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Laplacian Minimum Split Dominating Energy of Graphs

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ABSTRACT: For a graph G, a subset D of $V(G)$ is called a split dominating set if the induced graph $< V-D >$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split domination set. In this paper we introduce the Laplacian minimum split dominating energy $LE_s(G)$ of a graph G and computed Laplacian minimum split dominating energies of some standard graphs. Upper and lower bounds for $LE_s(G)$ are established.

Key Words: Split domination, split domination number, split domination energy, Laplacian minimum split dominating energy.

Contents

1. Introduction

Let $G = (V, E)$ be a graph with n vertices and m edges, and let $A = (a_{i,j})$ denote its adjacency matrix. The eigenvalues of the graph G are defined as the eigenvalues of its adjacency matrix $A(G)$, and are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. A graph G is said to be singular if at least one of its eigenvalues is zero, which implies that $\det(A) = 0$. Conversely, if none of the eigenvalues are zero, the graph is called nonsingular, in which case $\det(A) > 0$.

A graph G is called k-regular if every vertex in G has degree k. A complete graph, denoted by K_n, is a simple graph in which every pair of distinct vertices is connected by an edge. A star graph, denoted by $K_{1,n-1}$, also referred to as a claw or cherry, is a tree with a single central vertex joined to all other vertices. The friendship graph (or Dutch windmill graph or n-fan) F_n is a planar, undirected graph with $2n+1$ vertices and $3n$ edges. The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph.

A subset $D \subseteq V$ is called a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is called minimal if no proper subset of D is also a dominating set. The smallest possible size of a minimal dominating set in G is termed the domination number, denoted by $\gamma(G)$.

The notion of graph energy was introduced by I. Gutman [5]. Although it initially received limited attention, it has since gained widespread interest and has become a topic of active research. Over time, analogous energy concepts have also been developed for matrices other than the adjacency matrix. The energy $E(G)$ of a graph G is defined as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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I. Gutman and B. Zhou [6] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n. The elements of the Laplacian matrix are defined as

$$L_{ij} = \begin{cases} -1, & \text{if } v_i \, and \, v_j \, are \, adjacent, \\ 0, & \text{if } v_i \, and \, v_j \, are \, not \, adjacent, \\ d_i, & \text{if i=j.} \end{cases}$$

where d_i is the degree of the vertex v_i.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of Laplacian matrix G. Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \Big| \lambda_i - \frac{2m}{n} \Big|.$$

The basic properties Laplacian energy including various upper and lower bounds have been established in [11], [12], [14], [15] and it has found that remarkable chemical application, high resolution satellite image classification and segmentation using Laplacian graph energy and finding semantic structures in image hierarchies using Laplacian graph energy. The Laplacian energy provides a measure of the overall connectivity and robustness of a network. A low Laplacian energy typically indicates a highly connected network (less fragmentation), where as high Laplacian energy might indicate a more fragmented or less connected network. The Laplacian matrix is used to model molecular structures where atoms are represented as vertices and bonds as edges. In this context, Laplacian energy can be used to estimate the stability and reactivity of molecules. A molecule with a high Laplacian energy might be less stable or more reactive, while lower energy could suggest a more stable structure.

2. The Minimum Split Domination Energy of Graphs

Let G be simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. Let D be a subset of $V(G)$ which is said to be split dominating set [7] if the induced graph $< V - D >$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set. Any split dominating set with minimum cardinality is called a minimum split dominating set. Let D be a minimum split dominating set of a graph G. The minimum split dominating matrix of G is the $n \times n$ matrix defined by $A_s(G) = a_{ij}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E(G), \\ 1, & \text{if } i = j, v_i \in D, \\ 0, & \text{if } otherwise. \end{cases}$$

