact Solutions for the Oscillatory Flow of Micropolar Fluid Outwith a Fluid Sphere Using Slip

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ABSTRACT: In this paper, we investigate an axisymmetric rectilinear flow over a micropolar fluid sphere particle in an incompressible non-Newtonian fluid that oscillates with minuscule amplitudes. The fluid velocity field and microrotation components were shown to have analytical expressions. The velocity field was studied using modified Bessel functions and stream function under slip circumstances on the boundary. For rectilinear oscillations, the drag force acting on the particle was calculated. Values of the slip parameter, cross viscosity parameter, real drag, and imaginary drag for micro polarity parameters are retrieved. Both tabular and visual representations of numerical quantities are used. It was observed that there was a direct relation between slip parameter values, real drag and inverse relationship between slip parameter and imaginary drag for different cross viscosity and micro polarity values. Results for the drag force are compared for some particular cases. Which agree with values in literature.

Key Words: Oscillatory flow, micropolar fluid, slip condition, drag force, fluid sphere.

Contents

1. Introduction

Due to its wide applications in engineering and applied sciences, the study of oscillatory Stokes flows has attracted a lot of attention from researchers. These themes incorporated the biomechanics of blood flow, ultra-filtration, Brownian motion and other biological phenomena or chemical phenomena as rendered by [1].

The micropolar fluid is one of the most basic fluid models. It shows that the traditional framework of continuum mechanics is unable to adequately explain the motion of complicated liquids such as polymer

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solvents, polymer melts, emulsions, animal blood, liquid metals and fluid particles. The development of ideas to represent such intricate fluids has grown substantially.

[2] considered Micropolar fluid with internal structure a well-accepted theory and [3] reported in his work about fluid particles that can spin individually from the gyration and migration of the fluid as a whole. The review article by [4] and monograph of [5] have reported valuability of micropolar theory and brought in immense literature reviews on it. [6] have investigated a laminar flow of Micropolar fluid over a liquid bubble by non-zero spin, no-slip, using the analytic method force of drag as illustrated. [7] examined Stokes flow of a non-Newtonian liquid past an impervious sphere shielded through a narrow foil and the drag force was computed analytically. [8] analysed a non-Newtonian liquid flow beyond a permeable sphere via non-zero boundary for microrotation, no-slip condition and they obtained closed form solutions for drag force. They noticed that the force of drag on the body decreased as the rotation variable increased. [9] have considered an analytical steady of axisymmetric creeping flow of micropolar fluid around a permeable sphere with a narrow surface that consists of an impervious sphere with no-slip condition. They noticed that the drag of a permeable sphere is lesser compared with an impermeable sphere.

The slip boundary condition was presented by [10], in which the fluid's tangential velocity corresponding to a location on its surface is proportional to tangential stress. He insisted through slip for microrotation and velocity is more practically appropriate as a consequence of both conditions are activated to the coequal shallow and the slip is primarily related to the humor of the shallow and liquid. [11] investigated the effect of slip on the torque applied on rotary oscillating over spheroids and spheres and discovered that, in all instances, slip lessens the torque. Now-a-days there has been an enlarge significance in using such slip conditions for microfluidic flows which are brought to attention by authors [**?**,**?**,**?**].

The slow fixed rotation of a sphere with micropolar fluid passing over it, about its diameter was analysed by [15] and in continuation [16] extended the study of both rectilinear and rotatory oscillations. A spheroid's rectilinear oscillations in a non-Newtonian fluid was considered using spheroidal coordinates by [17]. [18] obtained a formula for the force applied on a sphere that was oscillating longitudinally in an incompressible non-Newtonian fluid. [19] worked on oscillating flow beyond a sphere for Reynolds numbers from $\pi/4$ to 2π using a series truncation method. [20] considered a stationary axisymmetric body which is placed in non-Newtonian liquid with rectilinear oscillatory flow. A widespread expression for drag force was obtained using an analytic method. They reduced the special cases of spheroid and sphere. [21] studied an oscillating sphere at limited Reynolds numbers using three methods. [22] examined the oscillatory flow of non-Newtonian liquid over a constant permeable sphere and computed the pressure field and its drag force with no-slip condition. [23] investigated the flow created in a concentric spherical container by integrating torsional and oblique vibrations of a sphere solved analytically. [24] computed the velocity field in terms of modified Bessel functions using the oscillatory flow generated with two spheres having equal frequency with a similar diameter and different angular speed. [25] studied that the Lorentz force causes a time-dependent laminar (Re 640) flow on an electrolytic fluid in the space between two concentric spheres and various forcing frequencies, yielding oscillatory Reynolds numbers ranging from 28 to 2820 by using no-slip static condition. [26] reported on numerical analysis of the flow and heat conduction aspect caused by a rigid drop moving inside a cylinder containing FENE-P viscoelastic fluids. [27] monograph of Happer and Brenner studied the low Reynolds numbers. [28] monogragh of Abramowitz and Stegun studied the mathematical functions.

[29] investigated the unsteady thermos-viscous fluid transport between two indefinitely stretched impermeable horizontal plates using artificial neural networks. [30] evaluated an unsteady thermo-viscous fluid with incompressible flow in a porous slab in a semi-infinite region via a flat plate that oscillates horizontally and has an impervious bottom byAn artificial neural network (ANN). [31]A numerical and analytical method to examine unsteady fluid flow around an oscillating sphere. With appropriate assumed boundary conditions, Bessel functions representing the fluid's temperature and angular velocity around an oscillating sphere have been used to derive analytical equations. [32] considered thermophoresis, Brownian motion, and first-order chemical react parameters, among other factors, this research examined the effects of heat generation of nanofluid movement over a stretching sheet. FDM with collocation polynomial technique (bvp4c) was used to solve the equations using MATLAB software. [**?**] studied on the train a machine learning technique through the historical data to distinguish the pattern of the independent

features in the data and predict whose risk factor is unknown as potentially hazardous asteroids (PHAs).

