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Balanced Graphs from Lexicographic Products of Open Neighborhood Graphs

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ABSTRACT: This paper investigates the structural balance properties of Lexicographic Product graphs formed by combining standard graph classes with their corresponding Open Neighborhood graphs. Specifically, we construct the Lexicographic Product $G' = G[N(G)]$, where G is a standard graph and $N(G)$ denotes its open neighborhood graph. We examine the balance of the resulting signed graphs under various edge sign assignments. For each graph class considered, we rigorously demonstrate that the resulting Lexicographic Product graph is both regular and structurally balanced. The sign assignment methodology is derived from adjacency relationships in both G and $N(G)$. Through detailed examples and structural proofs, we confirm that the signed Lexicographic Product graphs consistently exhibit balance, underscoring their significance in the study of signed and structured networks.

Key Words: Lexicographic product, balanced graph, signed graph, open neighborhood graph, regular graph.

Contents

1. Introduction

In graph theory, the study of signed graphs (where each edge is assigned either a positive or negative sign) plays a significant role in analyzing structural balance in complex networks. A signed graph is formally defined as a pair (G, σ), where $G = (V, E)$ is a Graph and $\sigma : E \to \{+1, -1\}$ is a sign function that maps each edge to a sign. This paper investigates the structural balance of Lexicographic Product graphs derived from classical graph families, namely, cycle graphs, complete graphs, and wheel graphs, along with their corresponding Open Neighborhood Graphs. The notion of balance in this context follows the principle that a signed graph is balanced if every cycle contains an even number of negative edges.

The Lexicographic Product of two graphs G_1 and G_2, denoted $G_1[G_2]$, forms a new graph that enhances structural complexity while preserving fundamental adjacency properties. This research focuses on constructing such products using standard graphs and their Open Neighborhood graphs, and further by assigning edge signs based on defined adjacency rules.The results proved in this paper are mainly construct the graphs which are both regular, balanced and making them highly relevant for modeling stable network structures in both theoretical and applied domains. The work contributes novel proofs and visual illustrations that confirm balance in each constructed case, reinforcing the practical utility of Lexicographic Products in signed graph theory.

The concept of structural balance in signed graphs was initially introduced by Harary [5], building on foundational research in social psychology that mathematically modeled networks of friends and enemies [12]. Zaslavsky further advanced signed graph theory by formalizing cycle-based criteria for balance and developing an algebraic framework to handle edge signs [11]. Regarding graph products, Imrich and Klavžar provided an in-depth study of various graph product operations, including the lexicographic product, which laid important groundwork for analyzing their structural properties [6]. Additionally, Godsil and Royle offered an algebraic perspective that expanded the theoretical understanding of the

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regularity and spectral characteristics of such graphs [4]. The author [8] examined the edge structure of lexicographic product using open neighborhood graphs in some regular complexity graphs.

Building on these foundational works, recent research has increasingly focused on applying graph theory to complex systems. For example, Kothimbire et al. [7] presented a thorough review of graph-theoretic methods for analyzing network structures, particularly emphasizing balance and connectivity. Wang [9] demonstrated the significance of signed and lexicographic graphs in social network analysis, illustrating their usefulness for modeling positive and negative interpersonal relationships. In geospatial analytics, Ghosh et al. [3] applied graph-theoretic tools to model and interpret spatial data structures. Furthermore, Dayap et al. [2] employed graph theory to evaluate collaboration and citation networks, underscoring the role of balance in sustaining coherent research communities. Collectively, these studies emphasize the growing relevance of lexicographic product and open neighborhood graphs not only as theoretical constructs but also as practical tools in network science. This paper contributes to this expanding body of work by proving the balance and regularity of lexicographic products formed from standard graphs and illustrating their applicability through examples in social and structural network analysis.

Definition 1.1 ***Open Neighborhood:***
For any finite, undirected, and simple graph $G = (V, E)$ where $|V| = p$ and $|E| = q$, the open neighborhood of a vertex $u \in V$ is defined as:

$$N(u) = \{v \in V : uv \in E\}.$$

The collection of all open neighborhoods in G is denoted as the set:

$$S = \{N(u_1), N(u_2), \ldots, N(u_p)\}.$$

Definition 1.2 ***Lexicographic Product:***
The lexicographic product of two graphs G and H, denoted $G[H]$, constructs a graph with vertex set:

$$V(G[H]) = V(G) \times V(H),$$

and edge set defined such that:

$$\big((u, v), (u', v')\big) \in E(G[H]) \iff (u = u' \text{ and } vv' \in E(H)) \text{ or } (uu' \in E(G)).$$

This product enriches the structural analysis of graph interactions and provides tools to model and evaluate multi-layered networks.