The characteristic polynomial of $A_s(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A_s(G))$. The minimum split dominating eigenvalues of the graph G are the eigenvalues of $A_s(G)$. Since $A_s(G)$ is real and symmetric, its eigenvalues are real numbers and are labelled in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The minimum split dominating energy of G is defined as

$$E_s(G) = \sum_{i=1}^{n} |\lambda_i|.$$

3. The Laplacian Minimum Split Dominating Energy of Graphs

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then the Laplacian minimum split dominating matrix of G is denoted by $LE_s(G)$ and is defined as follows $LE_s = D(G) - A_s(G)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values $LE_s(G)$ arranged in non increasing order. These eigen values are called Laplacian minimum split dominating eigen values of G. The Laplacian minimum split dominating energy of a graph G is defined as

$$LE_s(G) = \sum_{i=1}^{n} \Big| \lambda_i - \frac{2m}{n} \Big|.$$

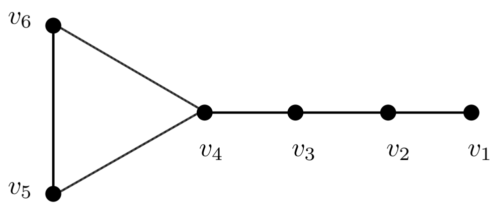


Figure 1: A Simple Graph

Example 3.1 *Consider a graph as shown in Figure 1. The possible minimum split dominating sets are: (i)* $D_1 = \{v_1, v_4\}$ *(ii)* $D_2 = \{v_2, v_4\}$ *(iii)* $D_3 = \{v_2, v_5\}$ *(iv)* $D_4 = \{v_2, v_6\}$

If the split dominating set is $D_1 = \{v_1, v_4\}$, then

$$A_{D_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$LE_sD_1(G) = D(G) - A_{D_1}(G) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G, \lambda) = \lambda^6 - 10\lambda^5 + 34\lambda^4 - 40\lambda^3 - 4\lambda^2 + 20\lambda + 3 = 0.$

$$Spec(G) = \begin{pmatrix} -0.5191 & -0.1535 & 1.2455 & 2.7125 & 3 & 3.7146 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $= \frac{2m}{n} = \frac{2 \times 6}{6} = 2.$
Hence, Laplacian minimum split dominating energy, $LE_sD_1(G) \approx 8.8542$.
If the minimum split dominating set is $D_2 = \{v_3, v_6\}$, then

$$A_{D_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$LE_sD_2(G) = D(G) - A_{D_2}(G) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial is given by
$f_n(G,\lambda) = \lambda^6 - 10\lambda^5 + 35\lambda^4 - 47\lambda^3 + 10\lambda^2 + 15\lambda - 0 = 0.$

$$Spec(G) = \begin{pmatrix} -0.4142 & 0 & 1.3820 & 2.4142 & 3 & 3.6180 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $= \frac{2m}{n} = \frac{2\times 6}{6} = 2$.
Hence, Laplacian minimum split dominating energy, $LE_sD_2(G) \approx 8.0644$.
Note that the Laplacian minimum split dominating energy of the graph G depends on its minimum split dominating set.

4. Laplacian Minimum Split Dominating Energy of Some Standard Graphs

Theorem 4.1 *For $n \geq 3$, the Laplacian minimum split dominating energy of a star graph $K_{1,n-1}$ is $\frac{(n-2)^2}{n} + \sqrt{n^2 - 2n + 5}$.*

Proof 4.1 *Let $K_{1,n}$ be a star graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ having the vertex v_1 at the center. The minimum split dominating set is $D = \{v_1\}$. Then,*

$$A_s(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \end{pmatrix}.$$

and

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \end{pmatrix}.$$

$$LE_s(K_{1,n-1}) = D(K_{1,n-1}) - A_s(K_{1,n-1}) = \begin{pmatrix} n-2 & -1 & -1 & ... & -1 & -1 \\ -1 & 1 & 0 & ... & 0 & 0 \\ -1 & 0 & 1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & 0 & 0 & ... & 1 & 0 \\ -1 & 0 & 0 & ... & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of $LE_s(K_{1,n-1})$ is given by,
$(\lambda - 1)^{n-2}(\lambda^2 - (n-1)\lambda - 1) = 0.$
The Laplacian minimum split dominating eigen values are:

