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Computing Double Domination Number of Double Graph of Some Regular Graphs

Udhayashree R.* and Rajeswari R.

ABSTRACT: A graph in which every vertex has the same valency or degree is known as a regular graph. The double graph D(G) of a regular graph is constructed by duplicating each vertex and connecting the corresponding duplicates iff the original vertices are adjacent. This process creates a motivating and composite graph structure, raising questions about the domination properties of D(G). In this paper we focus on the double domination of double graph of some regular graphs which is constructed using an algorithm (Llama New, 2016) [11]. The double domination number of regular graphs on even and odd order are determined.

Key Words: Regular graph, k-regular graph, double dominating set, double domination number, double graph.

Contents

1. Introduction

Domination plays a vital role in application of graph theory. Domination was studied in 1978 by R. B. Allan et al. and proved the relations between domination number and independent domination number [1]. Twin domination was studied in 2013 by Arumugam et al. [2]. The double graphs was studied deeply by Emanuele Munarini et al., in 2008 and discussed their properties [3]. In 2013, V. R. Kulli edited book on Advances in domination theory II, which speaks about the different types of domination and relations between them [4]. In 2015, domination colouring of prism graph was discussed by Manjula & R. Rajeswari [5]. In 2022, domination in different graphs by K. Mythily, et al. gave a detailed analyses of the concept of domination and its characteristics in graph theory [6]. Rajeswari R and Anbunathan discussed about arc domination and twin arc domination in cayley digraphs in 2021 [7]. And in 2019, domination of cayley digraph and its complement was calculated and generalized [8]. Shobha Shukla et al. worked on domination and its types in 2020 [9]. Domination in Cayley graph was studied by Tamizh Chelvam et al. [10] in 2017.

A graph in which every node has the same valency or degree is known as a regular graph. A graph in which each node is of degree k is called a k - regular graph or a regular graph of degree k.

In a graph $G = (V, E)$, if $S \subseteq V$ then S is a double dominating set of G, if every vertex in V is dominated by at least two vertices in S. The "double domination number" γ_{x2} is the minimum cardinality of a double dominating set of G.

The double graph $D(G)$ of a given graph G is constructed by making two copies of G (including the initial edge set of each) and adding edges $u_i v_j$ and $v_i u_j$ for every edge $u_i u_j$ of G.

* Corresponding author.
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In this chapter, we emphasis on the double domination of double graph of some regular graphs which is constructed using an algorithm (Llama New, 2016) [11]. Further, the double domination number of regular graphs on odd and even order is determined.

2. Algorithm for Construction of Regular Graph

Let a graph G has n nodes and let the degree of each node of G be k, where $0 < k < n$. The procedure to construct k-regular graph is explained below.
First all the nodes are placed in a circle.
If n is odd:
  For i in range $(1, k)$:
   If i is odd:
    Make edge for each node x steps away (i)
If n is even:
  If k is even:
   Count Even numbers $= \frac{k}{2}$
    For i in range (Count Even numbers):
     Make edge for each node x steps away $(i + 1)$
Else if k is odd:
  Count odd numbers $= \frac{k-1}{2} + 1$
   For i in range (Count odd numbers):
    Make edge for each node x steps away $\frac{n}{2} - i$

That is, a regular graph is constructed as explained below.

Case 1: n is odd and k must be even.
All the nodes are placed in a circle. Then the nodes are joined x positions away where x is all odd numbers between 1 to k.

Table 1: Procedure for construction of k - regular graph of odd order where k is even.

n	k	Procedure for construction of k - regular graph
11	2	Produce edge for each node x steps away 1
11	4	Produce edge for each node x steps away 1 Produce edge for each node x steps away 3
11	6	Produce edge for each node x steps away 1 Produce edge for each node x steps away 3 Produce edge for each node x steps away 5
11	8	Produce edge for each node x steps away 1 Produce edge for each node x steps away 3 Produce edge for each node x steps away 5 Produce edge for each node x steps away 7

Case 2: n is even and k is even.
All the nodes are placed in a circle. Then the nodes are joined x positions away where x is a range of numbers from 1 to $\frac{n}{2}$ and the range of these numbers is limited by the number of even numbers from 1 to k including k.

