Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 4** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78550

On the Existence of Optimal Binary LCD Codes Under Hierarchical Poset Metrics *

Rohini Baliram More and Venkatrajam Marka†

ABSTRACT: A linear code is referred to as a linear complementary dual (LCD) code when it has only a trivial intersection with its dual. LCD codes have gained prominence in research due to their application in cryptography, communication systems, and data storage.

"This article explores the binary LCD hierarchical poset code, wherein the dimension is determined by the rank of the Gramian of its generator matrix. By employing the canonical systematic form of the generator matrix of the hierarchical poset code, the corresponding Gramian matrix is specified by imposing certain conditions on the support of basis elements. Utilizing the Griesmer bound of linear code under Hamming metric and the canonical decomposition of the hierarchical poset code, an upper bound is established on the maximum distance of the hierarchical poset code with any hull dimension under specified conditions. Furthermore, the study investigates the existence of optimal binary LCD codes under a hierarchical poset metric when n is equivalent to a mod 8 where, a ranges from 0 to 7."

Key Words: Griesmer bound, hierarchical poset codes, LCD codes, optimal codes.

Contents

1. Introduction

Linear complementary dual (LCD) codes are those linear codes whose intersection with its dual codes is trivial. They are widely utilized in various fields, including data storage, communication systems, consumer electronics, and cryptography. In 1992, Massey introduced the idea of LCD codes to provide an optimum linear coding solution for the two user binary adder channel [1]. By using the hull dimension spectra of linear codes Sendrier [2] demonstrated that LCD codes satisfy the asymptotic Gilbert-Varshamov bound. In [3] Carlet et al. investigated an application of binary LCD codes against Side-Channel Attacks (SCA) and Fault Injection Attack (FIA), and presented several constructions of LCD codes. Additionally they have been employed to construct EAQECCs codes (see [4],[5]). In order to construct new LCD codes from existing LCD codes, several methods such as direct sum, direct product, puncturing, shortening, extension and matrix product have been used (see [6],[7],[8],[9]). By modifying some typical methods Shitao Li et al. [10] have introduced some methods for constructing LCD codes over small finite fields . Moreover they have constructed several new quaternary Hermitian LCD codes, binary and ternary Euclidean LCD codes which are useful to modify the established lower bounds on the largest minimum weights. Thus, conjecture proposed by Bouyuklieva [6] is disproved by two counterexamples.

* The first author, Rohini Baliram More, thanks the Department of Science and Technology of the Government of India for the research grant through DST INSPIRE Fellowship [IF No:190335].
† Corresponding author.
2010 *Mathematics Subject Classification:* 94B05, 94B60, 06A11.
Submitted August 21, 2025. Published October 09, 2025

In addition, as an application of quaternary Hermitian LCD codes, they identified some binary EAQECCs with new parameters. In Ranya D. Boulanouar et al. [11] presented first characterization of linear complementary dual skew constacyclic codes and discussed various construction of LCD skew negacyclic and skew cyclic codes [12]. In 2024, Yang Liu et al. [13] studied on the minimum distances of binary optimal LCD codes with dimension 5. In Conghui Xie et al. [14] constructed two new families of ternary LCD BCH codes with lengths $n = p^\gamma$ and $2p^\gamma$. Many researchers have investigated the construction of LCD codes ([15], [16], [17], [25], [26], [27]).

In this article we have worked on the existence of optimal binary LCD codes under hierarchical poset metrics. The motivation for this work stems from the growing demand for advanced coding schemes tailored to modern applications in secure communications and data integrity. While classical error-correcting codes are traditionally designed under the Hamming metric, various practical scenarios necessitate more flexible and application-specific distance measures. One such alternative is the poset metric, introduced by Brualdi et al. [19] which imposes a partial order on coordinate positions and generalizes the Hamming metric. Within this framework, hierarchical poset metrics form a particularly important subclass due to their structured nature. They allow for the canonical decomposition of codes and facilitate the analysis of essential parameters such as minimum distance and hull dimension. In parallel, Linear Complementary Dual (LCD) codes have attracted considerable attention in recent years because of their inherent resistance to side-channel and fault injection attacks, making them valuable for cryptographic and secure communication systems. Despite the individual importance of both hierarchical poset metrics and LCD codes, the literature lacks a systematic study of binary LCD codes under hierarchical poset metrics, particularly regarding their optimality. This observation serves as the primary motivation for the present study, which aims to explore the existence, structure, and optimality conditions of such codes specifically those of dimension two with respect to a Griesmer-type bound.

Inspired by the work done in [18], we have constructed two dimensional optimal binary linear complementary dual codes under hierarchical poset metrics. The structure of the paper is as follows: Section 2, reviews several basic results and properties of hierarchical poset code. In Section 3, hull of hierarchical poset code is defined. In Section 4, construction and bound are presented. Existence of two dimensional optimal binary linear complementary dual codes under hierarchical poset metric are given in Section 5. In Section 6 concluding remarks are mentioned.

2. Preliminaries

Brualdi et al. introduced the idea of poset metric in 1995 [19], which is a metric defined on a vector space $\mathbb{F}_q^n$ over a field $\mathbb{F}_q$ with a partial ordering ($\preceq$) imposed on a finite set P, containing elements 1 to n, i.e., $P = \{1, 2, , \cdots , n\}$.