$$Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{(n-1)-\sqrt{n^2-2n+5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_{1,n-1} = \frac{2(n-1)}{n}$.
Hence, the Laplacian minimum split dominating energy of $K_{1,n-1}$ *is*

$$\begin{aligned} LE_s(K_{1,n-1}) =& \left|1 - \frac{2(n-1)}{n}\right|(n-2) + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right| \\ &+ \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2} - \frac{2(n-1)}{n}\right| \\ =& \left|\frac{(-n+2)}{n}\right|(n-2) + \left|\frac{(n^2-5n+4)+\sqrt{n^2-2n+5}}{2n}\right| + \\ & \left|\frac{(n^2-5n+4)-\sqrt{n^2-2n+5}}{2n}\right|. \\ =& \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}. \end{aligned}$$

Therefore, $LE_s(K_{1,n-1}) = \frac{(n-2)^2}{n} + \sqrt{n^2-2n+5}$.

Theorem 4.2 *For* $n \geq 4$*, the Laplacian minimum split dominating energy of a complete bipartite graph is* $(n-1) + \sqrt{4n^2+1}$.

Proof 4.2 *Let* $K_{n,n}$ *be a complete bipartite graph with the vertex set* $V = \{v_1, v_2, \ldots, v_{2n}\}$*. The minimum split dominating set of a complete bipartite graph is* $D = \{v_1, v_2, ..., v_n\}$*. The minimum split dominating matrix* $A_s(K_{n,n})$ *and its characteristic polynomial are as follows:*

$$A_s(K_{n,n}) = \begin{pmatrix} 1 & 0 & 0 & ... & 1 & 1 \\ 0 & 1 & 0 & ... & 1 & 1 \\ 0 & 0 & 1 & ... & 1 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 1 & 1 & 1 & ... & 0 & 0 \\ 1 & 1 & 1 & ... & 0 & 0 \end{pmatrix}.$$

and

$$D(K_{n,n}) = \begin{pmatrix} n & 0 & 0 & ... & 0 & 0 \\ 0 & n & 0 & ... & 0 & 0 \\ 0 & 0 & n & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & n & 0 \\ 0 & 0 & 0 & ... & 0 & n \end{pmatrix}.$$

$$LE_s(K_{n,n}) = D(K_{n,n}) - A_s(K_{n,n}) = \begin{pmatrix} n-1 & 0 & 0 & ... & -1 & -1 \\ 0 & n-1 & 0 & ... & -1 & -1 \\ 0 & 0 & n-1 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ -1 & -1 & -1 & ... & n & 0 \\ -1 & -1 & -1 & ... & 0 & n \end{pmatrix}.$$

The characteristic polynomial of $LE_s(K_{n,n})$ *is given by,*
$f_s(K_{n,n}, \lambda) = (\lambda - (n-1))^{n-1}(\lambda - n)^{n-1}(\lambda^2 - (2n-1)\lambda - n)$.
The Laplacian minimum split dominating eigen values are:

$$Spec(K_{n,n}) = \begin{pmatrix} n-1 & n & \frac{(2n-1)+\sqrt{4n^2+1}}{2} & \frac{(2n-1)-\sqrt{4n^2+1}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Average degree of $K_{n,n} = \frac{2(n^2)}{2n} = n.$

Hence, the Laplacian minimum split dominating energy is

$$\begin{aligned} LE_s(K_{n,n}) =& \Big|(n-1)-n\Big|(n-1) + \Big|n-n\Big|(n-1) + \left|\frac{(2n-1)+\sqrt{4n^2+1}}{2} - n\right| \\ &+ \left|\frac{(2n-1)-\sqrt{4n^2+1}}{2} - n\right| \\ =&(n-1) + \left|\frac{(-n^2+2n-1)+\sqrt{4n^2+1}}{2n}\right| + \\ &\left|\frac{(-n^2+2n-1)-\sqrt{4n^2+1}}{2n}\right|. \\ =&(n-1) + \sqrt{4n^2+1}. \end{aligned}$$

