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Sharp Coefficient Estimates for Certain Subclasses of Univalent Functions Associated with Nephroid Domain

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ABSTRACT: In this article, first of all we introduce a newly defined class of analytic functions associated with Nephroid shaped domain and explores their coefficient properties. We determine sharp bounds for the initial coefficients including a sharp Fekete-Szegö inequality. The study also investigates exact bounds for Hankel determinants of different orders. We derive bounds for the inverse coefficients and logarithmic coefficients, and also sharpness of some of these estimates. Finally, this article demonstrates the sharpness of the Zalcman functional and the Krushkal inequality as a particular cases. Additionally, we examines how coefficient estimation, growth and distortion theorems reveal the relationship between an analytic functions structure and its behavior for different class by using the concept of caratheodory functions.

Keywords: Analytic functions, Fekete-Szegö inequality, coefficient problems, Nephroid-shaped domain, Hankel determinants, growth and distortion bounds.

Contents

1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ having normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Then, the function f can admits Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consists of the univalent functions in $\mathbb{D}$. Suppose that B_0 is the class of analytic functions, i.e., analytic function $w : \mathbb{D} \to \mathbb{D}$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$. The function $w \in B_0$ can be written as a power series of the form:

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}). \tag{1.2}$$

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For given analytic functions f and g in $\mathbb{D}$, we say f is subordinate to g in $\mathbb{D}$ and write $f \prec g$ if there exists $w \in B_0$ such that $f(z) = g(w(z))(z \in \mathbb{D})$. Moreover, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In 1959, Sakaguchi [22] introduced the class of starlike functions with respect to symmetric point as:

$$S^* := \left\{ f \in \mathcal{S} : \Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0; z \in \mathbb{D} \right\}.$$

These functions are also known as Sakaguchi functions which are close-to-convex as well as univalent. In 2004, making use of subordination between two analytic functions Ravichandran [21] introduced a unified class $S_s^*(\phi)$ as follows:

$$S_s^*(\phi) = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z); z \in \mathbb{D} \right\}$$

where $\phi(z) = 1 + \sum_{n=1}^{\infty} D_n z^n$ is univalent starlike function with respect to 1 which maps $\mathbb{D}$ onto a symmetric region with respect to real axis in the right half plane. A good amount of literature is available for finding the upper bounds of coefficient functional for several subclasses of the class $S_s^*(\phi)$. For details see [11] and reference within.

Motivated by aforementioned works, we introduce the following subclass of the class $\mathcal{A}$ associated with Nephroid domain.

Definition 1.1. *A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $C_{s,e}^{**}$ if the following condition holds true:*

$$C_{s,e}^{**} = \left\{ f \in \mathcal{S} : \frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec p(z) = 1 + z - \frac{z^3}{3}; z \in \mathbb{D} \right\}. \tag{1.3}$$

Remark 1. *First, we have to show that the function $p(z) := 1 + z - \frac{z^3}{3}$ is correctly chosen because $p'(z) = 1 - z^2$, hence $p'(0) = 1 \neq 0$. Also, it's easy to see that*

$$k(z) = \Re\frac{zp'(z)}{p(z) - p(0)} = \Re\frac{zp'(z)}{p(z) - 1} = \Re\frac{3z^2 - 3}{z^2 - 3} > 0 \quad (z \in \mathbb{D}).$$

This is shown in the Figure (1). Using this fact together with $p'(0) = 1 \neq 0$ it follows that $p(z) = 1 + z - \frac{z^3}{3}$ is also a starlike (univalent) function in $\mathbb{D}$ and because $p(\overline{z}) = \overline{p(z)}$, $z \in \mathbb{D}$, the domain $p(\mathbb{D})$ is symmetric with respect to the real axis (See Figure (2)).

Remark 2. *Next, to show that $C_{s,e}^{**}$ is non-empty for some appropriate choices of q. Let us consider the functions*

$$q(z) := z + 0.15z^2 \in \mathcal{A}$$

From the figure (3), $q(\mathbb{D}) \subset p(\mathbb{D})$ with the univalency of $p(z) = 1 + z - \frac{z^3}{3}$ seen previously for $C_{s,e}^{**}$ class, leads to $q(z) := z + 0.15z^2 \prec p(z)$.

It may be noted that the functions f_1, f_2, f_3 and f_4 defined by

$$f_1(z) = z + \frac{1}{4}z^2 - \frac{1}{48}z^4 + \cdots, \tag{1.4}$$

$$f_2(z) = z + \frac{1}{6}z^3 + \cdots, \tag{1.5}$$

$$f_3(z) = z + \frac{1}{16}z^4 + \cdots \tag{1.6}$$

$$f_4(z) = z + \frac{1}{20}z^5 + \cdots \tag{1.7}$$

are belong to the subclass $C_{s,e}^{**}$. Hence the class $C_{s,e}^{**}$ is non-empty.

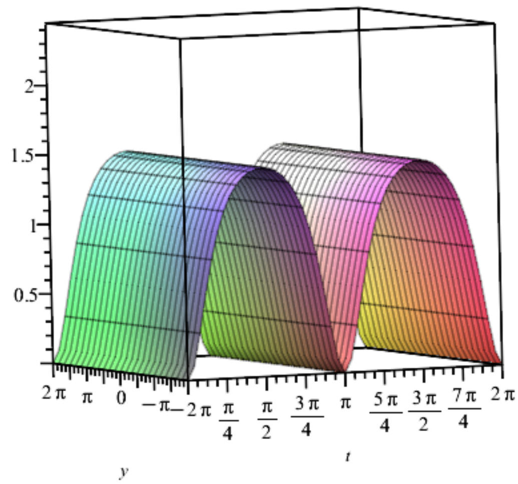

Figure 1: Univalency of $k(z)$

Definition 1.2. *A function $f \in \mathcal{A}$ given by (1.1) is said to be the member of the class $H(\phi)$ if the following subordination condition holds:*

$$\frac{2(zf'(z))'}{f'(z)+f'(-z)} \prec \phi(z) = \frac{1+z}{1-z}. \tag{1.8}$$

For each functions $f \in \mathcal{S}$ defined on $\mathbb{D}$, the famous one-quarter theorem of Koebe (see [7]) asserts that its inverse f^{-1} exists at least on a disk of radius $\frac{1}{4}$. If $f \in \mathcal{S}$, the function F which is the inverse of f has expression given by

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \quad (|w| < \frac{1}{4}). \tag{1.9}$$

Some of the initial coefficients of F are given by

$$\begin{aligned} A_2 = -a_2, \qquad A_3 = 2a_2^2 - a_3, \qquad A_4 = -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.10}$$

The inverse functions are studied by several authors in various subclasses of analytic functions. (see, for details [1] and reference therein). Recently, Sim and Thomas [23] obtained sharp upper and lower bounds on the difference of the moduli of successive inverse coefficients for the subclasses of univalent functions.

