Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 3** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78546

Sum Divisor Cordial Labeling in Certain Graphs

Jaggena Renukeswara Rao* and P. Vijayakumar

ABSTRACT: We prove that extended duplicate graphs of path, comb, twig, star and bistar graphs admit sum divisor cordial labeling.

Key Words:: Graph labeling, path, comb, twig, star, bistar, duplicate graph, sum divisor cordial graph.

Contents

1. Introduction

In 1967, Rosa introduced graph labeling [4]. A graph labeling is a mapping; when the map assigns integers to vertices or edges, it is called a vertex (resp. edge) labeling, and when it assigns to both, it is called a total labeling. Sampath Kumar E. in 1973 introduced duplicate graphs and proved certain characteristics of the same. The duplicate graph of $G = (V, E)$ with order p and size q is denoted by G'. The graph G' contains $2p$ vertices, and for each edge xy in G, G' will have two edges xy' and yx' [5]. A. Lourdusamy and F. Patrick in 2016 introduced the idea of sum divisor cordial labeling and verified its existence in some standard graphs [3]. Motivated by this study, we prove here that extended duplicate graphs of path, comb, twig, star, and bistar graphs are sum divisor cordial graphs.

2. Preliminaries

We present here the notations and definitions used in this article.

2.1. Notations

.
$G(V, E)$ - "A simple graph with p vertices and q edges"
SDCL - "Sum divisor cordial labeling"
$\mathcal{P}_m$ - "Extended duplicate graph of path graph"
$\mathcal{CB}_m$ - "Extended duplicate graph of comb graph"
$\mathcal{T}_m$ - "Extended duplicate graph of twig graph"
$\mathcal{S}_m$ - "Extended duplicate graph of star graph"
$\mathcal{BS}_{m,m}$ - "Extended duplicate graph of bistar graph"

Definition 2.1 *A function $f : V \to \{0, 1\}$ is a cordial labeling of $G(V, E)$, if each edge xy has the label $|f(x) - f(y)|$ such that $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph admitting cordial labeling is called cordial graph [8].*

* Corresponding author.
2010 *Mathematics Subject Classification:* 05C78.
Submitted August 21, 2025. Published October 16, 2025

Definition 2.2 *A function $f : V \to \{0,1\}$ is a sum cordial labeling of $G(V,E)$, if each edge xy gets the label induced by the function f^* defined by $f^*(uv) = [f(u)+f(v)](mod2)$ so that $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph G admitting sum cordial is said to be sum cordial graph [7].*

Definition 2.3 *A sum divisor cordial labeling(SDCL) of a simple graph $G(V,E)$ is a bijection $f : V \to \{1,2,...,p\}$ such that an edge xy is assigned the label 1 if if $2 divides f(u)+f(v)$ and 0 otherwise. The function f is called a SDCL if $|e_f(0)-e_f(1)| \leq 1$. A graph admitting sum divisor cordial labeling is called a sum divisor cordial graph [3].*

We refer [10] for the construction of extended duplicate graph of bistar, [12] for the construction of extended duplicate graph of star and [13] for the construction of extended duplicate graphs of path, comb and twig respectively.

3. Main Results

Theorem 3.1 *The graph $\mathcal{P}_m$, $m \geq 3$ admits SDCL.*

Proof: Let v_t, v'_t for $1 \leq t \leq m$ be the vertices and e_t for $1 \leq t \leq m$, e'_t for $1 \leq t \leq m-1$ be the edges of $\mathcal{P}_m$.

Case (i): For $m \equiv 0 \pmod 4$.
For $1 \leq k \leq \frac{m}{4}$, the $2m$ vertices of $\mathcal{P}_m$ are labeled as follows:

$v_{4k-3} \leftarrow 4k-3, \quad v_{4k-2} \leftarrow m+4k-3, \quad v_{4k-1} \leftarrow 4k, \quad v_{4k} \leftarrow m+4k-1,$

$v'_{4k-3} \leftarrow m+4k-3, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k, \quad v'_{4k} \leftarrow 4k-1.$

The induced function f^* defined by

$$\begin{aligned} f^*(uv) &= 1 \ if \ 2 \ divides \ f(u)+f(v) \\ &= 0, \ otherwise \end{aligned} \tag{3.1}$$

It assigns label 0 to $m-1$ edges: e_{2t-1} for $1 \leq t \leq \frac{m}{2}$, e'_{2t-1} for $1 \leq t \leq \frac{m}{2}$; and label 1 to m edges: e_{2t} for $1 \leq t \leq \frac{m}{2}$, e'_{2t} for $1 \leq t \leq \frac{m}{2}$, and e_m.

Case (ii): For $m \equiv 1 \pmod 4$.
The $2m$ vertices of $\mathcal{P}_m$ are labeled as follows.

For $1 \leq k \leq \frac{m+1}{4}$:
$v_{4k-3} \leftarrow 4k-3, \quad v'_{4k-3} \leftarrow m+4k-3.$

For $1 \leq k \leq \frac{m-1}{4}$:
$v_{4k-2} \leftarrow m+4k-1, \quad v_{4k-1} \leftarrow 4k, \quad v_{4k} \leftarrow m+4k.$
$v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k-2, \quad v'_{4k} \leftarrow 4k-1.$

The function f^* defined in (3.1) assigns label 0 to m edges e_t for $1 \leq t \leq m$ and label 1 to $m-1$ edges e'_t for $1 \leq t \leq m-1$.