Therefore, $LE_s(K_{n,n}) = (n-1) + \sqrt{4n^2+1}.$

Theorem 4.3 *For* $n \geq 3$*, Laplacian minimum split dominating energy of a friendship graph* F_3^n *is* $\frac{4n^2-2n+1}{2n+1} + 2\sqrt{n^2+1}.$

Proof 4.3 *Consider a friendship graph* F_3^n *with vertex set* $V = \{v_1, v_2, v_3, ..., v_{2n+1}\}$*, where the vertex* v_3 *is the center vertex of* F_3^n*. The minimum split dominating set of* F_3^n *is* $D = \{v_3\}$*.*

$$A_s(F_3^n) = \begin{pmatrix} 0 & 1 & 1 & ... & 0 & 0 \\ 1 & 0 & 1 & ... & 0 & 0 \\ 1 & 1 & 1 & ... & 1 & 1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 1 & ... & 0 & 1 \\ 0 & 0 & 1 & ... & 1 & 0 \end{pmatrix}.$$

and

$$D(F_3^n) = \begin{pmatrix} 2 & 0 & 0 & ... & 0 & 0 \\ 0 & 2 & 0 & ... & 0 & 0 \\ 0 & 0 & 2n & ... & 0 & 0 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & 2 & 0 \\ 0 & 0 & 0 & ... & 0 & 2 \end{pmatrix}.$$

$$LE_s(F_3^n) = D(F_3^n) - A_s(F_3^n) = \begin{pmatrix} 2 & -1 & -1 & ... & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 \\ -1 & -1 & 2n-1 & ... & -1 & -1 \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ . & . & . & ... & . & . \\ 0 & 0 & -1 & ... & 2 & -1 \\ 0 & 0 & -1 & ... & -1 & 2 \end{pmatrix}.$$

The characteristic polynomial is given by
$F_3^n(G,\lambda) = (\lambda-1)^{n-1}(\lambda-3)^n(\lambda^2-(2n)\lambda-1) = 0.$
The Laplacian minimum split dominating eigen values are

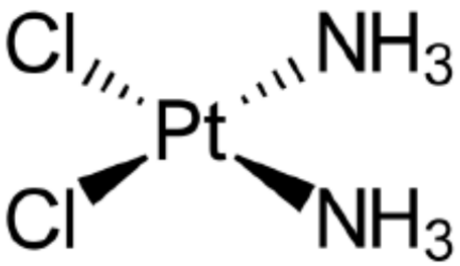
$$Spec(F_3^n) = \begin{pmatrix} 1 & 3 & \frac{2n+\sqrt{4n^2+4}}{2} & \frac{2n-\sqrt{4n^2+4}}{2} \\ (n-1) & n & 1 & 1 \end{pmatrix}.$$

Average degree of the graph $=\frac{2m}{n} = \frac{6n}{2n+1}$
Hence, the Laplacian minimum split dominating energy is

$$\begin{aligned} LE_s(F_3^n) =& \left|1-\frac{6n}{2n+1}\right|(n-1) + \left|3-\frac{6n}{2n+1}\right|n + \left|\frac{2n+\sqrt{4n^2+4}}{2}-\frac{6n}{2n+1}\right| \\ &+\left|\frac{2n-\sqrt{4n^2+4}}{2}-\frac{6n}{2n+1}\right| \\ =&\left|\frac{1-4n}{2n+1}\right|(n-1) + \left|\frac{3}{2n+1}\right|n + \left|\frac{(2n^2-5)+(2n+1)\sqrt{n^2+1}}{2n+1}\right| \\ &+\left|\frac{(2n^2-5)-(2n+1)\sqrt{n^2+1}}{2n+1}\right| \\ =&\frac{4n^2-2n+1}{2n+1}+2\sqrt{n^2+1}. \end{aligned}$$

Therefore, $LE_s(F_3^n) = \frac{4n^2-2n+1}{2n+1} + 2\sqrt{n^2+1}$.