For each function $f \in \mathcal{S}$, the logarithmic coefficients γ_n $(n \in \mathbb{N})$ are defined as:

$$F_f(z) = log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} R_n z^n \quad (z \in \mathbb{D}). \tag{1.11}$$

The number $R_n := R_n(f) \quad (n \in \mathbb{N})$ are called the logarithmic coefficients of f. Therefore, some of the

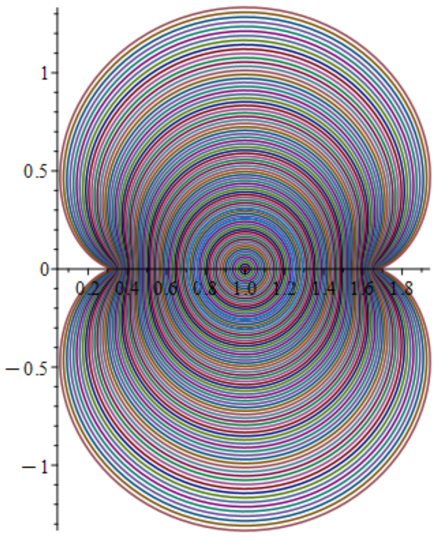

Figure 2: The images of $p\left(e^{it}\right)$, $t \in [0, 2\pi)$

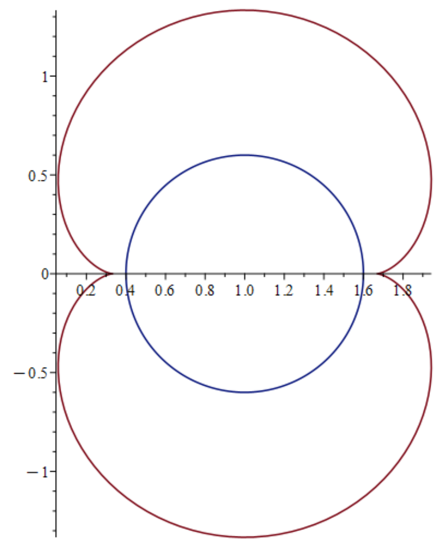

Figure 3: The images of $p\left(e^{it}\right)$ (red color) and $\mathfrak{q}\left(e^{it}\right)$ (blue color), $t \in [0, 2\pi)$

initial logarithmic coefficients are given as:

$$R_1 = \frac{a_2}{2}, \qquad R_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right), \qquad R_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right),$$
$$R_4 = \frac{1}{2}\left(a_5 - a_2 a_4 - \frac{1}{2}a_3^2 + a_2^2 a_3 - \frac{1}{4}a_2^4\right). \tag{1.12}$$

The importance of logarithmic coefficients R_n in the context of Bieberbach conjecture was pointed out by Milin in his conjecture (see [17]). For $f \in \mathcal{S}$ and $n \geq 2$, Milin has conjectured that

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|R_k|^2 - \frac{1}{k}\right) \leq 0,$$

which led De Branges by proving this conjecture to the proof of Bieberbach conjecture [5]. Very recently, the upper bounds of logarithmic coefficients of functions f in some subclasses of the class $\mathcal{S}$ have been obtained by various authors [1,29].

The Hankel determinants $H_{q,n}(f)$ of Taylor's coefficients of functions $f \in \mathcal{A}$ represented by (1.1) is defined for $q,\ n \in \mathbb{N}$ as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}, \end{vmatrix} (a_1 := 1).$$

By specializing the different values for q and n we can obtain Hankel determinants of different orders as follows:

1. For $q = 2$ and $n = 1$ we have

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

that's a special case of the well-known Fekete-Szegő functional [8]. For various subclasses of $\mathcal{A}$, the maximum value of $|H_{2,1}(f)|$ has been obtained by different authors (see, for example [25,26]).

2. For $q = n = 2$ we get

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

known as the second Hankel determinant. The upper bound for $|H_{2,2}(f)|$ has been investigated by several authors (see [16,18,19]).

3. For $q = 3$ and $n = 1$,

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_5\left(a_3 - a_2^2\right) - a_4\left(a_4 - a_2a_3\right) + a_3\left(a_2a_4 - a_3^2\right), \tag{1.13}$$

well-known as the third Hankel determinant. Babalola [3] was the first who studied this determinant for the subclasses of $\mathcal{S}$, namely $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$. Later, Zaprawa [27] enhanced Babalola's results and proved that

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{if} \quad f \in \mathcal{S}^*, \\ \dfrac{49}{540}, & \text{if} \quad h \in \mathcal{S}, \\ \dfrac{41}{60}, & \text{if} \quad h \in \mathcal{R}. \end{cases}$$

Further, Kwon et al. [13] improved the Zaprawa inequality for $h \in \mathcal{S}^*$ by achieving $|H_{3,1}(f)| \leq \dfrac{8}{9}$ and Zaprawa et al. [28] refined this bound even further by establishing that $|H_{3,1}(f)| \leq \dfrac{5}{9}$ for $h \in \mathcal{S}^*$, and none of these bounds are sharp. Finally, Kowalczyk et al. [12] and Lecko et al. [14] achieved the following sharp bounds of $|H_{3,1}(f)|$ for the set $\mathcal{K}$ and $\mathcal{S}^*\left(\dfrac{1}{2}\right)$, respectively, that are

$$|H_{3,1}(f)| \leq \begin{cases} \dfrac{4}{135}, & \text{if} \quad h \in \mathcal{K}, \\ \dfrac{1}{9}, & \text{if} \quad h \in \mathcal{S}^*\left(\dfrac{1}{2}\right). \end{cases}$$

Many research articles have been published in the recent years for estimating upper bounds for third order Hankel determinant, see for example, [2,24,27].

2. Preliminaries

We need the following lemmas in order to investigate our results.