Case (iii): For $m \equiv 2 \pmod 4$.
The $2m$ vertices of $\mathcal{P}_m$ are labeled as follows.

Fix $v_m \leftarrow m, \quad v'_m \leftarrow 2m-1.$

For $1 \leq k \leq \frac{m+2}{4}$:
$v_{4k-3} \leftarrow 4k-3, \quad v'_{4k-3} \leftarrow m+4k-2.$

For $1 \leq k \leq \frac{m-2}{4}$:
$v_{4k-1} \leftarrow 4k, \quad v_{4k-2} \leftarrow m+4k, \quad v_{4k} \leftarrow m+4k-1;$
$v'_{4k-1} \leftarrow m+4k-3, \quad v'_{4k} \leftarrow 4k-1, \quad v'_{4k-2} \leftarrow 4k-2.$

The function f^* defined in (3.1) assigns label 0 to $m-1$ edges e_t for $1 \leq t \leq m-2$, together with e_m, and label 1 to m edges e'_t for $1 \leq t \leq m-1$, together with e_{m-1}.

Case (iv): For $m \cong (3 \mod 4)$.
$2m$ vertices of $\mathcal{P}_m$ are labeled below
For $1 \leq k \leq \frac{m+1}{4}$
$v_{4k-3} \leftarrow 4k-3$, $v_{4k-2} \leftarrow m+4k-2$, $v_{4k-1} \leftarrow 4k$,
$v'_{4k-3} \leftarrow m+4k, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k-1.$

For $1 \leq k \leq \frac{m-3}{4}$:
$v_{4k} \leftarrow m+4k+1, \quad v'_{4k} \leftarrow 4k-1.$

The function f^* defined in (3.1) assigns label 0 to $m-1$ edges e_t for $1 \leq t \leq m-1$, and label 1 to m edges e'_t for $0 \leq t \leq m-1$, together with e_m.

Hence, the graph $\mathcal{P}_m$, $m \geq 3$, admits SDCL.

Example 3.1 ***Example***

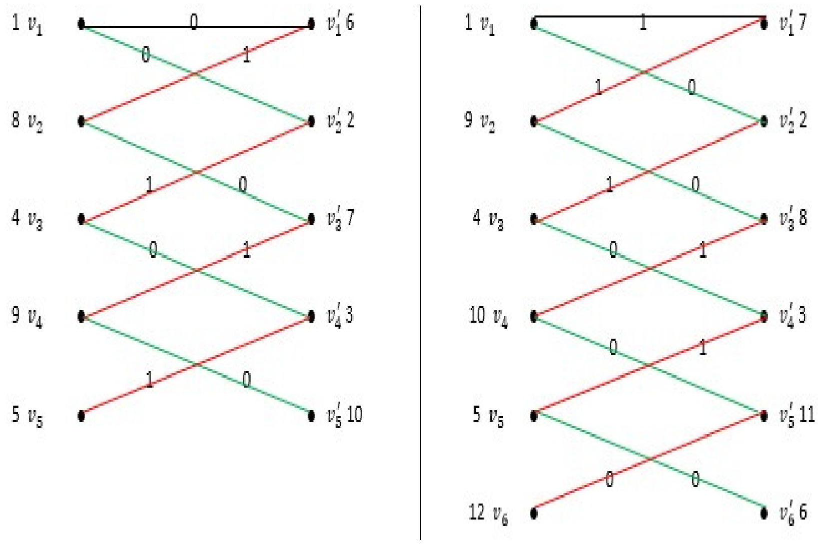

Figure 1: SDCL in $\mathcal{P}_5$ and $\mathcal{P}_6$

□

Theorem 3.2 *The graph $\mathcal{CB}_m$, $m \geq 3$ admits SDCL.*

Proof: Let v_t, v'_t for $1 \leq t \leq 2m$ be the set of vertices, and e_t for $1 \leq t \leq 2m$, e'_t for $1 \leq t \leq 2m-1$ be the set of edges of $\mathcal{CB}_m$.

Case (i): For odd m.
The $4m$ vertices of $\mathcal{CB}_m$ are labeled as follows.

Fix $v_1 \leftarrow 1, \quad v_{2m} \leftarrow 4m, \quad v'_1 \leftarrow 2m+4k, \quad v'_{2m} \leftarrow 2m.$
When $1 \leq k \leq \frac{m-1}{2}$,
$v_{4k} \leftarrow 4k$, $v_{4k+1} \leftarrow 4k+1$, $v_{4k-2} \leftarrow 2m+4k-2$ $v_{4k-1} \leftarrow 2m+4k-1$;
$v'_{4k} \leftarrow 2m+4k, \quad v'_{4k+1} \leftarrow 2m+4k+1, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow 4k-1.$

By the induced function f^* defined in (3.1), the $2m-1$ edges e_{2t} for $1 \leq t \leq m-1$ and e'_{2t} for $1 \leq t \leq m-1$ are labeled 1, and the $2m$ edges e_{2t-1} for $1 \leq t \leq m$ and e'_{2t-1} for $1 \leq t \leq m$ are labeled 0.