Table 2: Procedure for construction of k - regular graph of even order where k is even.

n	k	Procedure for construction of k - regular graph
12	2	Produce edge for each node x steps away 1
12	4	Produce edge for each node x steps away 1 Produce edge for each node x steps away 2
12	6	Produce edge for each node x steps away 1 Produce edge for each node x steps away 2 Produce edge for each node x steps away 3
12	8	Produce edge for each node x steps away 1 Produce edge for each node x steps away 2 Produce edge for each node x steps away 3 Produce edge for each node x steps away 4
12	10	Produce edge for each node x steps away 1 Produce edge for each node x steps away 2 Produce edge for each node x steps away 3 Produce edge for each node x steps away 4 Produce edge for each node x steps away 5

Case 3: n is even and k is odd

All the nodes are placed in a circle. Then the nodes are joined x positions away where x is a range of numbers from $\frac{n}{2}$ to 1 and the range of these numbers is limited by the number of odd numbers from 1 to k including k.

Table 3: Procedure for construction of k - regular graph of even order where k is odd.

n	k	Procedure for construction of k - regular graph
14	1	Produce edge for each node x steps away 7
14	3	Produce edge for each node x steps away 7 Produce edge for each node x steps away 6
14	5	Produce edge for each node x steps away 7 Produce edge for each node x steps away 6 Produce edge for each node x steps away 5
14	7	Produce edge for each node x steps away 7 Produce edge for each node x steps away 6 Produce edge for each node x steps away 5 Produce edge for each node x steps away 4
14	9	Produce edge for each node x steps away 7 Produce edge for each node x steps away 6 Produce edge for each node x steps away 5 Produce edge for each node x steps away 4 Produce edge for each node x steps away 3
14	11	Produce edge for each node x steps away 7 Produce edge for each node x steps away 6 Produce edge for each node x steps away 5 Produce edge for each node x steps away 4 Produce edge for each node x steps away 3 Produce edge for each node x steps away 2

3. Main Results

3.1. Double Domination Number of Double Graph of Regular Graph with Even Order

A graph in which every node is of same degree and is an even number, is known as a regular graph of even order. This section determines the double domination number of some regular graph G of even order.

Theorem 3.1 *If a graph G with n nodes where n>4 and n is even, is regular with degree n-2, then its double domination number of double graph is* $\gamma_{x2}= 3$.

Proof: *Let* $\{u_i/1\leq i\leq n\}$ *be the collection of nodes of the regular graph G. The edges of the* $n-2$ *regular graph is constructed using algorithm case 2 for each copy. The double graph of G is constructed by taking another copy a regular graph whose nodes are represented as* $\{v_i/\ 1\leq i\leq n\}$ *and connecting the nodes* u_iv_j *and* v_iu_j *for every edge* u_iu_j *of G.*

As G is $n-2$ *regular, the node* u_1 *dominates all the nodes of the first and second copy once, except* $\{u_{\frac{n}{2}+1}, v_{\frac{n}{2}+1}, v_1\}$. *Then* u_2 *dominates all the nodes of first and second copy once except* $\{u_{\frac{n}{2}+2}, v_{\frac{n}{2}+2}, v_2\}$. *So, the vertices are double dominated by* u_1 *&* u_2 *except* $\{u_{\frac{n}{2}+1}, v_{\frac{n}{2}+1}, v_1, u_{\frac{n}{2}+2}, v_{\frac{n}{2}+2}, and, v_2\}$ *which are all dominated once. To dominate those nodes, we choose* u_3*, which dominates all the above undominated nodes. Therefore, all the nodes in the double graph are double dominated by* u_1*,* u_2 *and* u_3 *(See Figure 1). Hence, double domination number of double graph of* $n-2$ *regular graph on n nodes, n even and n>4 is* 3.