Lemma 2.1. *(see [7]). If $w \in B_0$ is of the form (1.2) then the sharp estimate $|c_n| \leq 1$ holds for $n \geq 1$.*

Lemma 2.2. *(see [9]). Let $w \in B_0$ be of the form (1.2). Then for all $\lambda \in \mathbb{C}$, we have*

$$|c_2 + \lambda c_1^2| \leq \max\{1, |\lambda|\}.$$

Lemma 2.3. *(see [20]). Let $w \in B_0$ be a Schwarz function of the form (1.2). Then for any real numbers μ and ν such that*

$$|c_3 + \mu c_1c_2 + \nu c_1^3| \leq 1,$$

where $(\mu, \nu) \in D_1 \cup D_2$ with

$$D_1 = \{(\mu, \nu) \in \mathbb{R}^2, |\mu| \leq \frac{1}{2}, -1 \leq \nu \leq 1\},$$

and

$$D_2 = \left\{(\mu, \nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2,\ \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\right\}.$$

Lemma 2.4. *(see [6]). Let $w \in B_0$ be a Schwarz function of the form (1.2). Then*

$$|c_2| \le 1 - |c_1|^2, \qquad |c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|},$$

$$|c_4| \le 1 - |c_1|^2 - |c_2|^2, \qquad |c_5| \le 1 - |c_1|^2 - |c_2|^2 - \frac{|c_3|^2}{1+|c_1|}.$$

3. Initial coefficients estimates for the class $C_{s,e}^{**}$

In this section, we explore the sharp upper bounds for the first four initial coefficients, as well as the Fekete-Szegö functional $|a_3 - \mu a_2^2|$, for our defined class $C_{s,e}^{**}$. Our initial focus is on determining the upper bounds for these first four coefficients within the class $C_{s,e}^{**}$.

Theorem 3.1. *Let the function $f \in \mathcal{A}$ be of the form (1.1) belongs to the class $C_{s,e}^{**}$. Then*

$$|a_2| \le \frac{1}{4}, \quad |a_3| \le \frac{1}{6}, \quad |a_4| \le \frac{1}{16}, \quad |a_5| \le \frac{1}{20}. \tag{3.1}$$

All the coefficient estimates are sharp.

Proof: If the function $f \in \mathcal{A}$ of the form (1.1) belongs to the class $C_{s,e}^{**}$, then by Definition 1.1 there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{D})$ such that

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} = 1 + w(z) - \frac{w(z)^3}{3}. \tag{3.2}$$

From (1.1), it follows that

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} = 1 + 4a_2 z + 6a_3 z^2 + (-12a_2a_3 + 16a_4)\, z^3 + \left(-18a_3^2 + 20a_5\right) z^4 + \cdots \tag{3.3}$$

Since $w(z)$ is of the form (1.2), a simple routine calculation gives

$$1 + w(z) - \frac{w(z)^3}{3} = 1 + c_1 z + c_2 z^2 + \left(c_3 - \frac{c_1^3}{3}\right) z^3 + \left(-c_1^2 c_2 + c_4\right) z^4 + \cdots \tag{3.4}$$

Comparing the coefficients of (3.3) and (3.4) we get

$$a_2 = \frac{c_1}{4}, \tag{3.5}$$

$$a_3 = \frac{c_2}{6}, \tag{3.6}$$

$$a_4 = -\frac{1}{48}c_1^3 + \frac{1}{32}c_1c_2 + \frac{1}{16}c_3 = \frac{1}{16}\left(c_3 + \frac{1}{2}c_1c_2 - \frac{1}{3}c_1^3\right), \tag{3.7}$$

$$a_5 = -\frac{1}{20}c_1^2c_2 + \frac{1}{40}c_2^2 + \frac{1}{20}c_4. \tag{3.8}$$

Taking modulus on both sides of a_2, a_3 and applying Lemma 2.1 we get our desired estimation. Now taking modulus on both sides of (3.7) and then applying Lemma 2.3 in the resulting relation we get our desired estimate. Now taking modulus on both sides of (3.8) and then applying triangle inequality followed by application of Lemma 2.4 we get

$$|a_5| \le \frac{1}{20}|c_1|^2|c_2| + \frac{1}{40}|c_2|^2 + \frac{1}{20}(1 - |c_1|^2 - |c_2|^2) = \frac{1}{20}p^2q + \frac{1}{40}q^2 + \frac{1}{20}(1 - p^2 - q^2) = K_1(p,q)\,(say)$$

where $|c_1| = p$, $|c_2| = q$. The shape of the region of variability of (p,q) as a consequence of the Schwarz-Pick Lemma coincides with the set

$$\Lambda = \{(p,q) : 0 \le p \le 1, 0 \le q \le 1 - p^2\}. \tag{3.9}$$

Our aim is to determine the maximum value of $K_1(p,q)$ in the region Λ. The critical points of $K_1(p,q)$ satisfies the conditions

$$\frac{\partial K_1}{\partial p}=\frac{1}{10}pq-\frac{1}{10}p=0, \quad \frac{\partial K_1}{\partial q}=\frac{p^2}{20}-\frac{q}{20}=0.$$

Solving the above simultaneous equations we obtain (0,0) and $(1,1)$ are the critical points. But no one lies inside the interior of the region Λ. Now on the boundary of Λ, computation shows that

$$K_1(0,q)=\frac{1}{20}-\frac{q^2}{40}\leq\frac{1}{20} \qquad \textit{for} \qquad 0\leq q\leq 1,$$

$$K_1(p,0)=\frac{1}{20}-\frac{p^2}{20}=M_1(p)\leq M_1(0)=\frac{1}{20} \qquad \textit{for} \qquad 0\leq p\leq 1,$$

$$K_1\left(p,1-p^2\right)=-\frac{3}{40}p^4+\frac{1}{20}p^2+\frac{1}{40}=M_2(p)\leq M_2\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{30}.$$

Therefore, we obtain the best possible estimate of $|a_5|$ as $\frac{1}{20}$. So our claim is established. In this context, the coefficients a_2,a_3,a_4,a_5 are sharp for the Schwarz functions $w(z)$ as z,z^2,z^3,z^4 respectively and the corresponding extremal functions are provided in equations (1.4), (1.5), (1.6), and (1.7) respectively. This completes the proof of Theorem 3.1. □

Next theorem gives the bounds of Fekete-Szegö functional for the class $C^{**}_{s,e}$.

Theorem 3.2. *If $f\in C^{**}_{s,e}$ has the form (1.1), then for any complex number μ we have*

$$|a_3-\mu a_2^2|\leq\frac{1}{6}\ max\ \left\{1;\left|\frac{3\mu}{8}\right|\right\}. \tag{3.10}$$

Proof: Now, making use of (3.5) and (3.6) we get

$$a_3-\mu a_2^2=\frac{1}{6}\left[c_2-\left(\frac{3}{8}\mu\right)c_1^2\right]. \tag{3.11}$$

The estimation (3.10) follows from (3.11) by virtue of Lemma 2.2. This inequality is sharp for the Schwarz function $w(z)=z^2$ and the corresponding extremal function presented in (1.5). The proof of Theorem 3.2 is thus completed. □

Letting $\mu=1$ in the Theorem 3.2 gives the following result in the form of corollary:

Corollary 3.1. *If $f\in C^{**}_{s,e}$ has the form (1.1), then*

$$\left|a_3-a_2^2\right|=|H_{2,1}(f)|\leq\frac{1}{6}.$$

The equality holds for the Schwarz function $w(z)=z^2$ and the extremal function is given in (1.5).