Case (ii): For even m.

Fix $v_1 \leftarrow 1, \quad v_{2m} \leftarrow 2m, \quad v'_1 \leftarrow 2m+1, \quad v'_{2m} \leftarrow 4m.$

For $1 \leq k \leq \frac{m-2}{2}$:
$v_{4k} \leftarrow 4k, \quad v_{4k+1} \leftarrow 4k+1, \quad v_{4k-2} \leftarrow 2m+4k-2, \quad v_{4k-1} \leftarrow 2m+4k-1;$

$v'_{4k} \leftarrow 2m+4k, \quad v'_{4k+1} \leftarrow 2m+4k+1, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow 4k-1.$

The function f^* defined in (3.1) assigns label 1 to the $2m$ edges e_{2t} for $1 \leq t \leq m-1$, e_{2m-1}, e'_{2t} for $1 \leq t \leq m-1$, and e'_{2m-1}, and label 0 to the $2m-1$ edges e_{2t-1} for $1 \leq t \leq m$ and e'_{2t-1} for $1 \leq t \leq m$. respectively.
Hence, the graph $\mathcal{CB}_m$, $m \geq 2$ admits SDCL.

Example 3.2 ***Example***

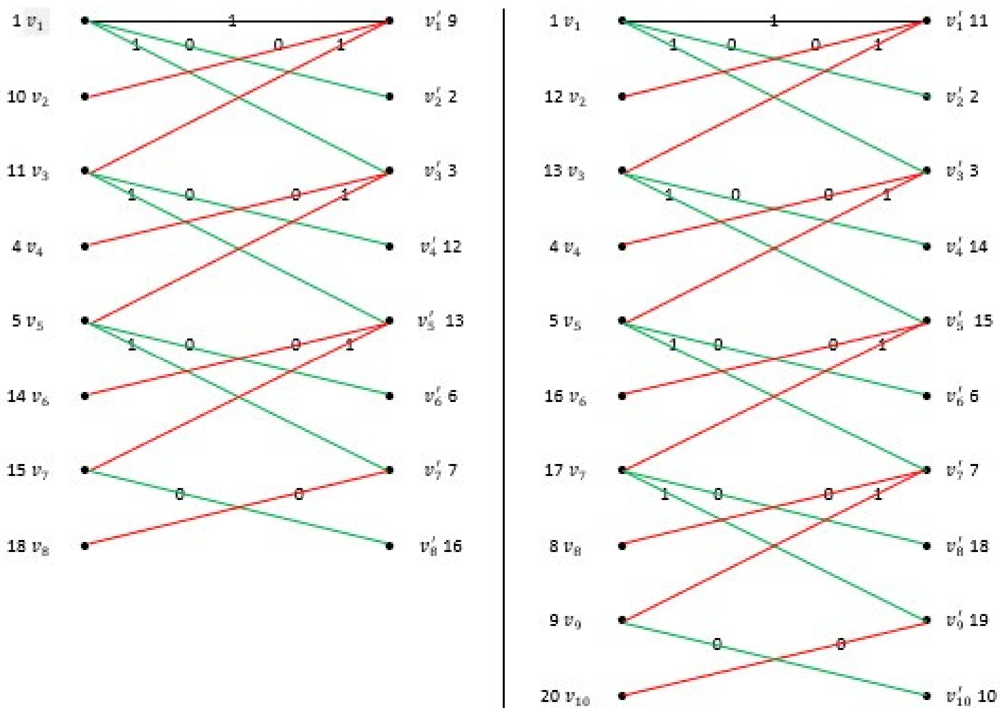

Figure 2: SDCL in $\mathcal{CB}_4$ and $\mathcal{CB}_5$

□

Theorem 3.3 *The graph $\mathcal{T}_m$, $m \geq 2$ admits SDCL.*

Proof: Let v_t, v'_t for $1 \leq t \leq 3m+2$ be the vertices, and e_t for $1 \leq t \leq 3m+2$, e'_t for $1 \leq t \leq 3m+1$ be the edges of $\mathcal{T}_m$.

Case (i): For $m \equiv 0 \pmod 4$.
The $6m+4$ vertices of $\mathcal{T}_m$ are labeled as follows.