An ideal I is a subposet of P having property that whenever $y \in I$ and $z \preceq y$ it implies that $z \in I$. The poset weight of any vector $y \in \mathbb{F}_q^n$ is determined as the cardinality of the smallest ideal of P containing support of y. Symbolically, $\varpi_P(y)=|\langle supp(y)\rangle|$ where, $supp(y) = \{j|y_j \neq 0\}$. Poset distance between any two vectors y and z in $\mathbb{F}_q^n$ is defined as $d_P(y, z) = \varpi_P(y - z)$. This distance satisfies all the properties of metric so it is known as *Poset metric*. A linear subspace C of $\mathbb{F}_q^n$ that possesses a poset metric having dimension k and minimum distance d_P is referred to as a *Poset code*, with parameters $[n, k, d_P]$.

We mention below some concepts which are required to define hierarchical poset. The height $h(b)$ of an element $b \in P$ is determined by the cardinality of the largest chain having b as the maximal element. The height $h(P)$ of a poset P is the maximal height of its elements. i.e., $h(P) = max\{h(b) : b \in [n]\}$. The i^{th} level Γ_i of a poset P is the collection of all its elements having height i i.e., $\Gamma_i = \{b \in [n] : h(b) = i\}$.

Definition 2.1 (Hierarchical poset [20]) *Let,*

$$[n] = \dot{\bigcup}_{i=1,2,\cdots,l}\Gamma_i \tag{2.1}$$

be a partition with $n_i = |\Gamma_i| > 0$. We Define $n = (n_1, n_2, \cdots , n_l)$ and $\mathcal{H} = (\Gamma_1, \cdots , \Gamma_l)$ to be hierarchical array and hierarchical spectrum respectively. Note that $n = n_1 + n_2 + \cdots + n_l$. A hierarchical poset with hierarchical spectrum $\mathcal{H}$ is the poset $P_\mathcal{H} = ([n], \preceq_\mathcal{H})$ where,

$$a' \preceq_\mathcal{H} b' \text{ if and only if } a' \in \Gamma_i, b' \in \Gamma_j, \text{ and } i < j. \tag{2.2}$$

Definition 2.2 (Hierarchical poset metric) *Consider* $P_{\mathcal{H}}=([n], \preceq_{\mathcal{H}})$ *is a hierarchical poset having hierarchical spectrum* $\mathcal{H} = (\Gamma_1, \Gamma_2, \cdots, \Gamma_l)$ *and* $z \in \mathbb{F}_q^n$ *with* $z = z_1 + z_2 + \cdots + z_l$ *where,* $supp(z_i) \subseteq \Gamma_i$*. If*

$$M(z) = \max\{i : z_i \neq 0\}, \tag{2.3}$$

then,

$$\langle supp(z)\rangle = (supp(z) \cap \Gamma_{M(z)}) \dot{\bigcup} (\bigcup_{i=1}^{M(z)-1} \Gamma_i) \tag{2.4}$$

and the fact that the union is disjoint which ensures that

$$\varpi_{\mathcal{H}}(z) = \left|supp(z_{M(z)})\right| + \sum_{i=1}^{M(z)-1} n_i \tag{2.5}$$

where, $(n_1, n_2, \cdots, n_l)$ *is the hierarchical array of* $\mathcal{H}$*.*

Definition 2.3 (Hierarchical poset code) *A linear code* $C \subseteq \mathbb{F}_q^n$ *is called an hierarchical poset code when we consider a metric on* $\mathbb{F}_q^n$ *induced by an hierarchical poset.*

Theorem 2.1 ([21],Proposition 2) *Let* $P = ([n], \preceq_{\mathcal{H}})$ *be a hierarchical poset having* l *levels with* $n_i = |\Gamma_i|$*. Also, assume that* $\{0\} \neq C \subseteq \mathbb{F}_q^n$ *is a linear code with* P*-canonical decomposition of* C *as* $C_1 \oplus C_2 \oplus \cdots \oplus C_l$ *. Then,*

$$d_P(C) = \sum_{i=1}^{y_1-1} n_i + d_H(C_{y_1}) \tag{2.6}$$

where, $y_1 = \min\{i \in [l] : C_i \neq 0\}$ *and* $d_H(C_{y_1})$ *is the minimum distance of* C_{y_1} *in the Hamming space* $\mathbb{F}_q^{n_{y_1}}$*.*

Theorem 2.2 ([21]) *Consider* $P = ([n], \preceq)$ *as a poset with* l *levels. Then,* P *is a hierarchical poset if and only if every linear code* $C \subseteq \mathbb{F}_q^n$ *satisfies* P*-canonical decomposition.*

3. Hull Of Hierarchical Poset Code

Consider C is a hierarchical poset code over $\mathbb{F}_q$ having length n. Then, its dual is defined as

$$C^{\perp} = \{v \in F_q^n : \langle v, c\rangle = 0 \ \ \forall c \in C\} \tag{3.1}$$

where, $\langle . \rangle$ denotes the Euclidean inner product. The *hull* of hierarchical poset code is defined as $Hull(C)$=$C \cap C^{\perp}$. The *LCD* hierarchical poset code is defined as $Hull(C) = \{0\}$.