4. Hankel determinants bounds for the class $C^{**}_{s,e}$

In this section, we investigate the upper bounds of Hankel determinants of order two and three for the functions that belong to the class $C^{**}_{s,e}$.

Theorem 4.1. *If the function $f\in\mathcal{A}$ given by (1.1) belongs to the class $C^{**}_{s,e}$, then*

$$|a_2a_4-a_3^2|\leq\frac{1}{36}. \tag{4.1}$$

This inequality is sharp.

Proof: From (3.5), (3.6) and (3.7) it follows that

$$a_2a_4-a_3^2=-\frac{1}{192}c_1^4+\frac{1}{128}c_1^2c_2+\frac{1}{64}c_1c_3-\frac{1}{36}c_2^2. \tag{4.2}$$

Now taking module on both sides of (4.2), applying triangle inequality and followed by Lemma 2.4 we get

$$\begin{aligned}|a_2a_4-a_3^2| &\leq \frac{1}{192}|c_1|^4+\frac{1}{128}|c_1|^2|c_2|+\frac{1}{64}|c_1||c_3|+\frac{1}{36}|c_2|^2\\ &\leq \frac{1}{192}|c_1|^4+\frac{1}{128}|c_1|^2|c_2|+\frac{1}{64}|c_1|\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right)+\frac{1}{36}|c_2|^2\\ &= \frac{1}{192}p^4+\frac{1}{128}p^2q+\frac{1}{64}p\left(1-p^2-\frac{q^2}{1+p}\right)+\frac{1}{36}q^2=K_2(p,q) \quad (say).\end{aligned}$$

where $|c_1|=p$, $|c_2|=q$. The critical points of $K_2(p,q)$ satisfies the conditions

$$\frac{\partial K_2}{\partial p}=\frac{p^3}{48}+\frac{pq}{64}+\frac{1}{64}-\frac{3p^2}{64}-\frac{q^2}{64\,(p+1)^2}=0,$$

and

$$\frac{\partial K_2}{\partial q}=\frac{p^2}{128}-\frac{pq}{32\,(p+1)}+\frac{q}{18}=0.$$

Solving the above simultaneous equations we obtain there are no critical points in the interior of Λ. Now on the boundary of Λ, computation shows that

$$K_2(0,q)=\frac{q^2}{36}\leq\frac{1}{36} \qquad for \qquad 0\leq q\leq 1,$$

$$K_2(p,0)=\frac{p^4}{192}+\frac{p\left(-p^2+1\right)}{64}\leq M_2\left(\frac{20765}{29906}\right)=\frac{1063}{155655} \qquad for \qquad 0\leq p\leq 1,$$

$$K_2\left(p,1-p^2\right)=\frac{11}{1152}p^4-\frac{37}{1152}p^2+\frac{1}{36}=M_2(p)\leq M_2(0)=\frac{1}{36}.$$

We have established our claim that $|a_2a_4-a_3^2|\leq\frac{1}{36}$. The inequality is sharp for the extremal function provided in equation (1.5). This completes the proof of Theorem 4.1. □

Theorem 4.2. *If the function $f\in\mathcal{A}$ given by (1.1) belongs to the class $C_{s,e}^{**}$, then*

$$|a_4-a_2a_3|\leq\frac{1}{16}.$$

This inequality attains its sharp bound for the extremal function stated in (1.6)

Proof: If $f\in C_{s,e}^{**}$ has the form (1.1), using the relations (3.5), (3.6) and (3.7) we get

$$|a_4-a_2a_3|=\left|-\frac{1}{96}c_1c_2-\frac{1}{48}c_1^3+\frac{1}{16}c_3\right|=\frac{1}{16}\left|c_3-\frac{1}{6}c_1c_2-\frac{1}{3}c_1^3\right|. \tag{4.3}$$

Applying Lemma 2.3 with $\mu=\frac{-1}{6}$ and $\nu=\frac{-1}{3}$ provides the necessary estimate. This inequality is sharp for the Schwarz function $w(z)=z^3$ and the extremal function is described in (1.6). This completes the proof of Theorem 4.2.

□

Theorem 4.3. *Let the function $f\in\mathcal{A}$ given by (1.1) be in the class $C_{s,e}^{**}$. Then,*

$$|H_{3,1}(f)|\leq\frac{4459}{720739}=0.0061867056.$$

Proof: If $f \in C^{**}_{s,e}$ has the form (1.1), using the relations (3.5)- (3.8) we get

$$\begin{aligned}\left|H_{3,1}(f)\right| &= |a_5\left(-a_2^2+a_3\right) - a_4\left(-a_2a_3+a_4\right) + a_3\left(a_2a_4-a_3^2\right)| \\ &= \left|\frac{31}{11520}c_1^4c_2 - \frac{127}{15360}c_1^2c_2^2 - \frac{1}{2160}c_2^3 - \frac{1}{320}c_4c_1^2 + \frac{1}{120}c_4c_2 - \frac{1}{2304}c_1^6 + \frac{1}{384}c_1^3c_3 + \frac{1}{768}c_1c_2c_3 - \frac{1}{256}c_3^2\right|. \end{aligned} \tag{4.4}$$

Now rearranging the terms and applying triangle inequality in (4.4) we get

$$\left|H_{3,1}(f)\right| \le \left|\frac{31}{11520}c_1^4c_2 - \frac{127}{15360}c_1^2c_2^2 - \frac{1}{2160}c_2^3 - \frac{1}{320}c_4c_1^2 + \frac{1}{120}c_4c_2 - \frac{1}{2304}c_1^6\right| + \frac{|c_3|}{256}\left|c_3 - \frac{256}{768}c_1c_2 - \frac{256}{384}c_1^3\right|. \tag{4.5}$$