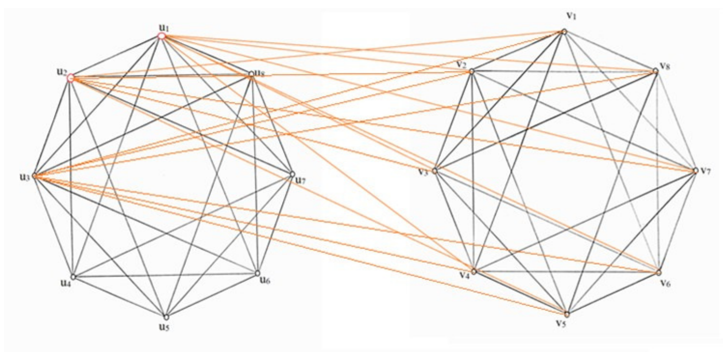

Figure 1: γ_{x2} Double domination number of double graph of regular graph of degree $n-2$ is 3

Theorem 3.2 *If a graph G with n nodes where n>4 and n is even, is regular with degree* $\frac{n}{2}$ *then its double domination number of double graph is 4.*

Proof: *Let* $\{u_i/1\leq i\leq n\}$ *be the collection of nodes of the regular graph. The edges of the* $\frac{n}{2}$ *regular graph is constructed using algorithm case 2 for even degree and algorithm case 3 for odd degree in each copy.*

The double graph of G is constructed by taking another copy a regular graph whose nodes are represented as $\{v_i/\ 1\leq i\leq n\}$ *and connecting the nodes* u_iv_j *and* v_iu_j *for every edge* u_iu_j *of G.*

Case i) *When* $n\equiv 2 \pmod 4$

When n is even and k is odd, As G is $\frac{n}{2}$ *regular, the nodes* u_1 *&* $u_{\frac{n}{2}+1}$ *dominates all the nodes of the corresponding graph and its copy once. To dominate all the vertices once again we need to take the vertices* u_2 *&* $u_{\frac{n}{2}+2}$. *So, these four vertices dominate all the vertices twice. Therefore, all the nodes in the double graph are double dominated by* u_1*,* u_2*,* $u_{\frac{n}{2}+1}$*,* $u_{\frac{n}{2}+2}$ *(See Figure 2) . Thus, the double domination number of double graph of* $\frac{n}{2}$ *regular on n nodes, with n as even and k as odd is 4.*

Case ii) *When* $n\equiv 0 \pmod 4$, *When n is even and k is even, As G is $\frac{n}{2}$ regular, the nodes u_1 & $u_{\frac{n}{2}+1}$ dominates all the nodes of the corresponding graph and its copy once and $u_{\frac{n}{4}+1}$,$u_{\frac{3n}{4}+1}$,$v_{\frac{n}{4}+1}$,$v_{\frac{3n}{4}+1}$ are dominated twice, except v_1, $v_{\frac{n}{2}+1}$. To dominate the undominated vertices, we choose $u_{\frac{n}{4}+1}$,$u_{\frac{3n}{4}+1}$ which dominates all the vertices twice. Therefore, all the nodes in the double graph are dominated by u_1, $u_{\frac{n}{2}+1}$, $u_{\frac{n}{4}+1}$,$u_{\frac{3n}{4}+1}$ (See Figure 3). Thus, the double domination number of double graph of $\frac{n}{2}$ regular on n nodes, with n as even and k as even is 4.*

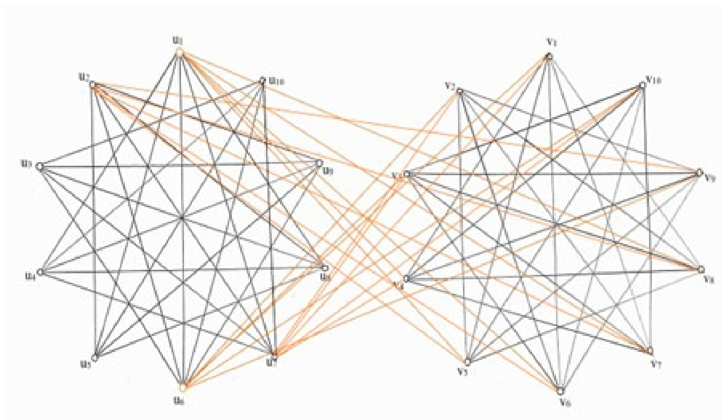

Figure 2: γ_{x2} Double domination number of double graph of regular graph of degree $\frac{n}{2}$ is 4