Definition 1.3 ***Signed Graph:***
A signed graph is a pair (G, σ), where $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The function $\sigma : E \to \{+1, -1\}$ is called the sign function*, and it assigns a sign (positive or negative) to each edge.*

2. Main Results

Theorem 1. Let C_n be a cycle graph with n vertices ($n \geq 3$), and let $N(C_n)$ be its Open Neighborhood Graph. Then the lexicographic product $G' = C_n[N(C_n)]$ is balanced.

Proof: Let C_n be the cycle graph with n vertices, as shown in Fig. 1.

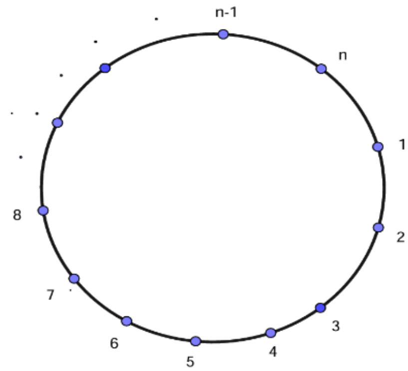

Figure 1: Cycle C_n

Let $N(C_n)$ be the Open Neighborhood Graph of a cycle graph with $2n$ vertices (see Fig. 2).

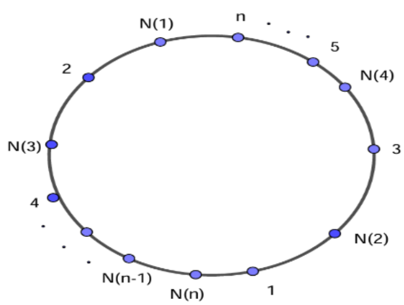

Figure 2: Open Neighborhood Graph $N(C_n)$

Now consider the Lexicographic Product Graph $G' = C_n[N(C_n)]$, which has the vertex set $V(G') = V(C_n) \times V(N(C_n)) = \{(v_i, u_j) \mid v_i \in V(C_n),\ u_j \in V(N(C_n))\}$,where $1 \leq i \leq n$, and $1 \leq j \leq 2n$.

$\therefore$ The total number of vertices in G' is $n \times 2n = (2n)^2$.

By the definition of the Lexicographic product, the vertices (v_{i_1}, u_{j_1}) and (v_{i_2}, u_{j_2}) are adjacent in G' if either

(i) v_{i_1} is adjacent to v_{i_2} in C_n, or

(ii) $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(C_n)$.

Using the above conditions, the Lexicographic Product Graph G' is constructed as shown in Fig. 3.

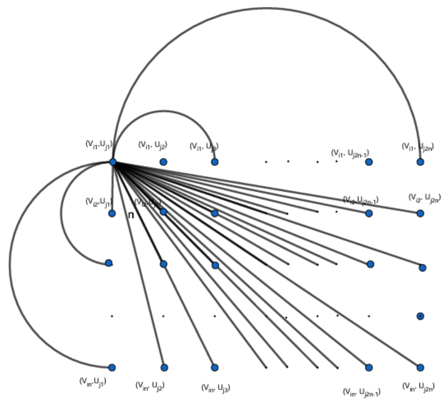

Figure 3: Lexicographic Product Graph of one vertex

In the above graph, we observe that the number of edges incident on each vertex is one from the cycle, and two from each of the $2n$ vertices in the Open Neighborhood Graph. For every vertex (v_{i_1}, u_{j_1}), there are

$$2 + 2n + 2n = 2 + 4n \text{ edges incident.}$$

$$\therefore d(v_{i_1}, u_{j_1}) = 2 + 4n.$$

Since in a cycle graph, every vertex has degree 2, the resulting graph G' is a $(4n + 2)$-regular graph. Now, we need to prove that G' is a balanced signed graph.

To do this, we assign signs ($+$ or $-$) to the edges as follows:

- Assign $+$ (or $-$) if the edge exists because v_{i_1} is adjacent to v_{i_2} in C_n for the vertices (v_{i_1}, u_{j_1}) and (v_{i_2}, u_{j_2}).
- Assign $-$ (or $+$) if $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(C_n)$.

Since the number of incident edges on each vertex in the above graph is even and the cycles formed have an even number of negative edges due to symmetric assignment of signs, the graph G' is obviously balanced.