Remark 3.1 *It is easy to observe that there does not exist any optimal binary LCD hierarchical poset codes for* $n = 1$ *and* 2*. Therefore, in this article, we consider binary LCD codes under hierarchical poset metric for length* $n > 2$*.*

Theorem 3.1 ([22], Propostion 3.1) *Consider* C *is a linear code having parameters* $[n, k, d]$ *and the corresponding generator matrix is* G*. Then,*

$$dim(Hull(C)) = k - rank(GG^{'}). \tag{3.2}$$

Theorem 3.1 can be adapted for hierarchical poset codes as follows:

Theorem 3.2 *Consider* C *is a hierarchical poset code having parameters* $[n, k, d_{\mathcal{H}}]$ *with* G *as its generator matrix. Then,*

$$dim(Hull(C)) = k - rank(GG^{'}). \tag{3.3}$$

Proof: Proof follows similar approach to that of Theorem 3.1. □

4. Constructions and Bounds

In this article, we consider the hierarchical poset of l level i.e., $P(n;\ n_1, n_2, \cdots, n_l)$ with $n = n_1 + n_2 + \cdots + n_l$. Let C be a binary hierarchical poset code. Define $\bar{C_0} = \{0\}$ and $\bar{C_i} = \{c \in C : M(c) \subset \Gamma_i\}$ for all $i \in \{1, \cdots, l\}$. Let $C_i = \bigcup_{j=0}^{i} \bar{C_j}$ which is vector subspace. Let $\Lambda(C) = \{y_1, y_2, \cdots, y_s\}$ be the set of levels for which $\bar{C_{y_j}} \neq 0$ and $d_{y_j} = dim(C_{y_j}) - dim(C_{y_{j-1}})$. For a two dimensional binary hierarchical poset code, we obtain $\Lambda(C) = \{y_1\}$ or $\Lambda(C) = \{y_1, y_2\}$.

Theorem 4.1 **([20], [23])** *Let P be a hierarchical poset of l level, i.e., $P(n;\ n_1, n_2, \cdots, n_l)$ with $n = n_1 + n_2 + \cdots + n_l$. Consider C is a binary hierarchical poset code having parameters $[n, k]$. Then, C is P equivalent to a code C_1 having a generator matrix $G_1 = (G_{1_{k,j}})$ consisting of blocks $G_{1_{k,j}}$ of size $d_{y_k} \times n_j$ such that $G_{1_{k,j}}$ is the zero matrix for all $j \neq y_k$ and, for $j = y_k$ it has the form $G_{1_{k,j}} = [I_{y_k}|A_{y_k}]$ where, $I_{d_{y_k}}$ is the identity matrix of size $d_{y_k} \times d_{y_k}$ and A_{y_k} is a matrix of size $d_{y_k} \times (n_{y_k} - d_{y_k})$. i.e., G_1 has the form:*

$$G_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & [I_{d_{y_s}}|A_{y_s}] & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & [I_{d_{y_1}}|A_{y_1}] & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{4.1}$$

Case 1: If $\Lambda(C) = \{y_1, y_2\}$, then canonical systematic form of generator matrix is:

$$G_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & [I_{d_{y_2}}|A_{y_2}] & 0 \\ 0 & \cdots & 0 & [I_{d_{y_1}}|A_{y_1}] & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{4.2}$$

where, $I_{d_{y_k}}$ is the identity matrix of size 1×1 and A_{y_k} is a matrix of size $1 \times (n_{y_k} - 1)$ as $d_{y_k} = 1$. The corresponding Gram matrix is

$$G_1 G_1' = \begin{bmatrix} [I_{d_{y_2}}|A_{y_2}] * [I_{d_{y_2}}|A_{y_2}]^t & 0 \\ 0 & [I_{d_{y_1}}|A_{y_1}] * [I_{d_{y_1}}|A_{y_1}]^t \end{bmatrix} \tag{4.3}$$

G_1 can be rewritten as follows:

$$G_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \beta_{y_2} & 0 \\ 0 & \cdots & 0 & \beta_{y_1} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

Consider,

$$\boldsymbol{R^1_{01}} = |supp(\beta_{y_1})|, \boldsymbol{R^1_{10}} = |supp(\beta_{y_2})| \quad \textit{and} \quad \boldsymbol{R^1_{00}} = \sum_{i=1}^{y_1 - 1} n_i \tag{4.5}$$

We consider the first R^1_{01} positions of β_{y_1} and the first R^1_{10} positions of β_{y_2} to be 1's.