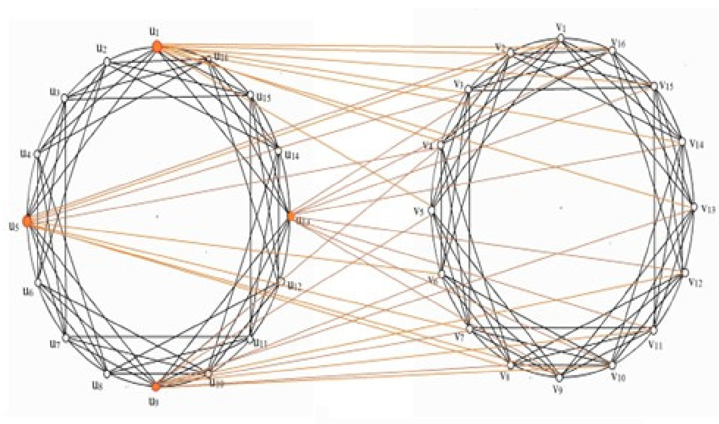

Figure 3: γ_{x2} Double domination number of double graph of regular graph of degree $\frac{n}{2}$ is 4

3.2. Double Domination Number of Double Grpah of Regular Graph with Odd Order

A graph in which every node is of same degree and is an odd number, is known as a regular graph of odd order. This section determines the double domination number of double graph of some regular graph G of odd order.

Theorem 3.3 *If a graph G with n nodes where $n \geq 9$ and n is odd, is regular with degree $n-3$ then its double domination number of double graph is 3.*

Proof: *Let $\{u_i/1 \leq i \leq n\}$ be the collection of nodes of the regular graph. The edges of the $n-3$ regular graph is constructed using algorithm case 1. The double graph of G is constructed by taking another* *copy a regular graph whose nodes are represented as $\{v_i/\, 1 \leq i \leq n\}$ and connecting the nodes u_iv_j and v_iu_j for every edge u_iu_j of G.*

As G is $n-3$ regular, the node u_1 dominates all the nodes of first and second copy once except $\{u_3, u_{n-1}, v_3, v_{n-1} and, v_3\}$. Then u_4 dominates all the nodes of first and second copy once except $u_3, u_6, v_2, v_6, and, v_4$. So, the nodes are double dominated by u_1 & u_4 except $\{u_3, u_{n-1}, v_3, v_{n-1}, v_3, u_2, u_6, v_2, v_6, and, v_4\}$ which are all dominated once. To dominate those nodes, we choose u_7, which dominates all the above undominated nodes. Therefore, all the nodes in the double graph are double dominated by u_1, u_4 and u_7 (See Figure 4). Hence, double domination number of double graph of $n-3$ regular graph on n nodes, n even and $n \geq 9$ is 3.

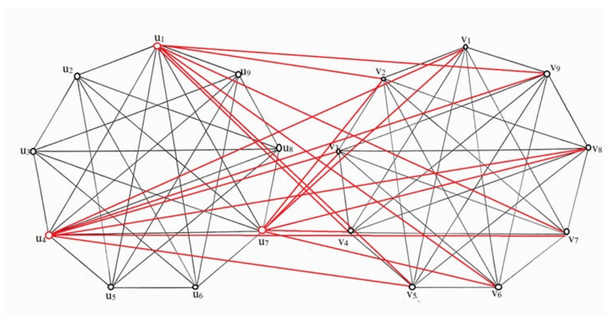

Figure 4: γ_{x2} Double domination number of double graph of regular graph of degree $n-3$ is 3

Theorem 3.4 *If a graph G with n nodes where $n > 5$, and $n \equiv 1 \pmod 4$ is regular with degree $\lfloor \frac{n}{2} \rfloor$, then its double domination number of a double graph of G is 5.*

Proof: *Let $\{u_i/1 \leq i \leq n\}$ be the collection of nodes of the regular graph. The edges of the $\lfloor \frac{n}{2} \rfloor$ regular graph is constructed using algorithm case 1. The double graph of G is constructed by taking another copy a regular graph whose nodes are represented as $\{v_i/\, 1 \leq i \leq n\}$ and connecting the nodes u_iv_j and v_iu_j for every edge u_iu_j of G.*

Let q be $\lceil \frac{n}{2} \rceil$. For every $1 \leq i \leq q$, the node u_1 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$, and for every $q < i \leq n$, the node u_1 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$.