Fix $v_1 \leftarrow 6m+4, \quad v_2 \leftarrow 1, \quad v'_1 \leftarrow 6m+2, \quad v'_2 \leftarrow 3m+3.$

For $1 \leq k \leq \frac{m}{4}$:
$v_{12k-9} \leftarrow 3m+12k-7, \quad v_{12k-8} \leftarrow 3m+12k-5, \quad v_{12k-7} \leftarrow 3m+12k-3;$

$v_{12k-6} \leftarrow 12k-10, \quad v_{12k-5} \leftarrow 12k-8, \quad v_{12k-4} \leftarrow 12k-6;$
$v_{12k-3} \leftarrow 3m+12k-4, \quad v_{12k-2} \leftarrow 3m+12k-2, \quad v_{12k-1} \leftarrow 3m+12k.$
$v_{12k} \leftarrow 12k-3, v_{12k+1} \leftarrow 12k-1, v_{12k+2} \leftarrow 12k+1.$
$v'_{12k-9} \leftarrow 12k-9, \quad v'_{12k-8} \leftarrow 12k-7, \quad v'_{12k-7} \leftarrow 12k-5;$
$v'_{12k-6} \leftarrow 3m+12k-10, \quad v'_{12k-5} \leftarrow 3m+12k-8, \quad v'_{12k-4} \leftarrow 3m+12k-6;$
$v'_{12k-3} \leftarrow 3m+12k-4, \quad v'_{12k-2} \leftarrow 3m+12k-2, \quad v'_{12k-1} \leftarrow 3m+12k;$
$v'_{12k} \leftarrow 3m+12k-1, \quad v'_{12k+1} \leftarrow 3m+12k+1, \quad v'_{12k+2} \leftarrow 3m+12k+3.$

The function f^* defined in (3.1) allots label 1 to the $2m-1$ edges e_{6t-4}, e_{6t-3}, e_{6t-2} for $1 \le t \le \frac{m}{2}$, e'_{6t-4}, e'_{6t-3}, e'_{6t-2} for $1 \le t \le \frac{m}{2}$, and e_{3m+2}; and label 0 to the $3m+2$ edges e_{6t-1}, e_{6t}, e_{6t+1} for $1 \le t \le \frac{m}{2}$, e'_{6t-1}, e'_{6t}, e'_{6t+1} for $1 \le t \le \frac{m}{2}$.

Case (ii): For $m \equiv 1 \pmod 4$.
The $6m+4$ vertices of $\mathcal{T}_m$ are labeled as follows.

Fix $v_1 \leftarrow 6m+2, \quad v_2 \leftarrow 1, \quad v'_1 \leftarrow 6m+4, \quad v'_2 \leftarrow 3m+6.$

For $1 \le k \le \frac{m+1}{4}$:
$v_{12k-9} \leftarrow 3m+12k-4, \quad v_{12k-8} \leftarrow 3m+12k-2, \quad v_{12k-7} \leftarrow 3m+12k;$
$v'_{12k-9} \leftarrow 12k-9, \quad v'_{12k-8} \leftarrow 12k-7, \quad v'_{12k-7} \leftarrow 12k-5.$

For $1 \le k \le \frac{m-1}{4}$:
$v_{12k-6} \leftarrow 12k-10, v_{12k-5} \leftarrow 12k-8, v_{12k-4} \leftarrow 12k-6;$
$v_{12k-3} \leftarrow 3m+12k-7, v_{12k-2} \leftarrow 3m+12k-5, v_{12k-1} \leftarrow 3m+12k-3;$
$v_{12k} \leftarrow 12k-3, v_{12k+1} \leftarrow 12k-1, v_{12k+2} \leftarrow 12k+1.$
$v'_{12k-6} \leftarrow 3m+12k-13, \quad v'_{12k-5} \leftarrow 3m+12k-11, \quad v'_{12k-4} \leftarrow 3m+12k-9;$
$v'_{12k-3} \leftarrow 12k-4, \quad v'_{12k-2} \leftarrow 12k-2, \quad v'_{12k-1} \leftarrow 12k;$
$v'_{12k} \leftarrow 3m+12k+2, \quad v'_{12k+1} \leftarrow 3m+12k+4, \quad v'_{12k+2} \leftarrow 3m+12k+6.$

The function f^* defined in (3.1) gives label 1 to the $3m+1$ edges e_{6t-4}, e_{6t-3}, e_{6t-2} for $1 \le t \le \frac{3m+2}{6}$, e'_{6t-4}, e'_{6t-3}, e'_{6t-2} for $1 \le t \le \frac{3m-4}{6}$, and e_{3m+1};
and label 0 to the $3m+2$ edges e_{6t-1}, e_{6t}, e_{6t+1} for $1 \le t \le \frac{3m-4}{6}$, e'_{6t-1}, e'_{6t}, e'_{6t+1} for $1 \le t \le \frac{3m-1}{6}$, and e_{3m-1}, e'_{3m-1}, e'_{3m}, e_1, e'_1.

Case (iii): For $m \cong 2$ $(mod\ 4)$
The $6m+4$ vertices of $\mathcal{T}_m$ are labeled as below.
Fix $v_1 \leftarrow 6m+4, v_2 \leftarrow 1, v'_1 \leftarrow 6m+3, \quad v'_2 \leftarrow 3m+5.$

For $1 \le k \le \frac{m+2}{4}$:
$v_{12k-9} \leftarrow 3m+12k-8, \quad v_{12k-8} \leftarrow 3m+12k-6, \quad v_{12k-7} \leftarrow 3m+12k-4;$
$v_{12k-6} \leftarrow 12k-10, \quad v_{12k-5} \leftarrow 12k-8, \quad v_{12k-4} \leftarrow 12k-6;$
$v'_{12k-9} \leftarrow 12k-9, \quad v'_{12k-8} \leftarrow 12k-7, \quad v'_{12k-7} \leftarrow 12k-5;$
$v'_{12k-6} \leftarrow 3m+12k-9, \quad v'_{12k-5} \leftarrow 3m+12k-7, \quad v'_{12k-4} \leftarrow 3m+12k-5.$