There have also been no meta-analyses on the oscillatory Micropolar flow on a micropolar fluid sphere with interfacial slip condition. This inspired us to add the probabilities into one research.

In this manuscript, our investigations are presented in four parts i.e.,

- Oscillatory flow of non-Newtonian beyond a non-Newtonian liquid sphere
- Oscillatory non-Newtonian liquid flow beyond a Newtonian fluid sphere
- Oscillatory flow of Newtonian fluid beyond a non-Newtonian liquid sphere
- Oscillatory Viscous liquid beyond a Viscous liquid sphere by slip over the boundary.

The numerical values and graphs are presented in the results and discussion segment, which is followed by conclusion.

[11] studied about a non -Newtonian nanoparticles in three dimensions under the new effects of activation power, nonlinear heat flux, and hydromagnetic characteristics.

2. Formulation and Solution of the Problems

2.1. Basic Equations of Micropolar Fluid

The momentum equation of a non-Newtonian fluid flow from [2] as

$$\frac{\partial \rho}{\partial t} + div(\rho V) = 0, \tag{2.1}$$

$$\rho \frac{dV}{dt} = \rho f - \nabla p + k\nabla \times w - (\mu + k)\, \nabla \times \nabla \times V + (\lambda + 2\mu + k)\, \nabla\, (div\ V)\,, \tag{2.2}$$

$$\rho J \frac{dw}{dt} = \rho I - 2kw + k\nabla \times V - \gamma \nabla \times \nabla \times w + (\alpha + \beta + \gamma)\nabla(div\ w)\,. \tag{2.3}$$

where p represents the pressure, μ the traditional viscosity, parameters k, λ, μ vortex viscosity coefficients and $\alpha,\ \beta,\ \gamma$ are gyroviscosity, ρ the density, w the microrotation field, V the velocity field, I microrotation driving forces based on gravity, f body forces based on gravity, J the gyration parameter gratifying the pursuing bias

$$3\alpha + \beta + \ \gamma \geq 0,\ 2\ \mu + \ k \geq 0,\ \ 3\lambda + 2\ \mu + \ k \geq 0 \quad ,\gamma \geq |\beta|\,,\ k \geq 0, \gamma \geq 0. \tag{2.4}$$

3. Oscillatory Flow of Non-Newtonian Beyond a Non-Newtonian Liquid Sphere

3.1. Modelling

Consider a non-Newtonian liquid sphere with radius R=a at rest in an oscillating non-Newtonian liquid flow. The oscillations are with small amplitudes. Far from the body, the flow intends to be uniform, stable and axisymmetric. Let μ_i, μ_e represents the viscosities of the liquid's interior (Region-2) and exterior (Region-1) of the liquid sphere. The geometry of the solution is dual micropolar liquid presented in Fig. 1.

Let $(R, \theta, \emptyset)$ represents for spherical polar locations beginning at the sphere's center and radius a through scale factors $h_1 = 1,\ h_2 = R$, $h_3 = Rsin\theta$ with basis unit vectors as , $(\overline{e}_r, \overline{e}_\theta, \overline{e}_\emptyset)$ and along Z-axis then flow order. Incorporate an oscillatory flow at a frequency of $U_\infty e^{i\omega t}\overline{k}$ along the symmetry axis $\theta = 0$ in an infinite region of an incompressible non-Newtonian liquid sphere that is stationary. Here, ω is the oscillation of frequency and all the flow objectives are autonomous of $\emptyset$. hen we define the acceleration vectors and microrotation as

The velocity component and micro rotation are assumed as

$$V = \left(\nabla \times \frac{\Psi\ \overline{e_\phi}}{h_3}\right) e^{i\omega t}, \tag{3.1}$$

$$W = \frac{C}{h_3} e^{i\omega t}\overline{e_\phi}\ . \tag{3.2}$$

Substituting the above expressions (3.1), (3.2) in (2.2), (2.3) we get

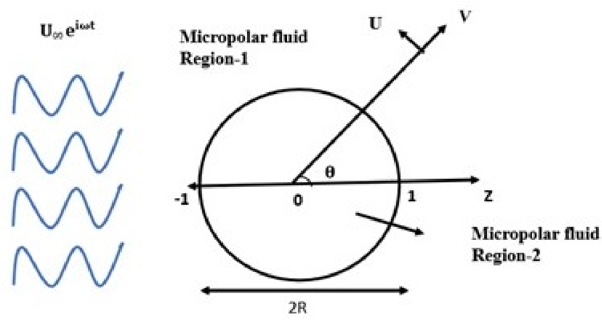

Figure 1: Geometry of oscillatory flow of Micropolar liquid beyond a Micropolar liquid sphere.

$$\rho i\omega V = -\nabla P + k\nabla \times W - (\mu + k)\ \nabla \times \nabla \times W, \tag{3.3}$$

$$i\rho j\omega W = -2\mathrm{k}W + k\nabla \times V - \gamma\ \nabla \times \nabla \times W. \tag{3.4}$$