Example: Let $n = 4$

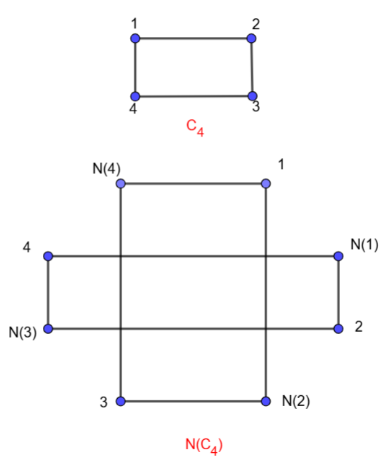

Figure 4: Cycle Graph C_4 and its Open Neighborhood Graph $N(C_4)$

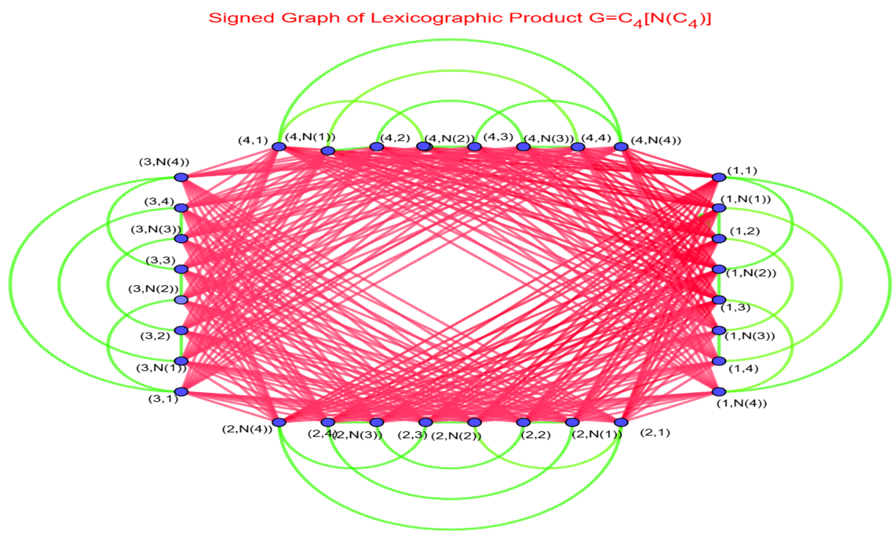

Figure 5: Signed Graph of Lexicographic Product of $G' = C_4[N(C_4)]$. Green color indicates +ve edges, Red color indicates −ve edges

Theorem 2. **Let K_n be a complete graph with n vertices ($n \geq 4$) and $N(K_n)$ be the Open Neighborhood Graph of K_n. Then the lexicographic product graph $G' = K_n[N(K_n)]$ is balanced.**

Proof. To prove this, consider the complete graph K_n with n vertices ($n \geq 4$), as shown in Figure 6.

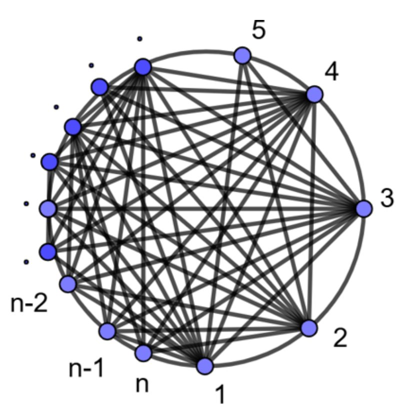

Figure 6: Complete Graph K_n

Then the graph $N(K_n)$ represents the Open Neighborhood Graph of K_n with $2n$ vertices, as shown in Figure 7.

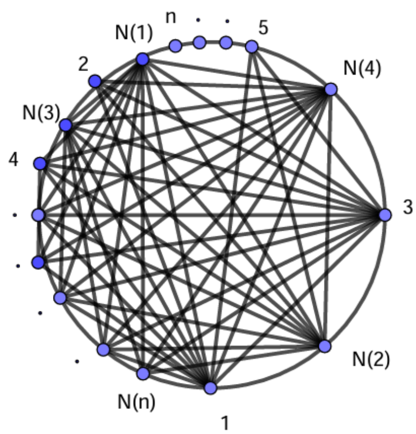

Figure 7: Open Neighborhood Graph $N(K_n)$

The lexicographic product graph is denoted by

$$G' = K_n\big[N(K_n)\big],$$

then the vertex set of G' is given by

$$V(G') = V(C_n) \times V(N(C_n)) = \{(v_i, u_j) \mid v_i \in V(C_n), u_j \in V(N(C_n))\},$$

where $1 \leq i \leq n$ and $1 \leq j \leq 2n$.