For $1 \le k \le \frac{m-2}{4}$:
$v_{12k-3} \leftarrow 3m+12k-3, v_{12k-2} \leftarrow 3m+12k-1, v_{12k-1} \leftarrow 3m+12k+1;$
$v_{12k} \leftarrow 12k-3, v_{12k+1} \leftarrow 12k-1, v_{12k+2} \leftarrow 12k+1.$
$v'_{12k-3} \leftarrow 12k-4, \quad v'_{12k-2} \leftarrow 12k-2, \quad v'_{12k-1} \leftarrow 12k;$
$v'_{12k} \leftarrow 3m+12k-2, \quad v'_{12k+1} \leftarrow 3m+12k, \quad v'_{12k+2} \leftarrow 3m+12k+2.$

Using the induced function f^* defined in (3.1), the $3m+2$ edges e_{6t-4}, e_{6t-3}, e_{6t-2} for $1 \leq t \leq \frac{3m-1}{6}$, e'_{6t-4}, e'_{6t-3}, e'_{6t-2} for $1 \leq t \leq \frac{3m-1}{6}$, and e_1, e'_1 receive label 1;

the $3m+1$ edges e_{6t-1}, e_{6t}, e_{6t+1} for $1 \leq t \leq \frac{3m-1}{6}$, e'_{6t-1}, e'_{6t}, e'_{6t+1} for $1 \leq t \leq \frac{3m-1}{6}$, and e_{3m+1} receive label 0.

Case (iv): For $m \cong 3$ $(mod\ 4)$
The $6m+4$ vertices of $\mathcal{T}_m$ are labeled as below.
Fix $v_1 \leftarrow 6m+4$, $v_2 \leftarrow 1$, $v'_1 \leftarrow 6m+3$, $\quad v'_2 \leftarrow 3m+5$.

For $1 \leq k \leq \frac{m+1}{4}$:
$v_{12k-9} \leftarrow 3m+12k-12, \quad v_{12k-8} \leftarrow 3m+12k-10, \quad v_{12k-7} \leftarrow 3m+12k-8;$
$v_{12k-6} \leftarrow 12k-10, \quad v_{12k-5} \leftarrow 12k-8, \quad v_{12k-4} \leftarrow 12k-6;$
$v_{12k-3} \leftarrow 3m+12k-5, \quad v_{12k-2} \leftarrow 3m+12k-3, \quad v_{12k-1} \leftarrow 3m+12k-1;$
$v'_{12k-9} \leftarrow 12k-9, \quad v'_{12k-8} \leftarrow 12k-7, \quad v'_{12k-7} \leftarrow 12k-5;$
$v'_{12k-6} \leftarrow 3m+12k-7, \quad v'_{12k-5} \leftarrow 3m+12k-5, \quad v'_{12k-4} \leftarrow 3m+12k-3.$

$v'_{12k-3} \leftarrow 12k-4, \quad v'_{12k-2} \leftarrow 12k-2, \quad v'_{12k-1} \leftarrow 12k.$

For $1 \leq k \leq \frac{m-3}{4}$:
$v_{12k} \leftarrow 12k-3, \quad v_{12k+1} \leftarrow 12k-1, \quad v_{12k+2} \leftarrow 12k+1;$
$v'_{12k} \leftarrow 3m+12k+1, \quad v'_{12k+1} \leftarrow 3m+12k+3, \quad v'_{12k+2} \leftarrow 3m+12k+5.$

Using the induced function f^* defined in (3.1), the $3m+2$ edges e_t for $1 \leq t \leq 3m+1$ and e'_1 receive label 1, while the $3m+1$ edges e'_t for $2 \leq t \leq 3m+1$ and e_{3m+2} receive label 0.

Hence, the graph $\mathcal{T}_m$, $m \geq 2$ admits SDCL.