Taking curl on (3.3) we get

$$(\mu + k)\ E_0^4\ \Psi + kE_0^2 C = i\rho\omega E_0^2 \Psi, \tag{3.5}$$

$$\text{where}\ \ E_0^2 \equiv \frac{\partial^2}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} - \frac{cot\theta}{R^2}\frac{\partial}{\partial \theta}. \tag{3.6}$$

Taking curl of curl to equation (3.4) we get

$$kE_0^4 \Psi = -(2k + i\rho j\omega)\left(E_0^2\ C\right) + \gamma\ \left(E_0^4\ C\right). \tag{3.7}$$

Substituting $E_0^2 C$ from equation (3.5) in equation (3.7) we get

$$E_0^2\left(E_0^2 - \frac{\delta_1^2}{a^2}\right)\left(E_0^2 - \frac{\delta_2^2}{a^2}\right)\Psi = 0. \tag{3.8}$$

where

$$\left.\delta_1^2 + \delta_2^2 = \frac{\{k\,(2\mu + k)\,i\rho\omega\,(j\mu + jk + \gamma)\}a^2}{\gamma\,(\mu + k)}, \quad \delta_1^2\delta_2^2 = \frac{i\rho\omega\,(2k + ji\rho\omega\gamma)\,a^4}{\gamma\,(\mu + k)}.\right\} \tag{3.9}$$

From equation (3.4)

$$(2k + \rho i\omega j)\,C = \gamma E_0^2 C - k\ E_0^2 \Psi. \tag{3.10}$$

Substituting equation (3.5) in equation (3.10), we get

$$(2k + \rho i\omega j)\,c = \frac{1}{k}E_0^2\left(\left(\gamma i\rho\omega - k^2\right)\Psi - (\mu + k)\,\gamma E_0^2 \Psi\right). \tag{3.11}$$

In the point of view the axisymmetric motion, the acceleration components represented by means of the stream function ψ as

$$u_r = \frac{1}{R^2 sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad u_\theta = -\frac{1}{R sin\theta}\frac{\partial\Psi}{\partial R}.$$

Using non-dimensional scheme
$R = ra,\ E_0^2 = \frac{E^2}{a^2},\ \Psi = \psi U_\infty a^2,\quad C = CU_\infty,\ c = \frac{k}{(\mu+k)}\ ,\ \sigma = \rho\omega a^2(\mu+k),\ J = \rho\omega j a^2/\gamma.$
Eq. (3.8) changes as

$$E^2\left(E^2-\delta_1^2\right)\left(E^2-\delta_2^2\right)\psi = 0, \tag{3.12}$$

$$cC = -\frac{i\sigma}{\delta_1^2\delta_2^2}E^2\left(E^2-\left(\delta_1^2+\delta_2^2\right)\right)\psi - E^2\psi. \tag{3.13}$$

The linearity and commutativity of E^2, $\left(E^2-\delta_1^2\right)$ and $\left(E^2-\delta_2^2\right)$ of (3.12) solution acquired by the position. The problem of

$$E^2\psi_0 = 0, \tag{3.14}$$

$$\left(E^2-\delta_1^2\right)\psi_1 = 0, \tag{3.15}$$

$$\left(E^2-\delta_2^2\right)\psi_2 = 0, \tag{3.16}$$

expressing as $\psi = \psi_0 + \psi_1 + \psi_2$.
Let ψ_e, ψ_i indicate the stream functions for the outer and the inner flow respectively i.e.,

$$\psi(r,x) = \begin{cases} \psi_e & \text{for} \quad R > a, \\ \psi_i & \text{for} \quad R < a, \end{cases}$$

The stream functions ψ_e, ψ_i and the microrotation components C_e, C_i satisfies the equation (3.12) and (3.13) these must be determined to the proper boundary and regularity constraints.at infinity.
The external stream function is obtained as

$$\psi_e = \left(r^2 + \frac{l_1}{r} + m_1\sqrt{r}K_{\frac{3}{2}}\left(\delta_{1e}r\right) + n_1\sqrt{r}K_{\frac{3}{2}}\left(\delta_{2e}r\right)\right)G_2(x)\,, \tag{3.17}$$

The internal stream function is

$$\psi_i = \left(\left(l_2 r^2 + m_2\sqrt{r}I_{\frac{3}{2}}\left(\delta_{1i}r\right) + n_2\sqrt{r}I_{\frac{3}{2}}\left(\delta_{2i}r\right)\right)\right)G_2(x)\,, \tag{3.18}$$

where $I_{\frac{3}{2}}$, $K_{\frac{3}{2}}$ are Bessel's function and $G_2(x)$ is a Gegenbauer functions [28].
The parameters $l_1, m_1, n_1, l_2, m_2, n_2$ from (3.17), (3.17) are computed through boundary conditions as:
(i) Regularity conditions:

$$\begin{aligned} &\lim_{r\to\infty}\psi_e = \frac{1}{2}Ur^2 sin^2\theta \quad (\text{Region}-1) \\ &\lim_{r\to 0}\psi_i = \ \text{finite}\ (\text{Region}-2). \end{aligned} \tag{3.19}$$

(ii) Normal velocity is zero on the boundary

$$\psi_e = \psi_i = 0 \quad \text{on} \quad r = 1. \tag{3.20}$$

(iii) Slip condition: Tangential velocity is proportional to the tangential shear stress along the clear surface [27].

$$\tau_{r\theta} = \beta\left(v_{\theta e} - v_{\theta i}\right) \quad \text{on} \quad r = 1. \tag{3.21}$$