Cancer is a disease which is caused by an uncontrolled division of abnormal cells in a part of the body. Now a days we can find many types of cancer like, bladder cancer lung cancer, brain cancer, melanoma, breast cancer, Non-Hodgkin lymphoma, cervical cancer, ovarian cancer. Etc., Cisplatin is a medicine which is widely used against the cancer disease. In this paper we are calculating the Laplacian minimum split dominating energy of the Cisplatin which is very much useful for further research and development in the treatment of cancer.
Structural formula: $Pt(NH_3)_2Cl_2$.



The Laplacian minimum split dominating matrix with the consideration of minimum split dominating set of $\{Pt(NH_3)_2Cl_2)\}$ is given by

$$A_s(G) = \begin{pmatrix} 0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\1&1&1&1&1 \end{pmatrix} \text{ and}$$

$$D(G) = \begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&4 \end{pmatrix}.$$

$$LE_s(G) = D(G) - A_s(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix}.$$

Characteristic equation is $\lambda^5 - \lambda^4 - 10\lambda^3 - 8\lambda^2 + 5\lambda + 5 = 0$ and the Laplacian minimum split dominating eigenvalues are,

$$Spec(G) = \begin{pmatrix} -1.6940 & -1 & -1 & 0.7480 & 3.9460 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore the Laplacian minimum split dominating energy is $LE_s(Pt(NH_3)_2Cl_2) = 11.6920$

5. Bounds on Laplacian Minimum Split Dominating Energy of Graphs

Theorem 5.1 *If D is a minimum split dominating set of a graph G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are minimum split dominating eigen values of $LE_s(G)$ then*

(i) $\sum_{i=1}^{n} \lambda_i = 2 \mid E \mid - \mid D \mid$.

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2 \mid E \mid + \sum_{i=1}^{n} (d_i - t_i)^2$ *where* $t_i = \begin{cases} 1, & \text{if } v_i \in D, \\ 0, & \text{if } v_i \notin D. \end{cases}$

Proof 5.1 ***(i)*** *By definition, the sum of the principal diagonal elements of $LE_s(G)$ is equal to* $\sum_{i=1}^{n} \lambda_i - \mid D \mid = 2 \mid E \mid - \mid D \mid$.

Also the sum of eigen values of $LE_s(G)$ is trace of $LE_s(G)$.

(ii) *The sum of squares of eigen values of $LE_s(G)$ is the trace of $LE_s(G)^2$*

Therefore $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij} l_{ji} = \sum_{i=1}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$

$$= 2\sum_{i<j} (l_{ij})^2 + \sum_{i=1}^{n} (l_{ji})^2$$

$$= 2 \mid E \mid + \sum_{i=1}^{n} (d_i - t_i)^2 \text{ where } t_i = \begin{cases} 1, & if v_i \in D \\ 0, & if v_i \notin D \end{cases}$$

$$= 2M, \text{ where } M = \mid E \mid + \frac{1}{2}\left(\sum_{i=1}^{n} (d_i - t_i)^2\right)$$

Theorem 5.2 *If G be a graph with n vertices, m edges and D is a minimum split dominating set of a graph G. Then $LE_s(G) \le \sqrt{2Mn} + 2m$.*

Proof 5.2 *Let G be a graph with n vertices and m edges and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of G. By using Cauchy's - Schwarz inequality*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1,\ b_i = \mid \lambda_i \mid$ *in above inequality then,*

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \mid \lambda_i \mid^2\Big)$$
$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \leq n\ 2M$$
$$\therefore\ \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big) \leq \sqrt{2Mn}.$$