Applying triangle inequality and Lemma 2.4 in 1st part and application of Lemma 2.3 with $\mu = -\frac{256}{768}$ and $\nu = -\frac{256}{384}$ in 2nd part of (4.5) we get

$$\begin{aligned}|H_{3,1}(f)| \le{}& \frac{31}{11520}|c_1|^4|c_2| + \frac{127}{15360}|c_1|^2|c_2|^2 + \frac{1}{2160}|c_2|^3 + \left(\frac{1}{320}|c_1|^2 + \frac{1}{120}|c_2|\right)(1-|c_1|^2-|c_2|^2) \\ &+ \frac{1}{2304}|c_1|^6 + \frac{1}{256}\left(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|}\right) \\ ={}& \frac{31}{11520}p^4q + \frac{127}{15360}p^2q^2 + \frac{1}{2160}q^3 + \left(\frac{1}{320}p^2+\frac{1}{120}q\right)(1-p^2-q^2) + \frac{1}{2304}p^6 \\ &+ \frac{1}{256}\left(1-p^2-\frac{q^2}{1+p}\right) = K_3(p,q) \quad (say). \end{aligned}$$

where $|c_1| = p$, $|c_2| = q$. The critical points of $K_3(p,q)$ satisfies the conditions

$$\frac{\partial K_3}{\partial p} = \frac{31p^3q}{2880} + \frac{127p\,q^2}{7680} + \frac{p\left(-p^2-q^2+1\right)}{160} - 2\left(\frac{p^2}{320}+\frac{q}{120}\right)p + \frac{p^5}{384} - \frac{p}{128} + \frac{q^2}{256\left(p+1\right)^2} = 0,$$

and

$$\frac{\partial K_3}{\partial q} = \frac{31p^4}{11520} + \frac{127p^2q}{7680} - \frac{q^2}{144} - \frac{p^2}{120} + \frac{1}{120} - 2\left(\frac{p^2}{320}+\frac{q}{120}\right)q - \frac{q}{128\left(p+1\right)} = 0.$$

Solving the above simultaneous equations we obtain $\left(\frac{2347}{24253}, \frac{3263}{7079}\right)$ is the only critical point. Therefore,

$$K_3(p,q) \le K_3\left(\frac{2347}{24253}, \frac{3263}{7079}\right) = \frac{4459}{720739} = 0.0061867056.$$

Now on the boundary of Λ, computation shows that

$$K_3(0,q) = \frac{1}{256} + \frac{q^3}{2160} + \frac{q\left(-q^2+1\right)}{120} - \frac{q^2}{256} \le \frac{1}{36} \qquad for \qquad 0 \le q \le 1,$$

$$K_3(p,0) = \frac{1}{256} + \frac{p^2\left(-p^2+1\right)}{320} + \frac{p^6}{2304} - \frac{p^2}{256} \le M_2\left(\frac{20765}{29906}\right) = \frac{1063}{155655} \qquad for \qquad 0 \le p \le 1,$$

$$\begin{aligned}K_3\left(p,1-p^2\right) &= \frac{1487}{138240}p^6 - \frac{599}{23040}p^4 - \frac{1}{256}p^3 + \frac{701}{46080}p^2 + \frac{1}{256}p + \frac{1}{2160} = M_2(p) \\ &\le M_2\left(\frac{32238}{52925}\right) = \frac{2521}{551123}. \end{aligned}$$

Thus we obtain the best bound of $|H_{3,1}(f)|$ to be $\frac{4459}{720739} = 0.0061867056$ So our claim is established.

□

5. Inverse coefficient bounds for the class $C_{s,e}^{**}$

In this section, we will estimate the upper bounds of the first four coefficients belonging to the class $C_{s,e}^{**}$.

Theorem 5.1. *If $f \in C_{s,e}^{**}$ is given by (1.1) and its inverse f^{-1} has the form (1.9), then*

$$|A_2| \leq \frac{1}{4}, \quad |A_3| \leq \frac{1}{6}, \quad |A_4| \leq \frac{23\sqrt{4830}}{15120}, \quad |A_5| \leq \frac{3999}{45184}. \tag{5.1}$$

The first two initial bounds are sharp.

Proof: Substitute the values of $a_2,\ a_3,\ a_4$ and a_5 from (3.5)-(3.8) into (1.10) we get

$$A_2 = -\frac{c_1}{4}, \tag{5.2}$$

$$A_3 = \frac{c_1^2}{8} - \frac{c_2}{6}, \tag{5.3}$$

$$A_4 = -\frac{11}{192}c_1^3 + \frac{17}{96}c_1c_2 - \frac{1}{16}c_3, \tag{5.4}$$

$$A_5 = \frac{3}{128}c_1^4 - \frac{39}{320}c_1^2c_2 + \frac{3}{32}c_1c_3 + \frac{7}{120}c_2^2 - \frac{1}{20}c_4. \tag{5.5}$$

The bounds of $|A_2|$ and $|A_3|$ can respectively obtain by virtue of Lemma 2.1 and Lemma 2.2. Now, taking modulus and applying triangle inequality in the relation (5.4) and followed by Lemma 2.4 we get

$$\begin{aligned}|A_4| &\leq \frac{11}{192}|c_1|^3 + \frac{17}{96}|c_1||c_2| + \frac{1}{16}|c_3| \leq \frac{11}{192}|c_1|^3 + \frac{17}{96}|c_1||c_2| + \frac{1}{16}\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right)\\ &= \frac{11}{192}p^3 + \frac{17}{96}pq + \frac{1}{16}\left(1 - p^2 - \frac{q^2}{1+p}\right) = K_4(p,q) \quad (say).\end{aligned}$$

where $|c_1| = p$, $|c_2| = q$. Assume that $K_4(p,q)$ has a maximum value at an interior point of Λ. Differentiating $K_4(p,q)$ with respect to p and q we get

$$\frac{\partial K_4}{\partial p} = -\frac{p}{8} + \frac{q^2}{16\,(p+1)^2} + \frac{17q}{96} + \frac{11p^2}{64},$$

and

$$\frac{\partial K_4}{\partial q} = -\frac{q}{8\,(p+1)} + \frac{17p}{96}.$$

Solving $\frac{\partial K_4}{\partial p} = 0$ and $\frac{\partial K_4}{\partial q} = 0$ we get $p = 0$ and $q = 0$. There are no solutions of $K_4(p,q)$ inside the interior of Λ. Therefore, it is not possible for the function $K_4(p,q)$ to attain maximum value within the region Λ. On the boundary of Λ, we get

$$K_4(0,q) = \frac{1}{16} - \frac{q^2}{16} \leq \frac{1}{16} \qquad for\ \ 0 \leq q \leq 1,$$

$$K_4(p,0) = \frac{1}{16} - \frac{1}{16}p^2 + \frac{11}{192}p^3 = M_3(p) \leq M_3(0) = \frac{1}{16} \qquad for\ \ 0 \leq p \leq 1,$$

$$K_4\left(p, 1-p^2\right) = -\frac{35}{192}p^3 + \frac{23}{96}p = M_4(p) \leq M_4\left(\frac{\sqrt{4830}}{105}\right) = \frac{23\sqrt{4830}}{15120}.$$