Case 2: If $\Lambda(C) = \{y_1\}$ then canonical systematic form of generator matrix is given by,

$$G_2 = \begin{bmatrix} 0 & \cdots & 0 & [I_{d_{y_1}}|A_{y_1}] & 0 \end{bmatrix} \tag{4.6}$$

where, $I_{d_{y_1}}$ is the identity matrix of size 2×2 and A_{y_1} is a matrix of size $2 \times (n_{y_1} - 2)$ as $d_{y_1} = 2$. The corresponding Gramian matrix is

$$G_2 G_2' = \left[[I_{d_{y_1}}|A_{y_1}] * [I_{d_{y_1}}|A_{y_1}]^t \right]. \tag{4.7}$$

G_2 can be rewritten as follows:

$$G_2 = \begin{bmatrix} 0 & \cdots & 0 & \beta_{2_{y_1}} & 0 \\ 0 & \cdots & 0 & \beta_{1_{y_1}} & 0 \end{bmatrix}. \tag{4.8}$$

Consider,

$$\boldsymbol{R^2_{01}} = |supp(\beta_{1_{y_1}})|, \boldsymbol{R^2_{10}} = |supp(\beta_{2_{y_1}})| \quad \textit{and} \quad \boldsymbol{R^2_{00}} = \sum_{i=1}^{y_1 - 1} n_i \tag{4.9}$$

We consider the first R^2_{01} positions of β_{y_1} and the first R^2_{10} positions of β_{y_2} to be 1's.

Theorem 4.2 (Griesmer bound [24]) *Consider q is a prime power. If there exist a linear code having parameters $[n, k, d_H]$ then,*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{q^i} \right\rceil . \tag{4.10}$$

Using 2.1, we can deduce $\bar{C}_{y_1}$ as a linear code over $\mathbb{F}_q^{n_{y_1}}$. Therefore,

$$n_{y_1} \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{q^i} \right\rceil . \tag{4.11}$$

Definition 4.1 *For prime power q, $h_{dim} \in \mathbb{Z}^+$ and positive integers n, k, consider,*

$$A_q(n, k, h_{dim}) = \max \big\{ d_P |\ \exists \text{ a hierarchical poset code having hull dimension } h_{dim} \text{ under stated condition} \big\}. \tag{4.12}$$

Lemma 4.1 *Consider q is a prime power, h_{dim} is a non-negative integer and n, $k \in \mathbb{Z}^+$ such that $1 \leq k \leq n$. Then,*

$$A_q(n, k, h_{dim}) \leq \left\lfloor \frac{(q-1)q^{k-1} n_{y_1}}{(q^k - 1)} \right\rfloor + \sum_{i=1}^{y_1 - 1} n_i \tag{4.13}$$

Proof: By applying Griesmer bound as in 4.10, we have,

$$n_{y_1} \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{q^i} \right\rceil \geq d_H \sum_{i=0}^{k-1} \frac{1}{q^i} = \frac{d_H (q^k - 1)}{(q-1)q^{k-1}} \tag{4.14}$$

It follows that

$$d_H \leq \left\lfloor \frac{(q-1)q^{k-1} n_{y_1}}{(q^k - 1)} \right\rfloor \tag{4.15}$$

$$\sum_{i=1}^{y_1 - 1} n_i + d_H \leq \left\lfloor \frac{(q-1)q^{k-1} n_{y_1}}{(q^k - 1)} \right\rfloor + \sum_{i=1}^{y_1 - 1} n_i d_P(C) \leq \left\lfloor \frac{(q-1)q^{k-1} n_{y_1}}{(q^k - 1)} \right\rfloor + \sum_{i=1}^{y_1 - 1} n_i . \tag{4.16}$$

This implies

$$A_q(n, k, h_{dim}) \leq \left\lfloor \frac{(q-1)q^{k-1} n_{y_1}}{(q^k - 1)} \right\rfloor + \sum_{i=1}^{y_1 - 1} n_i \tag{4.17}$$

□

Remark 4.1 *For binary LCD hierarchical poset code, the upper bound in Lemma 4.1 can be written as*

$$A_2(n, 2, 0) \leq \left\lfloor \frac{2n_{y_1}}{3} \right\rfloor + \sum_{i=1}^{y_1 - 1} n_i . \tag{4.18}$$

5. Existence of Optimal $[n, 2, d_P]_2$ LCD Hierarchical Poset Codes

This section, presents existence of some two dimensional optimal binary LCD hierarchical poset codes with $n > 2$ under stated conditions.

5.1. Existence of optimal binary LCD hierarchical poset codes for the $\Lambda(C) = \{y_1, y_2\}$.

Theorem 5.1 *If* $n \equiv 0\ (mod\ 8)$, $\sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v$, $n_{y_1} = 3v+2$ *and* $\sum_{i=y_1+1}^{l} n_i = 4v-2$, *then* $A_2(n, 2, 0) = \lfloor 3v+1 \rfloor$.