(iv) Shear Stress is continuous across the surface i.e.,

$$\tau_{r\theta e} = \tau_{r\theta i} \quad \text{on} \quad r = 1. \tag{3.22}$$

(v) The angular velocity zero on the boundary

$$C = 0 \quad \text{on} \quad r = 1 \;\; i.e., C_e = C_i = 0. \tag{3.23}$$

Microrotation Components C in Equation (3.13) for external and internal flow reduces to

$$cC_e = \sqrt{r}\left(m_1\left(i\sigma - \delta_1^2\right)K_{\frac{3}{2}}\left(\delta_{1en}r\right) + n_1\left(i\sigma - \delta_2^2\right)K_{\frac{3}{2}}\left(\delta_{2e}r\right)\right)G_2\left(x\right), \tag{3.24}$$

$$cC_i = \sqrt{r}\left(m_2\left(i\sigma - \delta_1^2\right)I_{\frac{3}{2}}\left(\delta_{1i}r\right) + n_2\left(i\sigma - \delta_2^2\right)I_{\frac{3}{2}}\left(\delta_{2i}r\right)\right)G_2\left(x\right). \tag{3.25}$$

By the boundary condition (3.19) - (3.25), we get six equations with six unknowns as

$$\left.\begin{aligned}
l_1 + m_1' + n_1' = -1, \quad l_2 + m_2' + n_2' = 0,\\
-6l_1 - m_1'\left(z_2\delta_{1e}^2 + 4 + 2\Delta_1\left(\delta_{1e}\right)\right) - n_1'\left(z_2\delta_{2e}^2 + 4 + 2\Delta_1\left(\delta_{2e}\right)\right)\\
+m_2'\left(\delta_{1i}^2 z + 4 + 2\Delta_3\left(\delta_{1i}\right)\right) + n_2'\left(\delta_{2i}^2 z + 4 + 2\Delta_2\left(\delta_{2i}\right)\right) = 0,\\
l_1\left(-6 - sz_1\right) + m_1'\left(-z_2\delta_{1e}^2 - 4 - 2\Delta_1\left(\delta_{1e}\right) - sz_1\Delta_1\left(\delta_{1e}\right)\right)\\
+n_1'\left(-z_2\delta_{2e}^2 - 4 - 2\Delta_1\left(\delta_{2e}\right) - sz_1\Delta_1\left(\delta_{2e}\right)\right)\\
+m_2' sz_1\Delta_2\left(\delta_{1i}\right) - 2l_2 sz_1 + n_2'\Delta_2\left(\delta_{2i}\right)sz_1 = \left(-2z_1\right)s,\\
m_1'\left(\mathrm{i}\sigma - \delta_{1\mathrm{e}}^2\right) + n_1'\left(\mathrm{i}\sigma - \delta_{2\mathrm{e}}^2\right) = 0.\\
m_2'\left(\mathrm{i}\sigma - \delta_{1\mathrm{i}}^2\right) + n_2'\left(\mathrm{i}\sigma - \delta_{2\mathrm{i}}^2\right) = 0.
\end{aligned}\right\} \tag{3.26}$$

where

$$m_1' = m_1 K_{\frac{3}{2}}(\delta_{1e}), \quad n_1' = n_1 K_{\frac{3}{2}}(\delta_{1e}),),$$

$$m_2' = m_2 I_{\frac{3}{2}}(\delta_{2i}), n_2' = n_2 I_{\frac{3}{2}}(\delta_{2i})$$

slip parameter$(s) = \frac{\beta a}{\mu}, \mu = \frac{\mu_i}{\mu_e}$,

$$\Delta_1\left(\delta_{1e}\right) = 1 + \frac{\delta_{1e}\; K_{\frac{1}{2}}(\delta_{1e})}{K_{\frac{3}{2}}(\delta_{1e})},$$

$$\Delta_1\left(\delta_{2e}\right) = 1 + \frac{\delta_{2e}\; K_{\frac{1}{2}}(\delta_{2e})}{K_{\frac{3}{2}}(\delta_{2e})},$$

$$\Delta_2\left(\delta_{1i}\right) = 1 + \frac{\delta_{1i} I_{\frac{1}{2}}(\delta_{1i})}{I_{\frac{3}{2}}(\delta_{1i})}, \Delta_2\left(\delta_{2i}\right) = 1 + \frac{\delta_{2i} I_{\frac{1}{2}}(\delta_{2i})}{I_{\frac{3}{2}}(\delta_{2i})},$$

Cross viscosity parameter

$$c_e = \frac{k_e}{\mu_e + k_e}, c_i = \frac{k_i}{\mu_i + k_i}$$

.

The case of oscillatory Newtonian fluid can be obtained by taking $\delta_2 \longrightarrow \infty$. The system of equations solving analytically gives parameter values as

$$\left.\begin{aligned}
&l_1 = -1 + n_1'(\epsilon_e - 1), \quad l_2 = -1 + n_2'(\epsilon_{ec} - 1),\\
&m_1' = -n_1'\epsilon, \quad m_2' = -n_2'\epsilon,\\
&n_1' = \frac{\left(-3z_1 s - 6\right)N_4 + 6N_2}{\eta}, \quad n_2' = \frac{-6N_1 + (3sz_1 + 6)N_3}{\eta},
\end{aligned}\right\} \tag{3.27}$$