From above we observe that $|A_4| \leq \frac{23\sqrt{4830}}{15120} = 0.1057181635$. Thus, we get the third inequality. Now taking modulus, applying triangle inequality and followed by application of Lemma 2.4 in (5.5) we get

$$\begin{aligned}|K_5| &\leq \frac{3}{128}|c_1|^4 + \frac{39}{320}|c_1|^2|c_2| + \frac{3}{32}|c_1|\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \frac{7}{120}|c_2|^2 + \frac{1}{20}(1 - |c_1|^2 - |c_2|^2)\\ &= \frac{3}{128}p^4 + \frac{39}{320}p^2q + \frac{3}{32}p\left(1 - p^2 - \frac{q^2}{1+p}\right) + \frac{7}{120}q^2 + \frac{1}{20}(1 - p^2 - q^2) = K_5(p,q) \quad (say).\end{aligned}$$

where $|c_1| = p$, $|c_2| = q$. Assume that $K_5(p,q)$ has a maximum value at an interior point of Λ. Differentiating $K_5(p,q)$ with respect to p and q we get

$$\frac{\partial K_5}{\partial p} = \frac{3p^3}{32} + \frac{39pq}{160} + \frac{3}{32} - \frac{3p^2}{32} - \frac{3q^2}{32\,(p+1)} + \frac{3p\left(-2p + \frac{q^2}{(p+1)^2}\right)}{32} - \frac{p}{10},$$

and

$$\frac{\partial K_5}{\partial q} = \frac{39p^2}{320} - \frac{3pq}{16\,(p+1)} + \frac{q}{60}.$$

It is not difficult to prove that $(0,0)$ is the only the critical point of K_5 in Λ. Hence it is enough to determine the greatest value of K_5 on the boundary of Λ. On the boundary of Λ, we get

$$K_5(0,q) = \frac{1}{20} + \frac{q^2}{120} \leq \frac{7}{120},$$

$$K_5(p,0) = \frac{3}{128}p^4 - \frac{3}{32}p^3 + \frac{3}{32}p - \frac{1}{20}p^2 + \frac{1}{20} = M_5(p) \leq M_5\left(\frac{20575}{45606}\right) = \frac{6538}{87781},$$

$$K_5\left(p, 1-p^2\right) = -\frac{353}{1920}p^4 + \frac{143}{960}p^2 + \frac{7}{120} = M_6(p) \leq M_5\left(\frac{16253}{25536}\right) = \frac{3999}{45184}.$$

so $|A_5| \leq \frac{3999}{45184}$. Thus, we get the fourth inequality. The first two coefficients inequality are sharp and the corresponding extremal function given in (1.4) and (1.5). This completes the proof of Theorem 5.1. □

6. Logarithmic Coefficients for the class $C^{**}_{s,e}$

In this section, we determine upper bounds estimates for the first four logarithmic coefficients of the functions f that belong to the class $C^{**}_{s,e}$.

Theorem 6.1. *If $f \in C^{**}_{s,e}$ given by (1.1), then*

$$|R_1| \leq \frac{1}{8}, \quad |R_2| \leq \frac{1}{12}, \quad |R_3| \leq \frac{1}{32}, \quad |R_4| \leq \frac{5923}{231479}.$$

The first three coefficient estimates are sharp.

Proof: Substituting the values of a_2,a_3,a_4 and a_5 from (3.5)-(3.8) in the relation (1.12) gives

$$R_1 = \frac{c_1}{8}, \tag{6.1}$$

$$R_2 = -\frac{c_1^2}{64} + \frac{c_2}{12}, \tag{6.2}$$

$$R_3 = -\frac{1}{128}c_1^3 - \frac{1}{192}c_1c_2 + \frac{1}{32}c_3, \tag{6.3}$$

$$R_4 = \frac{13}{6144}c_1^4 - \frac{91}{3840}c_1^2c_2 - \frac{1}{128}c_1c_3 + \frac{1}{180}c_2^2 + \frac{1}{40}c_4. \tag{6.4}$$

Applications of Lemma 2.1 to the relation (6.1) and Lemma 2.2 to (6.2) give the required bounds for $|R_1|$ and $|R_2|$. By using Lemma 2.3 with $\mu = \frac{-1}{6}$ and $\nu = \frac{-1}{4}$, the bounds for $|R_3|$ can be obtained. Now taking modulus on both sides of (6.4) and followed by application of triangle inequality yield

$$\begin{aligned} |R_4| &\leq \frac{13}{6144}|c_1|^4 + \frac{91}{3840}|c_1|^2|c_2| + \frac{1}{128}|c_1||c_3| + \frac{1}{180}|c_2|^2 + \frac{1}{40}|c_4| \\ &\leq \frac{13}{6144}|c_1|^4 + \frac{91}{3840}|c_1|^2|c_2| + \frac{|c_1|}{128}\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \frac{|c_2|^2}{180} + \frac{1}{40}(1 - |c_1|^2 - |c_2|^2) \\ &= \frac{13}{6144}p^4 + \frac{91}{3840}p^2q + \frac{p}{128}\left(1 - p^2 - \frac{q^2}{1+p}\right) + \frac{q^2}{180} + \frac{1}{40}(1 - p^2 - q^2) = K_6(p,q) \quad (say) \end{aligned}$$

where $|c_1| = p$, $|c_2| = q$. Now we assume that $K_6(p,q)$ has a maximum value at an interior point of Λ. Differentiating $K_6(p,q)$ with respect to p and q we get

$$\frac{\partial K_6}{\partial p} = \frac{13p^3}{1536} + \frac{91pq}{1920} + \frac{1}{128} - \frac{p^2}{128} - \frac{q^2}{128\,(p+1)} + \frac{p\left(-2p + \frac{q^2}{(p+1)^2}\right)}{128} - \frac{p}{20}$$

and

$$\frac{\partial K_6}{\partial q} = \frac{91p^2}{3840} - \frac{pq}{64\,(p+1)} - \frac{7q}{180}.$$

Solving $\frac{\partial K_6}{\partial p} = 0$ and $\frac{\partial K_6}{\partial q} = 0$ we obtain $\left(\frac{5465}{36859}, \frac{3289}{258257}\right)$ is the critical points such that

$$K_6(p,q) \le K_6\left(\frac{5465}{36859}, \frac{3289}{258257}\right) = \frac{5923}{231479} = 0.02558763429.$$

On the boundary of Λ, we get

$$K_6(0,q) = \frac{1}{40} - \frac{7q^2}{360} \le \frac{1}{40},$$

$$K_6(p,0) = \frac{13}{6144}p^4 + \frac{1}{40} - \frac{1}{128}p^3 + \frac{1}{128}p - \frac{1}{40}p^2 = K_6(p) \le K_6\left(\frac{7379}{50301}\right) = \frac{915}{35764},$$

$$K_6\left(p, 1-p^2\right) = -\frac{4501}{92160}p^4 + \frac{523}{11520}p^2 + \frac{1}{180} = K_6(p) \le K_6\left(\frac{26167}{38382}\right) = \frac{9779}{607164}.$$

So $|R_4| \le \frac{5923}{231479} = 0.02558763429$. Thus, we get the fourth inequality. The first three coefficients estimates are sharp and the corresponding extremal function are given by (1.4),(1.5) and (1.6) respectively. The proof of Theorem 6.1 is thus completed. □

7. The Zalcman functional for the class $C^{**}_{s,e}$

Lawrence Zalcman conjectured that the coefficients of every univalent functions $f \in \mathcal{S}$ given by (1.1) must satisfy the inequality

$$|a_n^2 - a_{2n-1}| \le (n-1)^2 \quad (n \ge 2). \tag{7.1}$$

and the equality holds only for the Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its rotations. The Zalcman functional has been studied by many researchers (see [4,10,15]).