Proof: We have, $n \equiv 0\ (mod\ 8)$ then $n = 8v$ for any $v \in \mathbb{N}$ and

$$\left\lfloor \frac{2n_{y_1}}{3} \right\rfloor + \sum_{i=1}^{y_1-1} n_i = (2v+1) + R_{00}^1 = 3v+1 \qquad \Big(\because \sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v, n_{y_1} = 3v+2\Big)$$

According to (4.18), the existence of an $[n,\ 2]$ binary LCD hierarchical poset code for $n \equiv 0\ (mod\ 8)$ can be proven by showing that there exists hierarchical poset code having parameters $[n, 2, 3v+1]_2$ and hull dimension zero. Assume that C is a $[n,\ 2]$ binary hierarchical poset code having a generator matrix G_1 specified in (4.2) such that (R_{01}^1, R_{10}^1)=$(2v+1, 2v+1)$ then C has parameters $[n, 2, d_P]_2$ where,

$$d_P(C) = \sum_{i=1}^{y_1-1} n_i + d_H(\bar{C}_{y_1}) = R_{00}^1 + R_{01}^1 = v + (2v+1) = 3v+1 \tag{5.1}$$

From (4.3), we have,

$$G_1G_1' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.2}$$

Therefore, by *Theorem 3.2*, the dimension of $Hull(C)$ is zero as $rank(G_1G_1') = 2$. □

Example: Let $v = 2$. Then:

$$n = v + (3v+2) + (4v-2) = 2 + 8 + 6 = 16, \quad \text{so } n \equiv 0 \pmod 8.$$

Let the poset levels be:

$$\Gamma_1 = \{1,2\}, \quad \Gamma_2 = \{3, \dots, 10\}, \quad \Gamma_3 = \{11, \dots, 16\}.$$

Let the code $C \subseteq \mathbb{F}_2^{16}$ be a hierarchical poset code with $\Lambda(C) = \{2,3\}$ and generator matrix:

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This satisfies:

$$R_{00}^1 = 2, \quad R_{01}^1 = R_{10}^1 = 5, \quad d_P(C) = 2 + 5 = 7.$$

The Gram matrix is:

$$G_1G_1^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(G_1G_1^T) = 2.$$

So $\dim(\text{Hull}(C)) = 0$. Hence, C is an LCD hierarchical poset code with optimal distance 7.

Theorem 5.2 *If* $n \equiv 1 (mod\ 8)$, $\sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v+1$, $n_{y_1} = 3v+2$ *and* $\sum_{i=y_1+1}^{l} n_i = 4v-2$, *then* $A_2(n, 2, 0) = \lfloor 3v+2 \rfloor$.

Proof: Given that, $n \equiv 1\ (mod\ 8)$ then $n = 8v+1$ for any $v \in \mathbb{N}$ and

$$\left\lfloor \frac{2n_{y_1}}{3} \right\rfloor + \sum_{i=1}^{y_1-1} n_i = (2v+1) + R_{00}^1 = 3v+2 \qquad \Big(\because \sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v+1, n_{y_1} = 3v+2\Big)$$

By (4.18), in order to prove the existence of an $[n,\ 2]$ binary LCD hierarchical poset code for $n \equiv 1\ (mod\ 8)$, it is sufficient to prove that there exist a $[n, 2, 3v+2]_2$ code having hull dimension zero. Consider C is a $[n,\ 2]$ binary hierarchical poset code having a generator matrix G_1 specified in (4.2) such that (R^1_{01}, R^1_{10})=$(2v+1, 2v+1)$. Then C has parameters $[n, 2, d_P]_2$ where,

$$d_P(C) = \sum_{i=1}^{y_1-1} n_i + d_H(\bar{C}_{y_1}) = R^1_{00} + R^1_{01} = (v+1) + (2v+1) = 3v+2 \tag{5.3}$$

From (4.3), we have,

$$G_1 G_1^{'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.4}$$

Therefore, by *Theorem 3.2*, the dimension of $Hull(C)$ is 0 as $rank(G_1 G_1') = 2$. □

Theorem 5.3 *If* $n \equiv 2\ (mod\ 8)$, $\sum_{i=1}^{y_1-1} n_i = R^1_{00} = v+1$, $\sum_{i=y_1+1}^{l} n_i = 5v, n_{y_1} = 2v+1$, *then* $A_2(n, 2, 0) = \lfloor 3v+2 \rfloor$.

Proof: The proof follows a similar approach to that of *Theorem 5.2* with the choices of (R^1_{01}, R^1_{10})=$(2v+1, 2v+1)$. □

Theorem 5.4 *If* $n \equiv 3$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R^1_{00} = 2v, \sum_{i=y_1+1}^{l} n_i = 5v \ \ and\ n_{y_1} = v+3,$$

then $A_2(n, 2, 1) = \lfloor 4v+1 \rfloor$.

Proof: The proof is analogous to that of *Theorem 5.2* with the choices of (R^1_{01}, R^1_{10})=$(2v+1, 2v+3)$. □

Theorem 5.5 *If* $n \equiv 4$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R^1_{00} = v, \sum_{i=y_1+1}^{l} n_i = 6v+4 \ \ and\ n_{y_1} = v,$$

then $A_2(n, 2, 0) = \lfloor 3v+1 \rfloor$.

Proof: The proof closely resembles the method used for *Theorem 5.2* with the choices of $(R^1_{01}, R^1_{10}) = (2v+1, 2v+1)$. □

Theorem 5.6 *If* $n \equiv 5$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = v, \sum_{i=y_1+1}^{l} n_i = 3v+2 \ \ and\ n_{y_1} = 4v+3,$$

then $A_2(n, 2, 0) = \lfloor 5v+1 \rfloor$.