In G', the total number of vertices is $n \times 2n = 2n^2$.

Using the definition of the lexicographic product, the vertices (v_{i_1}, u_{j_1}) and (v_{i_2}, u_{j_2}) are adjacent in G' if

(i) v_{i_1} is adjacent to v_{i_2} in K_n, or

(ii) $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(K_n)$.

By applying these adjacency conditions, the lexicographic product graph G' is illustrated in Figure 8.

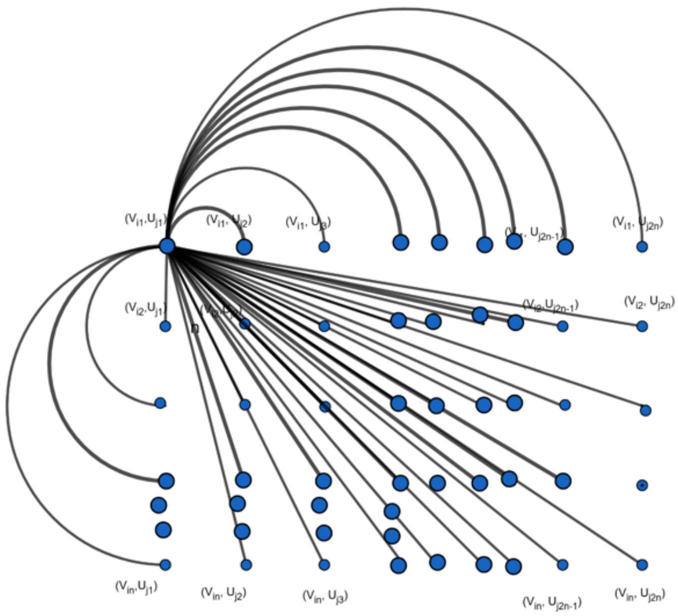

Figure 8: Lexicographic Product Graph $G' = K_n[N(K_n)]$

The structure of G' shows that the number of edges incident on each vertex is determined as follows. Each vertex (v_{i_1}, u_{j_1}) in G has (n - 1) + (2n)(n - 1) = (2n + 1)(n - 1) edges adjacent to other vertices.

$$\therefore \quad d\,(v_{i_1}, u_{j_1}) = (2n+1)(n-1).$$

Since in the complete graph, each vertex has degree $(n-1)$, this makes G' a $(2n+1)(n-1)$-regular graph.

The objective is now to establish that G' is balanced.

To achieve this, we assign the positive and negative signs systematically:

- A **positive edge** exists if v_{i_1} and v_{i_2} are adjacent in K_n, or

- A **negative edge** is assigned if $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(K_n)$.

By applying these two cases to all edges in G, it follows that the graph G' is balanced. Consequently, the lexicographic product graph $G' = K_n\big[N(K_n)\big]$ is balanced. Hence, the proof is complete.