Corollary 3.1 demonstrated that the above inequality (7.1) holds good for the class $C^{**}_{s,e}$ for $n = 2$. Now we prove the inequality (7.1) holds for $n = 3$.

Theorem 7.1. *If $f \in C^{**}_{s,e}$ has the form (1.1), then*

$$|a_3^2 - a_5| \le \frac{1}{20}. \tag{7.2}$$

This inequality is sharp.

Proof: Let $f \in C^{**}_{s,e}$. Then from the relations (3.6) and (3.8) we obtain

$$|a_3^2 - a_5| = \left|\frac{1}{360}c_2^2 + \frac{1}{20}c_2c_1^2 - \frac{1}{20}c_4\right|. \tag{7.3}$$

Applying triangle inequality in (7.3) and followed by application of Lemma 2.4 we get

$$\begin{aligned} |a_3^2 - a_5| &\le \frac{1}{360}|c_2|^2 + \frac{1}{20}|c_2||c_1|^2 + \frac{1}{20}(1 - |c_1|^2 - |c_2|^2) \\ &= \frac{1}{360}q^2 + \frac{1}{20}p^2q + \frac{1}{20}(1 - p^2 - q^2) = K_7(p,q)(say) \end{aligned}$$

where $|c_1| = p$, $|c_2| = q$. Now we assume that $K_7(p, q)$ has a maximum value at an interior point of Λ. Differentiating $K_7(p, q)$ with respect to p and q we get

$$\frac{\partial K_7}{\partial p} = -\frac{1}{10}p + \frac{1}{10}pq,$$

and

$$\frac{\partial K_7}{\partial q} = -\frac{17q}{180} + \frac{p^2}{20}.$$

Solving simultaneous equations $\frac{\partial K_7}{\partial p} = 0$ and $\frac{\partial K_7}{\partial q} = 0$ we obtain there is no critical point inside Λ.

On the boundary of Λ, we get

$$K_7(0, q) = \frac{1}{20} - \frac{17q^2}{360} \leq \frac{1}{20},$$

$$K_7(p, 0) = \frac{1}{20} - \frac{p^2}{20} \leq \frac{1}{20},$$

$$K_7\left(p, 1 - p^2\right) = \frac{1}{360} + \frac{17p^2}{180} - \frac{7p^4}{72} = K_6(p) \leq K_6\left(\frac{44320}{63593}\right) = \frac{9}{350}.$$

Thus $|a_3^2 - a_5| \leq \frac{1}{20}$. The estimation is sharp and the corresponding extremal function is given in (1.7). The proof of Theorem 7.1 is thus completed. □

8. Krushkal inequality for the class $C_{s,e}^{**}$

In this section, we will prove the well-known inequality:

$$|a_n^p - a_2^{p(n-1)}| \leq 2^{p(n-1)-n^p}, \tag{8.1}$$

for particular pair of values of $n = 4, p = 1$. We will investigate smaller upper bounds for the class $C_{s,e}^{**}$. The inequality (8.1) was originally introduced and proved by Krushkal for the class of normalized univalent function $f \in \mathcal{S}$ and integers $n \geq 3$, $p \geq 1$.

The following theorem gives upper bounds of l.h.s of (8.1) when $n = 4$ and $p = 1$ for the class $C_{s,e}^{**}$.

Theorem 8.1. *If the function $f \in C_{s,e}^{**}$ has the form (1.1), then*

$$|a_4 - a_2^3| \leq \frac{1}{16}.$$

This inequality attains its sharp bound.

Proof: Let the function $f \in \mathcal{A}$ be a member of the class $C_{s,e}^{**}$. Then from (3.5) and (3.7) we have

$$|a_4 - a_2^3| = \left|\frac{1}{16}c_3 + \frac{1}{32}c_1c_2 - \frac{7}{192}c_1^3\right| = \frac{1}{16}\left|c_3 + \frac{1}{2}c_1c_2 - \frac{7}{12}c_1^3\right|. \tag{8.2}$$

Now applying Lemma 2.3 in (8.2) with $\mu = \frac{1}{2}$ and $\nu = -\frac{7}{12}$ we get our desired result. This inequality is sharp for the extremal function given in (1.6). □

9. Coefficient Inequality, Growth and Distortion Bounds, Convex Combination for subclass $H(\phi)$

Estimating the coefficients of analytic functions helps us understand how these functions behave. Growth and distortion theorems make this clearer by showing how the function's output range is affected by how much the function stretches or shrinks. These theorems explain the connection between the function's internal structure (its coefficients) and how it changes when mapping points in the complex plane.

Theorem 9.1. *Let $f \in H(\phi)$, then*

$$\sum_{n=2}^{\infty} n^2|a_n| < 1. \tag{9.1}$$

Proof: Assume that $f \in \mathcal{A}$ of the form (1.1) is in the class $H(\phi)$. Then

$$\frac{2(zf'(z))'}{f'(z)+f'(-z)} = \frac{1+\sum_{n=2}^{\infty} n^2 a_n z^{n-1}}{1+\sum_{n=1}^{\infty}(2n+1)a_{2n+1}z^{2n}} = F(z) \ (say).$$

From the equation (1.8) of Definition 1.2 there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that

$$F(z) = \frac{1+w(z)}{1-w(z)} \implies |w(z)| = \left|\frac{F(z)-1}{F(z)+1}\right| < 1.$$

$$\left|\frac{F(z)-1}{F(z)+1}\right| = \frac{\sum_{n=2}^{\infty} n^2 a_n z^{n-1} + \sum_{n=1}^{\infty}(2n+1)a_{2n+1}z^{2n}}{2+\sum_{n=2}^{\infty} n^2 a_n z^{n-1} + \sum_{n=1}^{\infty}(2n+1)a_{2n+1}z^{2n}}.$$