Proof: This proof employs a methodology comparable to the one presented in *Theorem 5.2* with the choices of $(R^1_{01}, R^1_{10}) = (4v+1, 2v+1)$. □

Theorem 5.7 *If* $n \equiv 6$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v+1, \sum_{i=y_1+1}^{l} n_i = 6v+2 \text{ and } n_{y_1} = v+3,$$

then $A_2(n,2,0) = \lfloor 3v+2 \rfloor$.

Proof: The proof adopts a strategy similar to the one in *Theorem 5.2* with the choices of $(R_{01}^1, R_{10}^1) = (2v+1, 2v+1)$. □

Theorem 5.8 *If* $n \equiv 7$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^1 = v+1, \sum_{i=y_1+1}^{l} n_i = 3v+6 \text{ and } n_{y_1} = 4v,$$

then $A_2(n,2,0) = \lfloor 3v+2 \rfloor$.

Proof: The proof closely resembles the method used for *Theorem 5.2* with the choices of $(R_{10}^1, R_{01}^1) = (4v-1, 2v+1)$. □

5.2. Existence of optimal binary LCD hierarchical poset codes for the $\Lambda(C) = \{y_1\}$.

Theorem 5.9 *If* $n \equiv\ 0$ $(mod\ 8)$,

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v, \sum_{i=y_1+1}^{l} n_i = 4v-2 \text{ and } n_{y_1} = 3v+2,$$

then $A_2(n,2,0) = \lfloor 3v+1 \rfloor$.

Proof: Given that $n \equiv 0$ $(mod\ 8)$ then $n = 8v$ for any $v \in \mathbb{N}$ and

$$\left\lfloor \frac{2n_{y_1}}{3} \right\rfloor + \sum_{i=1}^{y_1-1} n_i = 2v+1+R_{00}^2 = 3v \qquad \left(\because \sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v, n_{y_1} = 3v+2\right)$$

By (4.18), in order to prove the existence of an $[n,\ 2]$ binary LCD hierarchical poset code for $n \equiv 0$ $(mod\ 8)$, it is sufficient to prove that there exists an $[n, 2, 3v+1]_2$ code having hull dimension zero. Consider, C is a $[n,\ 2]$ binary hierarchical poset code having a generator matrix G_2 specified in (4.6) such that (R_{01}^2, R_{10}^2)=$(2v+1, 2v+1)$. Then C has parameters $[n, 2, d_P]_2$ where,

$$d_P(C) = \sum_{i=1}^{y_1-1} n_i + d_H(\bar{C}_{y_1}) = R_{00}^2 + \min\{R_{01}^2,\ R_{10}^2\} = v+2v+1 = 3v+1 \tag{5.5}$$

From (4.3), we have,

$$G_2 G_2^{'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.6}$$

Therefore, by *Theorem 3.2*, the dimension of $Hull(C)$ is 0 as $rank(G_2 G_2') = 2$. □

Theorem 5.10 *If* $n \equiv 1$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v+1, \sum_{i=y_1+1}^{l} n_i = 4v-2 \text{ and } n_{y_1} = 3v+2,$$

then $A_2(n,2,0) = \lfloor 3v+2 \rfloor$.

Proof: We have, $n \equiv 1$ $(mod\ 8)$. Then $n = 8v+1$ for some $v \in \mathbb{N}$ and

$$\left\lfloor \frac{2n_{y_1}}{3} \right\rfloor + \sum_{i=1}^{y_1-1} n_i = 2v+1+R_{00}^2 = 3v+2 \qquad \left(\because \sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v+1, n_{y_1} = 3v+2\right)$$

By (4.18), in order to prove the existence of an $[n,\ 2]$ binary LCD hierarchical poset code for $n \equiv 1$ $(mod\ 8)$, it is sufficient to prove that there exists an $[n, 2, 3v+2]_2$ code having hull dimension zero. Consider C is a $[n,\ 2]$ binary hierarchical poset code having a generator matrix G_2 specified in (4.6) such that (R_{01}^2, R_{10}^2)=$(2v+1, 2v+1)$. Then C has parameters $[n, 2, d_P]_2$ where,

$$d_P(C) = \sum_{i=1}^{y_1-1} n_i + d_H(\bar{C}_{y_1}) = R_{00}^2 + \min\{R_{01}^2,\ R_{10}^2\} = v+1+2v+1 = 3v+2 \tag{5.7}$$

From (4.3), we have,

$$G_2 G_2^{'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.8}$$

Therefore, by *Theorem 3.2*, the dimension of $Hull(C)$ is 0 as $rank(G_2G_2') = 2$. □

Example: Let $v = 2$. Then:

$$n = 3+8+6 = 17, \quad \text{so } n \equiv 1 \pmod 8.$$

Define levels:

$$\Gamma_1 = \{1,2,3\}, \quad \Gamma_2 = \{4,\dots,11\}, \quad \Gamma_3 = \{12,\dots,17\}.$$

Let code $C \subseteq \mathbb{F}_2^{17}$ with $\Lambda(C) = \{2\}$ and generator matrix:

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So:

$$R_{00}^2 = 3, \quad R_{01}^2 = R_{10}^2 = 5, \quad d_P(C) = 3+\min(5,5) = 8.$$

And the Gram matrix:

$$G_2G_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \dim(\text{Hull}(C)) = 0.$$

Hence, C is a binary LCD hierarchical poset code of parameters $[17, 2, 8]_2$.