By Triangle inequality $\left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \left|\frac{2m}{n}\right|\ \ \forall i = 1, 2, \ldots, n$

$$i.e.,\ \left|\lambda_i - \frac{2m}{n}\right| \leq \mid \lambda_i \mid + \frac{2m}{n}\ \ \forall\, i = 1, 2, \ldots, n$$

$$\Big(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\Big) \leq \Big(\sum_{i=1}^{n} \lambda_i\Big) + \Big(\sum_{i=1}^{n} \frac{2m}{n}\Big)$$
$$\leq \sqrt{2Mn} + 2m$$
$$\therefore LE_{nsp}(G) \leq \sqrt{2Mn} + 2m$$

Theorem 5.3 *Let* G *be a graph with* n *vertices and* m *edges and* D *be a minimum split dominating set of* G*. Then* $LE_s(G) \leq \sqrt{2Mn + 4m(\mid D \mid - m)}$

Proof 5.3 *By using Cauchy's - Schwarz inequality*

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big)$$

Put $a_i = 1,\ b_i = \left|\lambda_i - \frac{2m}{n}\right|$ *in above inequality then,*

$$\Big(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|^2\Big).$$
$$i.e., [LE_s(G)]^2 = n\Big[\sum_{i=1}^{n} \lambda_i^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n} \sum_{i=1}^{n} \lambda_i\Big]$$
$$= n\Big[2M + \frac{4m^2}{n^2}.n - \frac{4m}{n}(2m - \mid D \mid)\Big]$$
$$= n\Big[2M + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m \mid D \mid}{n}\Big]$$
$$= 2Mn + 4m(\mid D \mid - m)$$
$$\therefore\ LE_s(G) \leq \sqrt{2Mn + 4m(\mid D \mid - m)}$$

Theorem 5.4 *Let* G *be a graph with* n *vertices and* m *edges and* D *is a minimum split dominating set of* G*. If* $D = \mid detLE_s(G) \mid$ *then*
$LE_s(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$

Proof 5.4 *Consider*

$$\Big[\sum_{i=1}^{n} \mid \lambda_i \mid\Big]^2 = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big).\Big(\sum_{j=1}^{n} \mid \lambda_j \mid\Big)$$
$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$\therefore \quad \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - \sum_{i=1}^{n} \mid \lambda_i \mid^2 \tag{5.1}$$

Applying Arithmtic and Geometric means for $n(n-1)$ terms, we have

$$\frac{\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid}{n(n-1)} \geq \Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$
$$i.e., \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \geq n(n-1)\Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$

Using (5.1) *we get,*

$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - \sum_{i=1}^{n} \mid \lambda_i \mid^2 \geq n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$
$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 - 2M \geq n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$
$$\Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \geq 2M + n(n-1)\Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$
$$\therefore \sum_{i=1}^{n} \mid \lambda_i \mid \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}}$$

We know that

$$\mid \lambda_i \mid - \Big|\frac{2m}{n}\Big| \leq \Big|\lambda_i - \frac{2m}{n}\Big| \ \forall i = 1, 2, 3, ..., n$$
$$\sum_{i=1}^{n} \mid \lambda_i \mid - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \Big|\lambda_i - \frac{2m}{n}\Big|$$
$$i.e., \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m \leq LE_s(G)$$
$$i.e., LE_s(G) \geq \sum_{i=1}^{n} \Big|\lambda_i\Big| - 2m$$
$$\geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$
$$\therefore LE_s(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m$$

6. Conclusion

In this paper, we introduced the concept of the Laplacian minimum split dominating energy $LE_s(G)$ associated with a graph G, building upon the notion of split dominating sets and the split domination number $\gamma_s(G)$. We successfully computed the Laplacian minimum split dominating energies for several standard classes of graphs, providing explicit values and insights. Furthermore, we established meaningful upper and lower bounds for $LE_s(G)$, contributing to the theoretical framework connecting split domination and spectral graph theory. These results open avenues for further research in understanding the interplay between domination parameters and Laplacian spectral properties of graphs.

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