Example 3.3 ***Example***

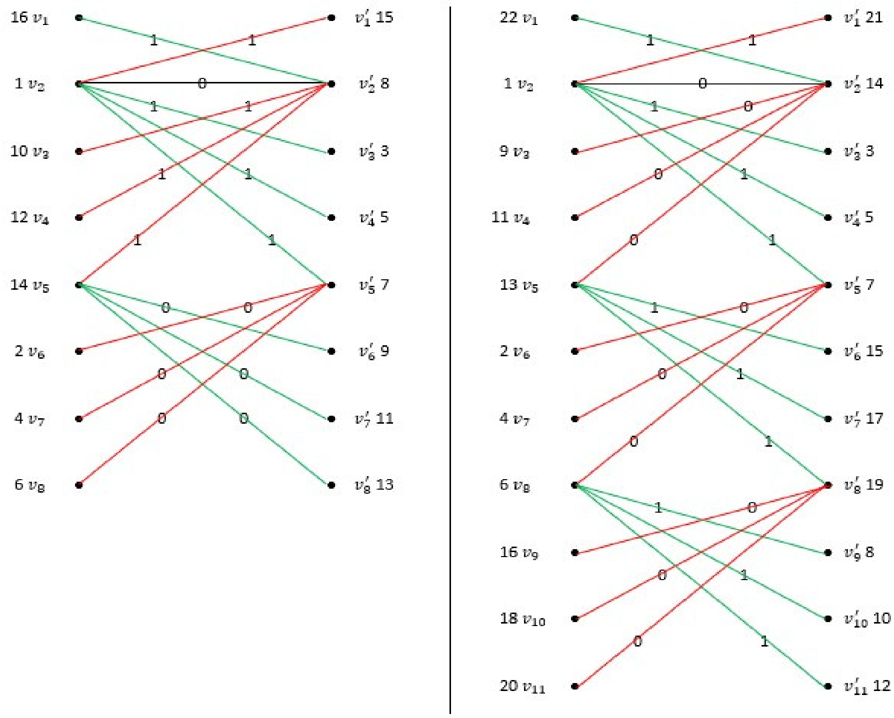

Figure 3: SDCL in $\mathcal{T}_2$ and $\mathcal{T}_3$

□

Theorem 3.4 *The graph $\mathcal{S}_m$, $m \geq 4$ admits SDCL.*

Proof: Let v_t, v'_t for $1 \leq t \leq m$ be the vertices and e_t for $1 \leq t \leq m$, e'_t for $1 \leq t \leq m-1$ be the edges of $\mathcal{S}_m$.

Case (i): For $m \equiv 0 \pmod 4$
For $1 \leq k \leq \frac{m}{4}$, the $2m$ vertices of $\mathcal{S}_m$ are labeled as follows:
$v_{4k-3} \leftarrow 4k-3, \quad v_{4k-2} \leftarrow m+4k-2, \quad v_{4k-1} \leftarrow 4k, \quad v_{4k} \leftarrow m+4k-1;$
$v'_{4k-3} \leftarrow m+4k-3, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k, \quad v'_{4k} \leftarrow 4k-1.$

The induced function f^* defined in (3.1) assigns label 0 to the $m-1$ edges e_{2t-1} $(1 \leq t \leq \frac{m}{2})$ and e'_{2t-1} $(1 \leq t \leq \frac{m}{2})$, and label 1 to m edges e_{2t-1} $(1 \leq t \leq \frac{m}{2})$, e'_{2t-1} $(1 \leq t \leq \frac{m}{2})$, and e_m.

Case (ii): For $m \equiv 1 \pmod 4$
For $1 \leq k \leq \frac{m+1}{4}$:
$v_{4k-3} \leftarrow 4k-3, \quad v'_{4k-3} \leftarrow m+4k-3.$
For $1 \leq k \leq \frac{m-1}{4}$
$v_{4k-2} \leftarrow m+4k-1$, $v_{4k-1} \leftarrow 4k$, $v_{4k} \leftarrow m+4k;$
$v'_{4k-2} \leftarrow m+4k-2, \quad v'_{4k-1} \leftarrow m+4k-2, \quad v'_{4k} \leftarrow 4k-1.$

The function f^* defined in (3.1) assigns label 0 to m edges e_t $(1 \leq t \leq m)$ and label 1 to $m-1$ edges e'_t $(1 \leq t \leq m-1)$.

Case (iii): For $m \equiv 2 \pmod 4$
The $2m$ vertices of $\mathcal{S}_m$ are labeled as follows:
Fix $v_m \leftarrow m, \quad v'_m \leftarrow 2m-1$.
For $1 \leq k \leq \frac{m+2}{4}$:
$v_{4k-3} \leftarrow 4k-3, \quad v'_{4k-3} \leftarrow m+4k-2$.

For $1 \leq k \leq \frac{m-2}{4}$
$v_{4k-2} \leftarrow m+4k$, $v_{4k-1} \leftarrow 4k$, $v_{4k} \leftarrow m+4k-1$;
$v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k-3, \quad v'_{4k} \leftarrow 4k-1$.

Now, f^* defined in (3.1) gives label 0 to m edges e_t $(1 \leq t \leq m-1)$ and e'_{m-1}, and label 1 to $m-1$ edges e'_t $(1 \leq t \leq m-1)$ and e_m.

Case (iv): For $m \equiv 3 \pmod 4$
The $2m$ vertices of $\mathcal{S}_m$ are labeled as follows:
For $1 \leq k \leq \frac{m+1}{4}$:
$v_{4k-3} \leftarrow 4k-3, \quad v_{4k-2} \leftarrow m+4k-2, \quad v_{4k-1} \leftarrow 4k$;
$v'_{4k-3} \leftarrow m+4k, \quad v'_{4k-2} \leftarrow 4k-2, \quad v'_{4k-1} \leftarrow m+4k-1$.

For $1 \leq k \leq \frac{m-3}{4}$:
$v_{4k} \leftarrow m+4k+1, \quad v'_{4k} \leftarrow 4k-1$.

Now, the function f^* defined in (3.1) allots label 0 to $m-1$ edges e_t $(1 \leq t \leq m-1)$ and label 1 to m edges e'_t $(1 \leq t \leq m-1)$ and e_m.