Theorem 5.11 *If* $n \equiv 2$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v+1, \sum_{i=y_1+1}^{l} n_i = 5v \text{ and } n_{y_1} = 2v+1,$$

then $A_2(n, 2, 0) = \lfloor 3v+2 \rfloor$.

Proof: The proof is similar to demonstrated in *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(2v+1, 2v+1)$. □

Theorem 5.12 *If* $n \equiv 3$ *(mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = 2v, \sum_{i=y_1+1}^{l} n_i = 5v \text{ and } n_{y_1} = v+3,$$

then $A_2(n, 2, 0) = \lfloor 4v+3 \rfloor$.

Proof: The proof follows similar approach to that of *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(2v+3, 2v+3)$. □

Theorem 5.13 *Consider a natural number $n > 4$. If $n \equiv 4$ (mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v, \sum_{i=y_1+1}^{l} n_i = 6v+4 \text{ and } n_{y_1} = v,$$

then $A_2(n,2,0) = \lfloor 3v+1 \rfloor$.

Proof: The proof adopts a strategy similar to the one in *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(2v+1, 2v+1)$.

□

Theorem 5.14 *If $n \equiv 5$ (mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v, \sum_{i=y_1+1}^{l} n_i = 3v+2 \text{ and } n_{y_1} = 4v+3,$$

then $A_2(n,2,0) = \lfloor 5v+3 \rfloor$.

Proof: The proof aligns with the methodology used in *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(4v+3, 4v+3)$.

□

Theorem 5.15 *If $n \equiv 6$ (mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v+1, \sum_{i=y_1+1}^{l} n_i = 6v+2 \text{ and } n_{y_1} = v+3,$$

then $A_2(n,2,0) = \lfloor 3v+2 \rfloor$.

Proof: The proof follows the same reasoning as that of *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(2v+1, 2v+1)$.

□

Theorem 5.16 *If $n \equiv 7$ (mod 8),*

$$\sum_{i=1}^{y_1-1} n_i = R_{00}^2 = v+1, \sum_{i=y_1+1}^{l} n_i = 3v+6 \text{ and } n_{y_1} = 4v,$$

where, $v \in \mathbb{N}$ then $A_2(n,2,0) = \lfloor 5v+2 \rfloor$.

Proof: The proof adopts a similar strategy to that of *Theorem 5.10* with the choices of (R_{01}^2, R_{10}^2)=$(4v+1, 4v+1)$ for some $v \in \mathbb{N}$.

□

6. Conclusion

Optimal binary LCD code under hierarchical poset metric have been systematically constructed for all $n \in \mathbb{N} \setminus \{1,2\}$ subject to specific conditions. Correspondingly, the precise value of $A_2(n,2,0)$ has been established under stated condition. This study can be expanded in several ways, one of which is to construct optimal LCD codes using hierarchical poset metrics over other finite fields. Another approach is to construct optimal LCD codes by using other poset metrics, such as the NRT metric and crown poset metric.

Acknowledgments

The first author, Rohini Baliram More, thanks the Department of Science and Technology of the Government of India for the research grant through DST INSPIRE Fellowship [IF No:190335].

Declarations

- Conflict of interest: Both authors declare that they do not have any conflicts of interest.

References

1. James L. Massey, Linear codes with complementary duals, Discrete Mathematics, Volumes 106–107, 1992, Pages 337-342, ISSN 0012-365X, https://doi.org/10.1016/0012-365X(92)90563-U.

2. Nicolas Sendrier, Linear codes with complementary duals meet the Gilbert–Varshamov bound, Discrete Mathematics, Volume 285, Issues 1–3, 2004, Pages 345-347, ISSN 0012-365X, https://doi.org/10.1016/j.disc.2004.05.005.

3. Claude Carlet, Sylvain Guilley. Complementary dual codes for counter-measures to side-channel attacks. Advances in Mathematics of Communications, 2016, 10(1): 131-150. doi: 10.3934/amc.2016.10.131

4. Ching-Yi Lai and Alexei Ashikhmin, "Linear Programming Bounds for Entanglement-Assisted Quantum Error-Correcting Codes by Split Weight Enumerators," in IEEE Transactions on Information Theory, vol. 64, no. 1, pp. 622-639, Jan. 2018, doi: 10.1109/TIT.2017.2711601.

5. Liangdong Lu, Ruihu Li, Luobin Guo and Qiang Fu, "Maximal entanglement entanglement-assisted quantum codes constructed from linear codes". Quantum Information Processing 14, 165–182 (2015). https://doi.org/10.1007/s11128-014-0830-y

6. Stefka Bouyuklieva, "Optimal binary lcd codes", Designs, Codes and Cryptography, 89, 2445–2461 (2021). https://doi.org/10.1007/s10623-021-00929-w

7. Guodong Wang, Shengwei Liu, and Hongwei Liu, "New constructions of optimal binary LCD codes" Finite Fields and their Applications,Volume 95, 2024, 102381, ISSN 1071-5797, https://doi.org/10.1016/j.ffa.2024.102381.