Example: If $n = 4$

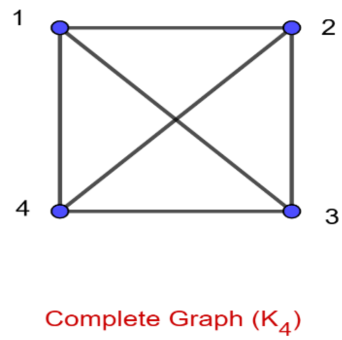

Figure 9: Complete Graph K_4

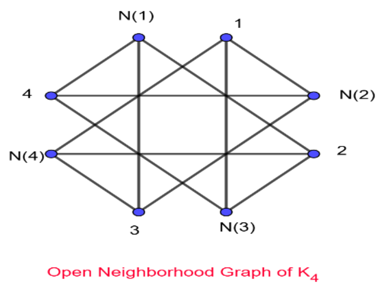

Figure 10: Open Neighborhood Graph $N[K_4]$

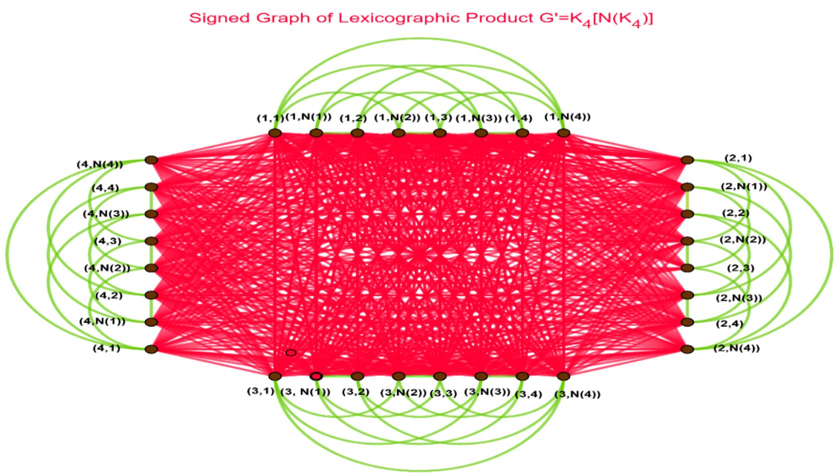

Figure 11: Signed graph of the lexicographic product $G' = K_4[N(K_4)]$. Green color indicates positive edges, red color indicates negative edges.

Theorem 3: Let W_n be a Wheel Graph of n vertices ($n \geq 5$) and $N(W_n)$ be its Open Neighborhood Graph. Then the Lexicographic Product Graph G' is balanced.

Proof: Let us denote the Wheel graph W_n as the graph formed by connecting a single central vertex c to all vertices of a cycle C_{n-1}. Hence, W_n consists of n vertices: one central vertex and $n - 1$ cycle vertices, as shown in Fig. 12.

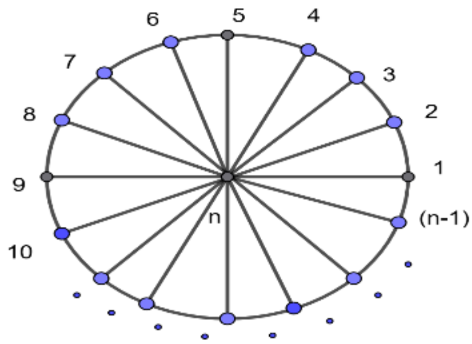

Figure 12: Wheel Graph W_n

Then $N(W_n)$ is the Open Neighborhood Graph of W_n with $2n$ vertices, as illustrated in Fig. 13.

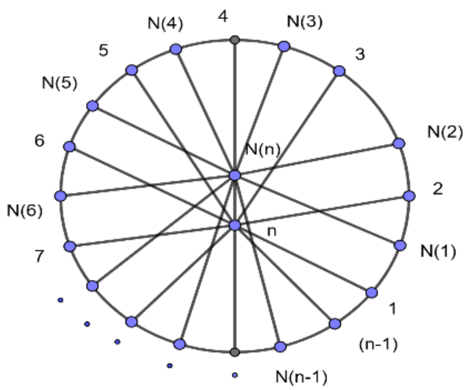

Figure 13: Open Neighborhood Graph $N(W_n)$

Now, define the Lexicographic product graph $G' = W_n[N(W_n)]$, where the vertex set is defined as:

$$V(G') = V(W_n) \times V(N(W_n)) = \{(v_i, u_j) \mid v_i \in V(W_n),\ u_j \in V(N(W_n))\},\ \text{where } 1 \leq i \leq n,\ 1 \leq j \leq 2n.$$

Since there are $n \times 2n = 2n^2$ vertices in G'. By the definition of Lexicographic Product, the vertices (v_{i_1}, u_{j_1}) and (v_{i_2}, u_{j_2}) are adjacent in G' if:

- v_{i_1} is adjacent to v_{i_2} in W_n, or
- $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(W_n)$.

In the lexicographic product G', the number of side vertices is $2n(n-1)$, and the number of center vertices are $2n$.

Applying these adjacency conditions, the Lexicographic Product Graph G' is illustrated in Fig. 14.

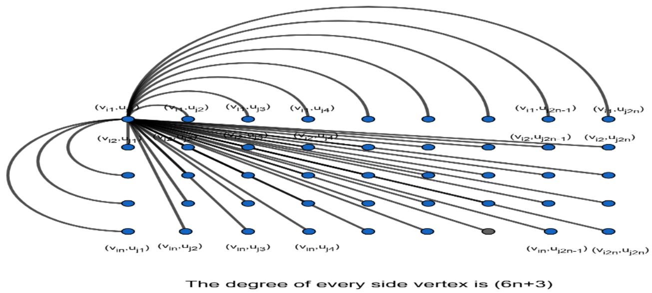

Figure 14: The degree of each side vertex in Lexicographic Product Graph

The above graph G' shows that the number of edges incident on each side vertex is