Here $\eta = N_1 N_4 - N_2 N_3$,

$$N_1 = 2\left(1 - \epsilon\right) + z_2\left(\delta_{1e}^2\epsilon - \delta_{2e}^2\right) + 2\Delta_3 + sz_1\left(1 - \epsilon + \Delta_3\right),$$

$$N_2 = -sz_1\left(2\left(1-\epsilon_{in}\right)+\Delta_4\right),$$

$$N_3 = 2\left(1-\epsilon\right)+z_2\left(\delta_{1e}^2\epsilon-\delta_{2e}^2\right)+2\Delta_3$$

$$N_4 = 2\left(1-\epsilon\right)+z_2\left(\delta_{1i}^2\epsilon-\delta_{2i}^2\right)+2\Delta_3$$

$$\Delta_3 = \left(\Delta_1\left(\delta_{1e}\right)\epsilon_{-}\Delta_1\left(\delta_{2e}\right)\right), \quad \Delta_4 = \left(\Delta_2\left(\delta_{1i}\right)\epsilon-\Delta_2\left(\delta_{2i}\right)\right)$$

$$z = \frac{(2-C_i)^2k_0 + C_iC_0(2-C_e)}{(2-C_e)C_o(2-C_i)}, \ \epsilon = \frac{i\sigma-\delta_2^2}{i\sigma-\delta_1^2} \ \ [z_1 = \frac{2(1-c_e)}{2-c_e} \quad z_2 = \frac{2}{2-c_e}$$

Thus, the stream function values (3.17) and (3.18) are obtained.

4. Drag force

The force dragF_z of an flow of oscillatory of the liquid beyond a body is given as [20].

$$F_z = i\rho\omega UV_0 + 4\pi i\rho\omega \lim_{r\to\infty}\frac{r\left(\psi_e-\psi_\infty\right)}{sin^2\theta}\ , \tag{4.1}$$

Substituting (3.17), (3.18), we get

$$F_z = i\rho\omega UV_0 + 2\pi i\rho\omega \lim_{r\to\infty}\left(l_1 + m_1r^{\frac{3}{2}}K_{\frac{3}{2}}\left(\delta_{1e}r\right)+n_1r^{\frac{3}{2}}K_{\frac{3}{2}}\left(\delta_{2e}r\right)\right)$$

As $r\longrightarrow\infty$, $m_1 = n_1 = 0$ (3.19), V_0 = volume of the spherical body = $\frac{4}{3}\pi r^3$.

$$F_z = \frac{4}{3}\pi i\rho\omega U_\infty a^2\left(-z_0+\frac{3}{2}l_1a\right)e^{i\omega t}, \qquad where \ \ z_0 = l_1+m_1'+n_1' \tag{4.2}$$

$F_z = 2\pi i\rho\omega U_\infty a^3l_1e^{i\omega t}$
Now (4.2) is expressed in the form
$F_z = M\omega U_\infty(-T1-iT)l_1e^{i\omega t}$.
Where M is the mass of the fluid laid-out by the sphere (M) = $2\pi\rho\omega U_\infty a^3l_1$,
Real drag(T) =$2\pi\rho\omega U_\infty a^3l_1 sin\omega t$ Imaginary drag (T1) = $-2\pi\rho\omega U_\infty a^3l_1 cos\omega t$, $\omega = \frac{\sigma(\mu+k)}{\rho a^2}$.
For various quantities of the frequency factor σ, microrotation factor δ, and cross viscosity factor c, the drag indices T and T1 are computed.
As $\delta_2\longrightarrow\infty$, $\mu\longrightarrow\infty$,$s\longrightarrow\infty$ then it turns to a no-slip solid sphere with oscillating viscous liquid, which corresponds to the drag force calculated by [16].

5. Oscillatory Non-Newtonian Fluid Flow Beyond a Newtonian Fluid Sphere

In this section, we considered a non-Newtonian oscillating fluid flow across a viscous fluid sphere that is kept stationary. The previously estimated momentum equations (3.12) and (3.13) hold good here also. The stream functions for equation (3.12) are

$$\psi_e = \left(r^2+\frac{l_1}{r}+m_1\sqrt{r}K_{\frac{3}{2}}\left(\delta_{1e}r\right)+n_1\sqrt{r}K_{\frac{3}{2}}\left(\delta_{2e}r\right)\right)G_2\left(x\right), \tag{5.1}$$

$$\psi'_i = (l_2r^2+m_2\sqrt{r}I_{\frac{3}{2}}\left(\delta_{1i}r\right))G_2\left(x\right), \tag{5.2}$$

6. Modelling

The parameters of (5.1) and (5.2) are computed through (3.19) - (3.25). We get

$$l_1 + m_1' + n_1' = -1, \quad l_2 + m_2' = 0,$$
$$6l_1 + m_1' \left(z_2\delta_{1e}^2 + 4 + 2\Delta_1(\delta_{1e})\right) + n_1' \left(z_2\delta_{2e}^2 + 4 + 2\Delta_1(\delta_{2e})\right) - \mu m_2' \left(\delta_{1i}^2 + 4 + 2\Delta_2(\delta_{1i})\right) = 0,$$

$$l_1(6 + sz_1) + m_1' \left(z_2\delta_{1e}^2 + 4 + 2\Delta_1(\delta_{1e}) + sz_1\Delta_1(\delta_{1e})\right) + n_1' \left(z_2\delta_{2e}^2 + 4 + 2\Delta_1(\delta_{2e}) + sz_1\Delta_1(\delta_{2e})\right) - m_2' sz_1 \Delta_2(\delta_{1i}) + 2l_2 sz_1 = (2z_1)s,$$

$$m_1' \left(\mathrm{i}\sigma - \delta_{1\mathrm{e}}^2\right) + n_1' \left(\mathrm{i}\sigma - \delta_{2\mathrm{e}}^2\right) = 0.$$