8. Claude Carlet, Sihem Mesnager, Chunming Tang, and Yanfeng Qi, "New characterization and parametrization of LCD codes" IEEE Transactions on Information Theory, vol. 65, no. 1, pp. 39-49, Jan. 2019, doi: 10.1109/TIT.2018.2829873.

9. Hualu Liu, Xiusheng Liu and Long Yu, "New binary and ternary lcd codes from matrix-product codes", Linear and Multilinear Algebra, 70(5):809–823, 2022.

10. Shitao Li, Minjia Shi, and Huizhou Liu "Several constructions of optimal LCD codes over small finite fields", Cryptography and Communications, 16, 779–800 (2024). https://doi.org/10.1007/s12095-024-00699-x

11. Ranya Boulanouar, Aicha Batoul and Delphine Boucher "An overview on skew constacyclic codes and their subclass of LCD codes", Advances in Mathematics of Communications 15, no. 4 (2021): 611-632.

12. Djoko Suprijanto and Hopein Christofen Tang " Skew cyclic codes over $Z_4 + vZ_4$ with derivation: structual properties and computational results" Communications in Combinatorics and Optimization, Vol. 10, No. 3 (2025), pp. 497-517 https://doi.org/10.22049/cco.2024.28837.1744

13. Yang Liu, Ruihu Li, Qiang Fu, and Hao Song "On the minimum distances of binary optimal LCD codes with dimension 5", AIMS Mathematics, 9(7):19137–19153, 2024, doi: 10.3934/math.2024933.

14. Conghui Xie, Hao chen and Chengju Li, "New Ternary LCD Cyclic and BCH Codes" Advances in Mathematics of Communication, 2025

15. Makoto Araya, Masaaki Harada, and Ken Saito "Quaternary Hermitian Linear Complementary Dual Codes", IEEE Transactions on Information Theory, vol. 66, no. 5, pp. 2751-2759, May 2020, doi: 10.1109/TIT.2019.2949040.

16. Lin Sok. "On Hermitian LCD codes and their Gray image", Finite Fields and their Applications, 62:101623, 2020, https://doi.org/10.1016/j.ffa.2019.101623.

17. Lin Sok, Minjia Shi, and Patrick Sole, "Constructions of optimal LCD codes over large finite fields" Finite Fields and their Applications, 50:138–153, 2018, https://doi.org/10.1016/j.ffa.2017.11.007.

18. Rohini Baliram More and Venkatrajam Marka, "Optimal binary hierarchical poset code having hull dimension one", Advances and Applications in Discrete Mathematics, 41(3), pp. 239–259, 2024, doi: 10.17654/0974165824018.

19. Richard A. Brualdi, Janine Smolin Graves, and K. Mark Lawrence, "Codes with a poset metric", Discrete Mathematics, 147(1-3):57–72, 1995, https://doi.org/10.1016/0012-365X(94)00228-B.

20. Marcelo Firer, Marcelo Muniz S. Alves, Jerry Anderson Pinheiro and Luciano Panek, "Poset Codes : Partial Orders , Metrics and Coding Theory", volume 1. 2018, https://doi.org/10.1007/978-3-319-93821-9

21. Roberto Assis Machado, Jerry Anderson Pinheiro, and Marcelo Firer "Characterization of Metrics Induced by Hierarchical Posets" IEEE Transactions on Information Theory, vol. 63, no. 6, pp. 3630-3640, June 2017, doi: 10.1109/TIT.2017.2691763.

22. Kenza Guenda, Somphong Jitman, and T. Aaron Gulliver, "Constructions of good entanglement-assisted quantum error correcting codes", Designs, Codes, and Cryptography, 86, 121–136 (2018). https://doi.org/10.1007/s10623-017-0330-z

23. Luciano Viana Felix and Marcelo Firer, "Canonical- systematic form for codes in hierarchical poset metrics", Advances in Mathematics of Communications, 6(3):315–328, 2012, doi:10.3934/amc.2012.6.315

24. W Cary Huffman and Vera Pless, "Fundamental of Error-Correcting Codes", Cambridge,University Press, 2003.

25. Mohammed El Badry, Abdelfattah Haily and Ayoub Mounir, "On LCD skew group codes", Design Codes and Cryptography, 93, 1487–1499 (2025). https://doi.org/10.1007/s10623-024-01561-0.

26. Shikha Yadav, Indibar Debnath and Om Prakash, "Some constructions of l-Galois LCD codes" Advances in Mathematics of Communications, 2025, 19(1): 227-244. doi: 10.3934/amc.2023053.

27. Sanjit Bhowmick, Satya Bagchi and Ramkrishna Bandi, "On LCD codes over Z_4, Indian Journal of Pure and Applied Mathematics, 2024, https://doi.org/10.1007/s13226-024-00563-x

Rohini Baliram More
Department of Mathematics,
School of Advanced Sciences,
VIT-AP University, Amaravati,
India.
E-mail address: rm25spc@gmail.com

and

Venkatrajam Marka
Department of Mathematics,
School of Advanced Sciences,
VIT-AP University, Amaravati,
India.
E-mail address: mvraaz.nitw@gmail.com