Then to dominate other undominated vertices u_4 was taken. For every $1 \leq i \leq q+2$, the node u_4 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$, and for every $q+3 < i \leq n$, the node u_4 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$.

So now all the vertices are dominated once except $\{u_{q+1}, u_{q+3}, v_{q+1}, v_{q+3}\}$. To dominate those vertices and to dominate all the other vertices twice, choose u_5, u_8, and, u_9. So, all the vertices are dominated twice. Therefore, all the nodes in the double graph are dominated twice by u_1, u_4, u_5, u_8, u_9 (See Figure 5). Thus, the double domination number $\gamma_{x2}(D(G))$ of $\lfloor \frac{n}{2} \rfloor$ regular graph on n nodes, $n > 5$, $n \equiv 1 \pmod 4$ is 5.

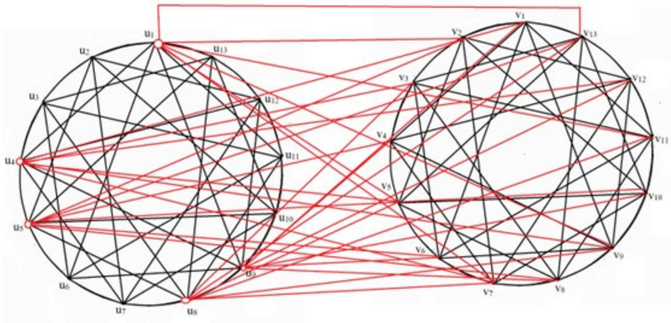

Figure 5: γ_{x2} Double domination number of double graph of regular graph of degree $\lfloor \frac{n}{2} \rfloor$ is 5

Theorem 3.5 *If a graph G with n nodes where $n > 5$, and $n \equiv 3 \pmod 4$ is regular with degree $\lceil \frac{n}{2} \rceil$, then its double domination number of a double graph of G is 4.*

Proof: *Let $\{u_i/1 \le i \le n\}$ be the collection of nodes of the regular graph. The edges of the $\lceil \frac{n}{2} \rceil$ regular graph is constructed using algorithm case 1. The double graph of G is constructed by taking another copy a regular graph whose nodes are represented as $\{v_i/\ 1 \le i \le n\}$ and connecting the nodes $u_i v_j$ and $v_i u_j$ for every edge $u_i u_j$ of G.*

Let q be $\lceil \frac{n}{2} \rceil$. For every $1 \le i \le q$, the node u_1 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$, and for every $q < i \le n$, the node u_1 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$.
Then to dominate other undominated vertices u_2 was taken. For every $1 \le i \le q+1$, the node u_2 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$, and for every $q+1 < i \le n$, the node u_2 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$.
So, all the vertices are dominated once.
To dominate all the vertices twice, we take u_3. For every $1 \le i \le q+2$, the node u_3 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$, and for every $q+2 < i \le n$, the node u_3 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$.
Then to dominate other undominated vertices u_4 was taken. For every $1 \le i \le q+3$, the node u_4 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$, and for every $q+3 < i \le n$, the node u_4 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$.
So, now all the vertices are dominated twice.
Therefore, all the nodes in the double graph are double dominated by $\{u_1, u_2, u_3, u_4\}$ (See Figure 6). Thus, the double domination number $\gamma_{x2}(D(G))$ of $\lceil \frac{n}{2} \rceil$ regular graph on n nodes , $n > 5$, $n \equiv 3 \pmod 4$ is 4.