$$(n-1)(2n)(6n+3) \tag{1}$$

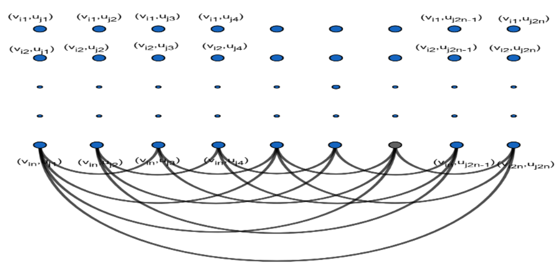

Figure 15: The degree of center vertex in Lexicographic Product Graph

The above Figure 15 shows that the number of edges incident on the center vertex.
In $N(W_n)$, the center vertex is also adjacent to other vertices. So, the number of such edges is

$$(2n)(n-1)(2n) + (n-1) = (n-1)(2n)(2n+1) \tag{2}$$

Adding equations (1) and (2), we have:

$$(2n)(n-1)(6n+3) + (2n)(n-1)(2n+1) = 2n(n-1)(6n+3+2n+1)$$

$$= 2n(n-1)(8n+4) = 8n(n-1)(2n+1)$$

Now, we have to determine whether G' is balanced.
To prove this, we assign the signs $+$ and $-$ to the edges in the following cases:

- If a $+$ sign is assigned, the edge exists between v_{i_1} and v_{i_2}, which are adjacent in W_n.
- If a sign $-$ (or $+$) is assigned, the edge exists between the vertices (v_{i_1}, u_{j_1}) and (v_{i_2}, u_{j_2}) such that $v_{i_1} = v_{i_2}$ and u_{j_1} is adjacent to u_{j_2} in $N(W_n)$.

Using the above conditions for every edge in G' systematically, we conclude that the graph G' is balanced.
Hence, the lexicographic product graph $G' = W_n[N(W_n)]$ is balanced.

Corollary 1.
Let S_n be the star graph with $n+1$ vertices and let $N(S_n)$ be the Open Neighborhood Graph of S_n. Then the lexicographic product graph $G' = S_n[N(S_n)]$ is a balanced signed graph.

Corollary 2.
Let P be a planar graph in which every face, including the outer face, is bounded by a cycle of length at least 3. Let $N(P)$ denote the Open Neighborhood Graph of P. Consider the lexicographic product $G' = P[N(P)]$. Then the resulting lexicographic product graph G' forms a balanced signed graph.

3. Application

Balanced Graph from Lexicographic Product Graph of Open Neighborhood Graphs in Social Network:
Consider a signed graph by assigning a function $\sigma : E \to \{+1, -1\}$ to the edges.
Let the social network G' consist of four individuals A, B, C, D with relationships:

Person	Friends (+)	Rivals (−)
A	B, D	C
B	A, C	D
C	B	A
D	A	B

The Lexicographic Product Graph of Open **Neighborhood** Graphs offer valuable information about community structures and interpersonal relationships. Such applications are particularly relevant in the analysis of social networks [7,9], geospatial systems [3], and the dynamics of research collaboration [2].
Adjacency Matrix (Signed):

	A	B	C	D
A	0	+1	-1	+1
B	+1	0	+1	-1
C	-1	+1	0	0
D	+1	-1	0	0

Step 1: Construct Open Neighborhood Graph $N(G)$:
- For each vertex, create edges among its direct neighbors.

Step 2: Form Lexicographic Product $G[N(G)]$:
- Each vertex becomes associated with a neighborhood.
- Vertices are pairs like $(A, \text{neighbor of } N(A))$, etc.

Step 3: Assign signs:
- Direct relationships inherit their real signs.
- Indirect relationships via neighborhoods are signed based on structural balance assumptions.

Step 4: Check Balance:
- Examine cycles in $G[N(G)]$.
- Every triangle (three-person group) must satisfy the balance: either all friendships or two rivalries.

Balanced parts suggest communities where internal conflicts are minimized.
Detected Communities:
- {A, B, D} form a community where friendships dominate.
- {C} is more loosely connected due to the higher rivalry with A.

4. Conclusion

In this paper we proved some results in Lexicographic Products of standard graphs with their Open Neighborhood Graphs invariably produce balanced signed graphs, and this paper expands our knowledge of signed graphs. The authors provide a strong framework for recognizing balance across different graph types by using methodical sign assignments and meticulous structural analysis. These findings support broader applications of network theory, especially in domains where structural regularity and balance are crucial to ensuring system stability and coherence.

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