solving above equations we get

$$\left. n_1' = \frac{(3z_1 s + 6)a_4 - 6a_2}{a_5}, \quad m_2' = \frac{-6a_1 + (3sz_1 + 6)a_3}{a_5}, \quad \right\} \tag{6.1}$$

where $a_5 = a_1a_4 - a_2a_3$,

$$a_1 = (sz_1 + 2)\left((1 - \epsilon - \Delta_3) - z_2\left(\delta_{1e}^2\epsilon - \delta_{2e}^2\right)\right),$$

$$a_2 = -sz_1(2 + \Delta_2(\delta_{1i})),$$

$$a_3 = 2(\epsilon - 1 - \Delta_3) - z_2\left(\delta_{1e}^2\epsilon - \delta_{2e}^2\right)$$

$$a_4 = -\mu\left(\delta_{1i}^2 + 4 + 2\Delta_{1i}\right)$$

Thus, the stream functions values (5.1) and (5.2) are obtained.

7. Oscillatory Flow of Viscous Fluid Outwith a Non-Newtonian Fluid Sphere

In this section, we studied at a non-Newtonian liquid oscillating across a viscous fluid sphere that was stationary. The previously estimated equations (3.12) and (3.13) hold good here also.
The solutions of equation (3.12) and (3.13) here are

$$\psi'_e = \left(r^2 + \frac{l_1}{r} + m_1\sqrt{r}K_{\frac{3}{2}}(\delta_{1e}r)\right)G_2(x), \tag{7.1}$$

$$\psi_i = (l_2 r^2 + m_2\sqrt{r}I_{\frac{3}{2}}(\delta_{1i}r) + n_2\sqrt{r}I_{\frac{3}{2}}(\delta_{2i}r))G_2(x), \tag{7.2}$$

solution By the boundary condition (3.19) - (3.25), we get five equations with five unknowns as

$$l_1 + m_1' = -1, \quad l_2 + m_2' + n_2' = 0,$$
$$-6l_1 + m_1' \left(\delta_{1e}^2 + 4 + 2\Delta_1(\delta_{1e})\right) + m_2' \left(z\delta_{1e}^2 + 4 + 2\Delta_2(\delta_{1i})\right) + n_2' \left(z\delta_{2i}^2 + 4 + 2\Delta_2(\delta_{1i})\right) = 0,$$

$$l_1(4 + s) + m_1' \left(2 + \delta_{1e}^2 + 4 + 2\Delta_1(\delta_{1e}) + sz_1\Delta_1(\delta_{1e})\right) + 2l_2 s - n_2'\Delta_2(\delta_{2i})s - m_2' sz_1\Delta_2(\delta_{1i}) = 2s + 2,$$

$$m_2' \left(\mathrm{i}\sigma - \delta_{1\mathrm{i}}^2\right) + n_2' \left(\mathrm{i}\sigma - \delta_{2\mathrm{i}}^2\right) = 0.$$

Solving analytically, we get

$$l_1 = -1 - m', \; m_2' = -n_2'\epsilon, \; l_2 = n_2'(\epsilon - 1), \; m_1' = \frac{(3s + 6)w_4 + 6w_2}{w_5}, \; n_2' = \frac{-(3s + 6)w_3 + 6w_1}{w_5}$$

$$w_5 = w_1 w_4 - w_2 w_3$$
$$w_1 = \delta_{1e}^2 + (2+s)(\Delta_1(\delta_{1e} - 1)),$$
$$w_2 = s(2(\epsilon - 1) + \Delta_4),$$
$$w_1 = \delta_{1e}^2 + 2(\Delta_1(\delta_{1e} - 1)),$$
$$w_4 = (\epsilon\delta_{1i}^2 - \delta_{2i}^2)z + (\epsilon - 1) + 2\Delta_4).$$

Thus, the stream functions values (7.1) and (7.2) are obtained.

8. Oscillatory Viscous Fluid Outwith a Viscous Fluid Sphere

An oscillatory viscous fluid flow over a Newtonian fluid sphere which is stationary in the stream is assumed. The geometry of the dual flow of viscous fluid problem is given in Figure 2.

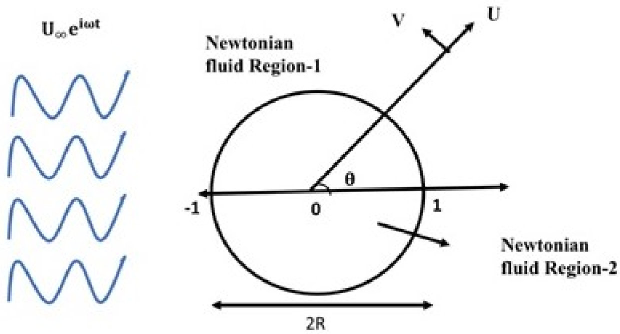

Figure 2: Geometry of oscillatory flow of viscous fluid beyond a viscous fluid sphere.

when $k \longrightarrow 0$, the oscillatory viscous fluid's momentum equation is accomplished in (3.3) and further simplification as

$$E^2 \left(E^2 - \delta_1^2\right) \psi = 0, \quad \delta_1^2 = \frac{\rho i \omega}{\mu} \tag{8.1}$$

The external and internal stream functions satisfying (7.1) are given by ψ_e, ψ_i respectively.
By separation of variables method, we get stream functions as

$$\psi'_e = \left(r^2 + \frac{l_1}{r} + m_1 \sqrt{r} K_{\frac{3}{2}}\left(\delta_{1e} r\right)\right) G_2\left(x\right), \tag{8.2}$$

$$\psi'_i = (l_2 r^2 + m_2 \sqrt{r} I_{\frac{3}{2}}\left(\delta_{1i} r\right)) G_2\left(x\right), \tag{8.3}$$