Hence, the graph $\mathcal{S}_m$, $m \geq 4$, admits SDCL.

Example 3.4 *Consider the star graphs $\mathcal{S}_5$ and $\mathcal{S}_6$. They admit sum divisor cordial labeling (SDCL) as described in the previous sections.*

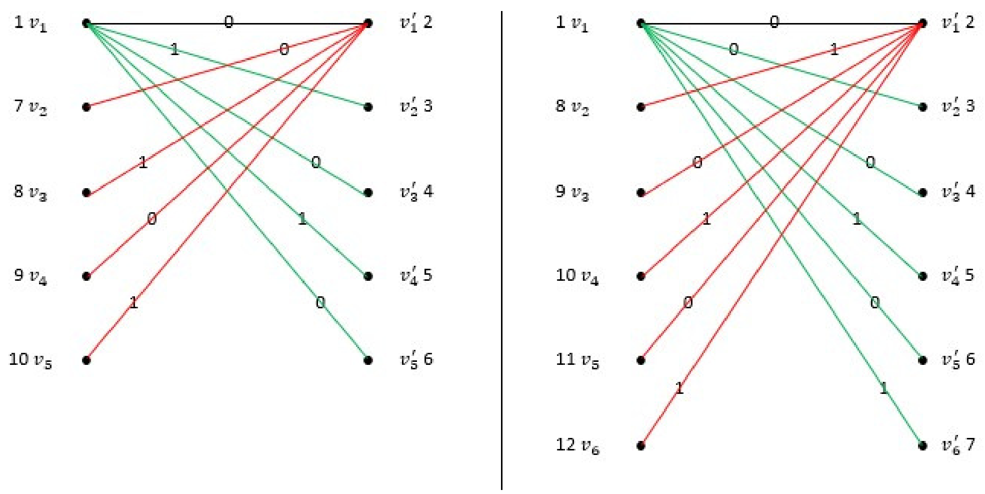

Figure 4: SDCL in $\mathcal{S}_5$ and $\mathcal{S}_6$

□

Theorem 3.5 *The graph $\mathcal{BS}_{m,m}$, for $m \geq 2$ admits SDCL.*

Proof: Let v_t, v_t' for $1 \leq t \leq 2m+2$ be the vertices, and e_t for $1 \leq t \leq 2m+2$, e_t' for $1 \leq t \leq 2m+1$ be the edges of $\mathcal{BS}_{m,m}$.

The $4m+4$ vertices of $\mathcal{BS}_{m,m}$ are labeled as follows:

$$v_1 \leftarrow 1, \quad v_1' \leftarrow 2m+3$$

For $1 \leq k \leq m$:

$$v_{k+1} \leftarrow 2m+k+3, \quad v_{k+1}' \leftarrow k+1$$

For $1 \leq k \leq m+1$:

$$v_{m+k+2} \leftarrow m+k+2, \quad v_{m+k+1}' \leftarrow 3m+k+4$$

Case (i): For odd m

The induced function f^* defined in (3.1) allots label 1 to $2m+1$ edges The induced function f^* defined in (3.1) assigns label 1 to

$$e_{2t}, \ (1 \leq t \leq m+1), \quad e_{2t}', \ (1 \leq t \leq m),$$

and label 0 to

$$e_{2t-1},\ (1 \le t \le m+2), \quad e'_{2t-1},\ (1 \le t \le m+1).$$

Case(ii): For even m
The function f^* defined in (3.1) assigns label 1 to $2m+1$ edges

$$e_{2t},\ (1 \le t \le m+1), \quad e'_{2t},\ (1 \le t \le m), \quad e'_{2m+1},$$

and label 0 to $2m+2$ edges

$$e_{2t-1},\ (1 \le t \le \tfrac{m+2}{2}), \quad e_{m+2t},\ (1 \le t \le \tfrac{m}{2}), \quad e'_{2t-1},\ (1 \le t \le \tfrac{m+2}{2}), \quad e'_{m+2t},\ (1 \le t \le \tfrac{m}{2}).$$

Hence, the graph $\mathcal{BS}_{m,m}$, for $m \ge 2$, admits SDCL.