After simplication the, above expression becomes

$$\begin{aligned}\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{\frac{1}{2}\sum_{n=1}^{\infty}[(n+1)^2(1-(-1)^n)+n(n+1)(1+(-1)^n)]a_{n+1}z^n}{2+\sum_{n=1}^{\infty}\frac{1}{2}[(n+1)^2(1-(-1)^n)+(n+1)(n+2)(1+(-1)^n)]a_{n+1}z^n}\right| \\ &= \frac{\frac{1}{2}\sum_{n=1}^{\infty}|(n+1)^2(1-(-1)^n)+n(n+1)(1+(-1)^n)||a_{n+1}||z^n|}{2+\sum_{n=1}^{\infty}\frac{1}{2}|(n+1)^2(1-(-1)^n)+(n+1)(n+2)(1+(-1)^n)||a_{n+1}||z^n|} \\ &\leq \frac{\frac{1}{2}\sum_{n=1}^{\infty}|(n+1)^2(1-(-1)^n)+n(n+1)(1+(-1)^n)||a_{n+1}|}{2-\sum_{n=1}^{\infty}\frac{1}{2}|(n+1)^2(1-(-1)^n)+(n+1)(n+2)(1+(-1)^n)||a_{n+1}|} < 1\end{aligned}$$

which implies,

$$\begin{aligned}&\frac{1}{2}\sum_{n=1}^{\infty}|(n+1)^2(1-(-1)^n)+n(n+1)(1+(-1)^n)||a_{n+1}| \\ &\qquad \leq 2-\sum_{n=1}^{\infty}\frac{1}{2}|(n+1)^2(1-(-1)^n)+(n+1)(n+2)(1+(-1)^n)||a_{n+1}|.\end{aligned}$$

On simplification gives

$$\begin{aligned}&\frac{1}{2}\sum_{n=1}^{\infty}|(n+1)^2(1-(-1)^n)+(n+1)(n+2)(1+(-1)^n)||a_{n+1}| \leq 1 \\ &\implies \sum_{n=1}^{\infty}(n+1)^2|a_{n+1}| \leq 1 \\ &\implies \sum_{n=2}^{\infty} n^2|a_n| \leq 1\end{aligned}$$

This leads to the desired result (9.1). □

Corollary 9.1. *If $f \in H(\phi)$, then*

$$|a_n| \leq \frac{1}{n^2}, n \geq 2.$$

and the equality holds for the function given by

$$f(z) = z + \frac{1}{n^2}z^n.$$

Theorem 9.2. *Let $f \in H(\phi)$, then*

$$r - \frac{1}{4}r^2 \leq |f(z)| \leq r + \frac{1}{4}r^2, \tag{9.2}$$

where $|z| = r < 1$ and the result is sharp for the following

$$f(z) = z + \frac{1}{4}z^2$$

Proof: Consider $f \in H(\phi)$ and by using (9.1) we have

$$2^2 \leq n^2 \quad (n > 1)$$

$$\implies 2^2 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n^2 |a_n| \leq 1$$

$$\implies \sum_{n=2}^{\infty} |a_n| \leq \frac{1}{4} \tag{9.3}$$

Let $f = z + \sum_{n=2}^{\infty} a_n z^n$. Since $|z| = r$, we have

$$|f(z)| = |z + \sum_{n=2}^{\infty} a_n z^n|$$

$$\leq r + r^2 \sum_{n=2}^{\infty} |a_n|$$

$$\implies |f(z)| \leq r + \frac{1}{4}r^2$$

Similarly,

$$|f(z)| \geq r - \frac{1}{4}r^2$$

Hence, Theorem 9.2 is thus completed. □

Theorem 9.3. *Let $f \in H(\phi)$, then*

$$1 - \frac{r}{2} \leq |f'(z)| \leq 1 + \frac{r}{2}, \tag{9.4}$$

where $|z| = r < 1$ and the result is sharp for the following

$$f(z) = z + \frac{1}{4}z^2$$

Proof: Consider $f \in H(\phi)$ and by using (9.3) we have

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1}{4} \implies \sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{2}$$

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$|f'(z)| = |1 + \sum_{n=2}^{\infty} n a_n z^{n-1}| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} = 1 + \sum_{n=2}^{\infty} n|a_n|r^{n-1} \leq 1 + r\sum_{n=2}^{\infty} n|a_n|$$
$$\leq 1 + \frac{r}{2},$$

Similarly,

$$|f'(z)| \geq 1 - \frac{r}{2}$$

Hence, the proof of Theorem 9.3 is thus completed. □

Theorem 9.4. *Let $f \in H(\phi)$, then the subclass is closed under convex combination.*

Proof:
We must show that if the functions

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n, \; a_{n,i} \geq 0, i = 1, 2$$

are in the class $H(\phi)$ then the function h defined by

$$h(z) = \sum_{i=1}^{2} \lambda_i f_i(z), \qquad \sum_{i=1}^{n} \lambda_i = 1$$

is also in the class $H(\phi)$. By definition of h, we have

$$h(z) = \sum_{i=1}^{2} \lambda_i (z + \sum_{n=2}^{\infty} a_{n,i} z^n) = \sum_{i=1}^{2} \lambda_i z + \sum_{i=1}^{2} \lambda_i \sum_{n=2}^{\infty} a_{n,i} z^n = z + \sum_{n=2}^{\infty} (\sum_{i=1}^{2} \lambda_i a_{n,i}) z^n$$

From Theorem 9.1, we have

$$\sum_{n=2}^{\infty} n^2 \left[\sum_{i=1}^{2} \lambda_i a_{n,i} \right] = \sum_{i=1}^{2} \lambda_i \left[\sum_{n=2}^{\infty} n^2 a_{n,i} \right] \leq \sum_{i=1}^{2} \lambda_i = 1$$

Thus $h \in H(\phi)$. This completes the proof of Theorem 9.4. □

Concluding Remark: In conclusion, this study has successfully established several key properties for a newly defined class of analytic functions. The derivation of precise initial coefficient bounds, along with a sharp Fekete-Szegö inequality, provides fundamental insights into the geometric characteristics of this class. Additionally, the calculation of exact bounds for specific Hankel determinants offers a deeper understanding of the relationships between these coefficients. While some bounds for inverse and logarithmic coefficients have been shown to be sharp, further research into achieving sharpness for the remaining estimates could be a promising direction for future studies. Finally, verifying the sharpness of the Zalcman functional and the Krushkal inequality for particular values enhances our comprehension of these essential functionals within this specific class. The findings presented in this article significantly contribute to the broader field of geometric function theory by introducing and thoroughly analyzing this novel class of analytic functions. Moreover, the established coefficient inequalities, growth and distortion bounds, along with the demonstration of closure under convex combinations for different class provide a detailed overview of the geometric properties of this class.

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