The arbitrary constants in (8.2), (8.3) are computed through boundary conditions (3.19) – (3.23): With (3.19) - (3.23) we get four equations with four unknowns as

$$l_1 + m_1' = -1, \quad l_2 + m_2' = 0,$$
$$-6l_1 + m_1' \left(\delta_{1e}^2 + 4 + 2\Delta_1\left(\delta_{1e}\right)\right) + m_2' \left(z\delta_{1e}^2 + 4 + 2\Delta_2\left(\delta_{1i}\right)\right) + n_2' \left(z\delta_{2i}^2 + 4 + 2\Delta_2\left(\delta_{1i}\right)\right) = 0,$$
$$l_1\left(4+s\right) + 2l_2 + m_1' \left(\delta_{1e}^2 + 2 + \Delta_1\left(\delta_{1e}\right) + (s+2)\right) - s m_2' \Delta_2(\delta_{1i}) = 2s + 2,$$

where $m_1' = m_1 k_{\frac{3}{2}}(\delta_1)$, $m_2' = m_1 I_{\frac{3}{2}}(\delta_1)$, slipparameter (s)$=\frac{\beta a}{\mu}$, $\mu = \frac{\mu_i}{\mu_e}$

$$\Delta_1(\delta_{1e}) = 1 + \frac{\delta_{1e}\ K_{\frac{1}{2}}(\delta_{1e})}{K_{\frac{3}{2}}(\delta_{1e})},$$

$$\Delta_2(\delta_{1i}) = 1 + \frac{\delta_{2e}\ I_{\frac{1}{2}}(\delta_{1i})}{I_{\frac{3}{2}}(\delta_{1i})},$$

Cross viscosity parameter

$$c_e = \frac{k_e}{\mu_e + k_e}, c_i = \frac{k_i}{\mu_i + k_i}$$

.

Solving equations we get

$$l_1 = -1 - m_1', \quad l_2 = -m_2',$$

$$l_1\mu(\delta_{1i}^2 + 4 + 2\Delta_2(\delta_{1i}))$$

+

$$l_2\mu(\delta_{1i}^2 + 4 + 2\Delta_2(\delta_{1i})) = 2(2 + \Delta_1(\delta_{1e})) + \delta_{1e}^2,$$

$$l_1(s + 2 - \delta_{1e}^2 - \Delta_1(\delta_{1e}(s+2)) + l_2 s(2 + \Delta_2(\delta_{1i})) = (s+2)(2 + \Delta_1(\delta_1 e)) + \delta_{1e}^2,$$

here $m_1' = \frac{(3s+6)B_4 - 6B_2}{B_5}$, $m_2' = \frac{6B_1 - (3S+6)}{B_3}$
$B_5 = B_1 B_4 - B_2 B_3$,
where $B_1 = (s+2)(\Delta_1(\delta 1e) - 1) + \delta_{1e}^2$ $B_2 = s(-2 - \Delta_2(\delta_{1i}))$, $B_3 = -2 + 2\Delta_1(\delta_{1e}) + \delta_{1e}^2$, $B_4 = -\mu(\delta_{1i}^2 + 4 + 2\Delta_2(\delta 1i))$

9. Drag force on oscillatory Newtonian Fluid Flow Beyond a Newtonian Fluid Sphere

The drag force F_z of oscillatory flow of fluid past a body is [20]

$$F_z = i\rho\omega U V_0 + 4\pi i\rho\omega \lim_{r\to\infty} \frac{r(\psi_e - \psi_\infty)}{sin^2\theta}, \tag{9.1}$$

$$F_z = i\rho\omega U V_0 a + 2\pi i\rho a\omega U_\infty \lim_{r\to\infty}\left(l_1 + m_1 r^{\frac{3}{2}} K_{\frac{3}{2}}(\delta_{1e} r)\right)$$

As $r \longrightarrow \infty$, $m_1 = 0$ (3.19), $V_0 =$ volume of the body $= \frac{4}{3}\pi r^3$.

$$F_z = \frac{4}{3}\pi i\rho\omega U_\infty a^2\left(-h_0 + \frac{3}{2}l_1 a\right)e^{i\omega t}, \qquad where\ \ h_0 = 1 + l_1 + m_1' \tag{9.2}$$

The expression for the drag then becomes

$$F_z = 2\pi i\rho\omega U_\infty a^3 l_1 e^{i\omega t} \tag{9.3}$$

here (9.3) can be expressed as

$$F_z = M\omega U_\infty(-T1 - iT)l_1 e^{i\omega t}.$$

Here M is the mass of the fluid laid-out sphere (M) $= 2\pi\rho\omega U_\infty a^3 l_1$, Real drag(T) $=2\pi\rho\omega U_\infty a^3 l_1 sin\omega t$ Imaginary drag (T1) $= -2\pi\rho\omega U_\infty a^3 l_1 cos\omega t$, $\omega = \frac{\sigma(\mu+k)}{\rho a^2}$.

where M is the mass of the fluid laid-out sphere In this instance, T and $T1$ values are calculated for various values of the frequency factor σ, microrotation factor s, and cross viscosity factor c.

As $\mu \longrightarrow \infty, s \longrightarrow \infty$, it converts to the no-slip rigid sphere case, which matches the force of the drag computed by [16]

Table 1: Slip (s) vs Real Drag (T) at vary of cross viscosity (c) points at fixed values to frequency parameters μ=10, k=0.1, ρ=0.6, ω=0.6, t=0.6.