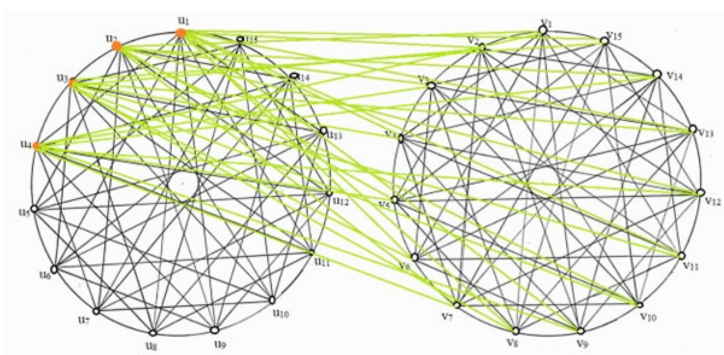

Figure 6: γ_{x2} Double domination number of double graph of regular graph of degree $\lceil \frac{n}{2} \rceil$ is 4

Theorem 3.6 *If a graph G with n nodes where $n > 9$, and $n \equiv 1 \pmod 4$ is regular with degree $\lceil \frac{n}{2} \rceil + 1$,* *then its double domination number of a double graph of G is 4.*

Proof: *Let $\{u_i/1{\leq}i{\leq}n\}$ be the collection of nodes of the regular graph. he edges of the $\lceil\frac{n}{2}\rceil$ regular graph is constructed using algorithm case 1. The double graph of G is constructed by taking another copy a regular graph whose nodes are represented as $\{v_i/\ 1{\leq}i{\leq}n\}$ and connecting the nodes u_iv_j and v_iu_j for every edge u_iu_j of G.*

Let q be $\lceil\frac{n}{2}\rceil$. For every $1 \leq i \leq q+1$, the node u_1 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$, and for every $q+1 < i \leq n$, the node u_1 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$.

Then to dominate other undominated vertices u_4 was taken. For every $1 \leq i \leq q+4$, the node u_4 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$, and for every $q+4 \leq i \leq n$, the node u_4 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$.

So, all the vertices are dominated once.

To dominate all the vertices twice, we choose u_7, u_{10}. For every $1 \leq i \leq q+7$, the node u_7 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$, and for every $q+6 \leq i \leq n$, the node u_7 dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$.

For every $1 \leq i \leq q+10$, the node u_{10} dominates u_i and v_i, whenever $i \equiv 1 \pmod 2$, and for every $q+9 \leq i \leq n$, the node u_10 dominates u_i and v_i, whenever $i \equiv 0 \pmod 2$.

So, here all the vertices are dominated twice.

Therefore, all the nodes in the double graph are double dominated by $\{u_1, u_4, u_7, u_{10}\}$ (See Figure 7).

Thus, the double domination number $\gamma_{x2}(D(G))$ of $\lceil\frac{n}{2}\rceil + 1$ regular graph on n nodes , $n > 9$, $n \equiv 1 \pmod 4$ is 4.

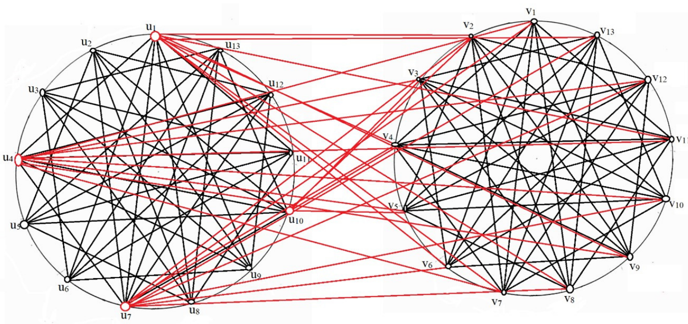

Figure 7: γ_{x2} Double domination number of double graph of regular graph of degree $\lceil\frac{n}{2}\rceil + 1$ is 4

4. Conclusion

The double domination number of double graph of regular graphs $\gamma_{x2}(D(G))$ on even order and odd order are determined. In Further research we have planned to study different type of dominations for the above graphs and relation between them, and the application of double domination will be analysed.

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Udhayashree R.
Department of Mathematics,
Sathyabama Institute of Science and Technology,
Chennai - 600119, Tamil Nadu, India.
E-mail address: `udhayakutty1996@gmail.com`

and

Rajeswari R.
Department of Mathematics,
Sathyabama Institute of Science and Technology,
Chennai - 600119, Tamil Nadu, India.
E-mail address: `rajeswarivel1998@gmail.com`