Example 3.5 ***Example***

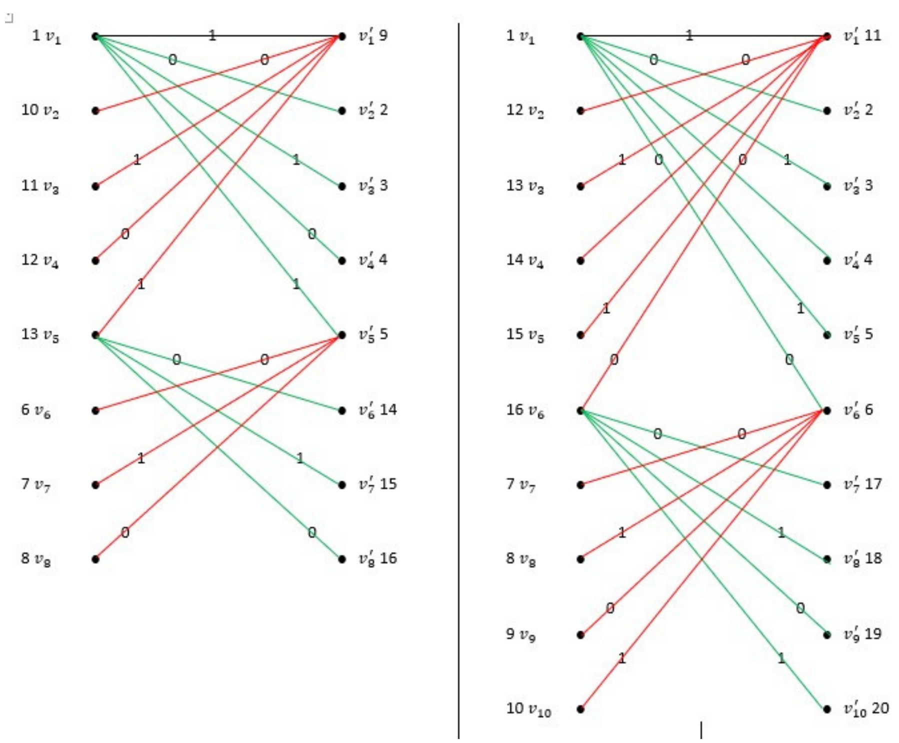

Figure 5: SDCL in $\mathcal{BS}_{3,3}$ and $\mathcal{BS}_{4,4}$

□

4. Conclusion

We have proved that the graphs $\mathcal{P}_m$, $\mathcal{CB}_{m,m}$, $\mathcal{T}_m$, $\mathcal{S}_m$ and $\mathcal{BS}_{m,m}$ admit SDCL.

References

1. Cahit, I., *Cordial graphs; a weaker version of graceful and harmonious graphs*, Ars Combin., Vol. 23, pp. 201-207, 1987.
2. Gallian, J. A., *A Dynamic Survey of Graph Labeling*, Electronic Journal of Combinatorics, #DS6, 2024.
3. Lourdusamy, A., and Patrick, F., *Sum Divisor Cordial Graphs*, Proyecciones Journal of Mathematics, Vol. 35, No. 1, pp. 119–136, March 2016. .
4. Rosa, A., *On certain valuations of the vertices of a graph, Theory of Graphs (Internet, Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris, Vol. 23, pp. 349-355, 1967.
5. Sampathkumar.E, *On duplicate graphs*. J. Indian Math. Soc 37 (1973), 285–293.
6. Selvam. B and Thirusangu. K,*Mean labelling in extended duplicate graph of twig*, Indian Journal of Science and Technology., Vol. 8, No. 36 DOI: 10.17485/ijst/2015/v8i36/79389, 2015.
7. Shiama. J,*Sum cordial labeling for some graphs*, Int. J. Math. Archive., Vol. 3, pp.3271-3276, (2012).
8. Thirusangu, K., Ulaganathan, P.P. and Vijaya Kumar, P., *Some Cordial Labeling of Duplicate Graph of Ladder Graph* , Annals of Pure and Applied Mathematics , Vol. 8, No 2, pp. 43-50, 2014.
9. Thirusangu, K., Ulaganathan, P.P. and Selvam, B.,*Cordial labeling in duplicate graphs*, Int. J. Computer Math. Sci. Appl., Vol. 4, No 1, pp. 179-186, 2010.
10. Ulaganathan. P.P, Selvam. B and Vijaya Kumar. P., *Signed Product Cordial labeling in duplicate graphs of Bistar, Double Star and Triangular Ladder Graph* , International Journal of Mathematics Trends and Technology (IJMTT) , Vol. 33, No 1, pp. 19-24, 2016.
11. Varatharajan. R, Navanaeethakrishnan, S and Thirusangu, K., *Divisor Cordial Graphs*, International J.Math. Combin., Vol. 4, pp. 15-25, (2011).
12. Vijaya Kumar, P., Ulaganathan, P.P. and Thirusangu, K., *Some Cordial Labeling in Extended Duplicate Graph of Star Graph*, International Journal of Applied Engineering Research, Vol. 10, No 80, pp. 171-174, 2015.
13. Vijayakumar, P., Thulukkanam, K., and Thirusangu, T., *Skolem Difference Mean Labeling in Duplicate Graphs of Some Path Related Graphs*, South East Asian Journal of Mathematics and Mathematical Sciences, Vol. 14, No. 3, pp. 63–78, 2018.
14. Vijaya Kumar. P, Ulaganathan. P.P and Thirusangu. K., *Mean Labeling in Some Duplicate Graphs*, International Journal of Pure and Applied Mathematics Vol. 117 No 11, pp. 201-210, March 2017.

Jaggena Renukeswara Rao,
Department of Mathematics,
Gandhi Institute of Engineering and Technology University, Odisha, Gunupur,
India.
E-mail address: jaggenarenukeswara.rao@giet.edu

and

P. Vijayakumar,
Department of Mathematics,
Gandhi Institute of Engineering and Technology University, Odisha, Gunupur,
India.
E-mail address: vijaysnthosh2002@gmail.com