$T \setminus s$	**4**	**8**	**12**	**16**
2	-0.2483	-0.2483	-0.2483	-0.2483
4	-0.2851	-0.2851	-0.2851	-0.2851
6	-0.2969	-0.2969	-0.2969	-0.2969
8	-0.3027	-0.3027	-0.3027	-0.3027
10	-0.3062	-0.3062	-0.3062	-0.3062
12	-0.3085	-0.3085	-0.3085	-0.3085
14	-0.3101	-0.3101	-0.3101	-0.3101
16	-0.3113	-0.3113	-0.3113	-0.31133

10. Results and Discussions

For an oscillatory flow of micropolar beyond a micropolar fluid sphere, the real (T) and imaginary part ($T1$) of drag force variations for slip parameters (s) with varying micro polarity e (ϵ values with cap and no-cap zones) i.e., $\epsilon = \frac{(i\sigma-\delta_2^2)}{(i\sigma-\delta_1^2)}$ at fixed values to frequency parameters μ =10,k=0.1,ρ =0.6, ω=0.6,t=0.6 are presented numerically and are shown in Fig. 3 and Fig. 4.

Fig. 3 shows the relationship between real drag (T) and the slip parameter (s) for various cross viscosity(c) values. Real drag (T) was shown to decrease simultaneously with an increase in slip parameter (s) values, whereas cross viscosity (c) variations had no effect on real drag (T) values. Table 1 displays the numerical values.

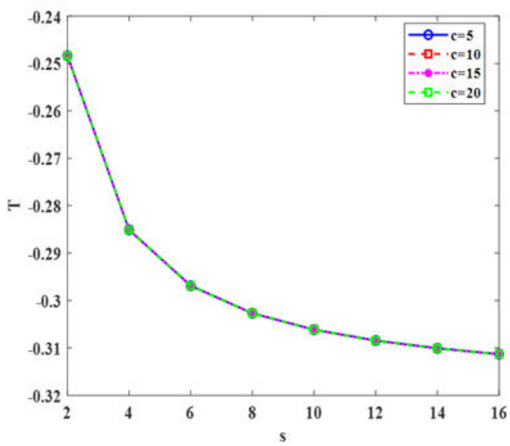

Figure 3: Slip (s) vs Real Drag (T) at vary of cross viscosity (c) points at fixed to frequency parameters μ=10, k=0.1, ρ=0.6, ω=0.6, t=0.6.

Fig. 4 shows the relationship between imaginary drag (T1) and the slip parameter (s) for various cross viscosity(c) values. It is found that an rise in slip (s) values causes an fall in Imaginary drag (T1) values, whereas cross viscosity (c) variations had no effect on imaginary drag (T1) values. Table 2 displays the numerical values.

11. Conclusion:

In this paper, we aimed to find an analytic solution for the following

1. Oscillatory flow of micropolar outwith a micropolar fluid sphere,
2. Oscillatory micropolar fluid flow outwith a viscous liquid sphere,

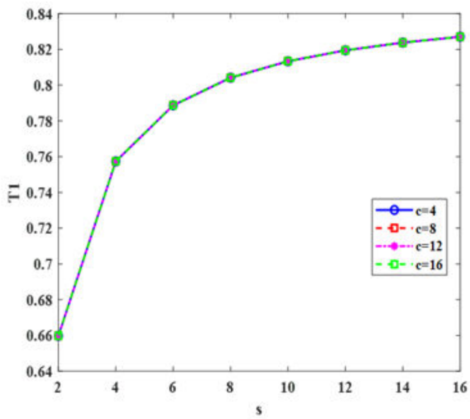

Figure 4: Slip (s) vs Imaginary Drag(T1) at vary of cross viscosity (c) points at fixed to frequency parameters μ=10, k=0.1, ρ=0.6, ω=0.6, t=0.6.

Table 2: Slip parameter (s) vs Imaginary Drag(T1) at vary of cross viscosity (c) points at fixed values to frequency parameters μ=10, k=0.1, ω=0.6,t=0.6 ρ=0.6.

$T' \setminus s$	**4**	**8**	**12**	**16**
2	0.6598	0.6598	0.6598	0.6598
4	0.7574	0.7574	0.7574	0.7574
6	0.7887	0.7887	0.7887	0.7887
8	0.8042	0.8042	0.8042	0.8042
10	0.8134	0.8134	0.8134	0.8134
12	0.8195	0.8195	0.8195	0.8195
14	0.8238	0.8238	0.8238	0.8238
16	0.8271	0.8271	0.8271	0.8271

3. Oscillatory flow of Newtonian fluid outwith a non-Newtonian fluid sphere,
4. Oscillatory Newtonian fluid outwith a Newtonian fluid sphere,

The axisymmetric rectilinear of a liquid sphere in an incompressible non-Newtonian medium is based on the idea of micro amplitude fluctuations.

The fluid sector and microrotation elements were found to have precise outcomes.A few specific situations are minimised in order to estimate the drag force application on the particle for rectilinear oscillations, which is consistent with the data that is currently accessible in the literature.Results are inferred by extracting and graphically displaying differences in real drag (T), imaginary drag (T1) with respect to micro polarity (e), slip parameter(s), and cross viscosity (c).At different cross viscosity (c) values, it is seen that true drag (T) values decrease as slip parameter values increase.Additionally, for a range of cross viscosity (c) values, an increase in slip values results in an increase in imaginary drag (T1).

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