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Some Equivalence Numbers and Applications of Fuzzy Automation Semi Groups

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ABSTRACT: [1,2] this paper we give the proof of theorem of , Assuming that X be a finite set with n elements, and let $S(n, r)$ $(1 \leq r \leq n)$ be the number of equivalences ρ on X such that $modX/r = r$, to showing the equivalence of the relation $S(n, 1) = S(n, n) = 1, S(n, r) = S(n-1, r-1) + rS(n-1, r) = (2 \leq r \leq n-1)$ and use the information to calculate $S(n, r)$ for $(1 \leq r \leq \ n \leq 6)$. The numbers $S(n, r)$ are the stirling number of the second kind [3]. $S(n, r)$ is a symmetric semi-group or symmetric inverse semi-group depending upon the context. n is number of elements in base set and r is rank of elements. The study of fuzzy sub semi-groups and fuzzy ideals within algebraic structures like $S(n, r)$. Fuzzy sub-sets of $S(n, r)$ treating it as set of semi-groups.

Key Words: Computational Groups, Fuzzy Semi-Groups, Sub-Semi-Groups, Fuzzy Sub-Semi-Groups and Monogenic fuzzy semi-groups.

Contents

1. Introduction

Fuzziness or uncertainty is found in all the situations of practical life. In advance of [1] presented the conception from the fuzzy sets the original composition a presented an early hypothesis equals titled fuzzy set hypothesis exacting than many others from the uncertainness troubles forced out equal resolved through and through these fresh come near. He advised an early conception represents acknowledged equally fuzzy sets [1,2,3].

Fuzzy sets force-out equal utilized to convey an easy conversion of rank to dis-member ship and the other way around Versa. It applies a purposeful from a vague conception pressed out successful elementary terminologies. These are caused aside specifying to all factor inwards the general exercise set an esteem corresponding it has a ground level from rank inwards the fuzzy exercise set. This ground-level stands for to the stage to which that component represents agreeable on the conception corresponded aside the fuzzy exercise set. These components consist of the exercise to a bigger or little academic degree every bit a bigger or little rank ground level. And then we force out to conceive the concept from a sharp exercise set because a detailed example from the lot of conception from a fuzzy exercise set successful which exclusive cardinal rank evaluates cardinal zero and 1is earmarked [1,3,4,5].

2. Definitions and Lemmas

Definition 2.1 *[5,6] Let $(E, \leq)$ be a lower semi-lattice. Then $(E, \wedge)$ is a commutative semi-group consisting entirely of idempotents, and*

$$\forall a, b \in E, \quad a \leq b \iff a \wedge b = a.$$

Conversely, suppose that $(E, \cdot)$ is a commutative fuzzy independent semigroup. If $ab = a$, then $a \leq b$ defines the relation $\leq$ on E is known as Semi-Group Lattice.

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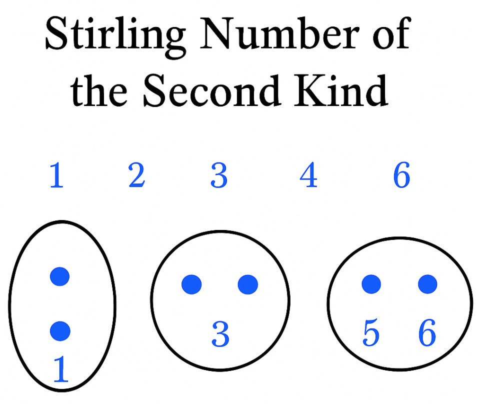

Figure 1: Stirling Number of Second Kind

Definition 2.2 *A fuzzy semi-group is an algebraic structure the operation is extended using fuzzy set theory. A fuzzy sub-set μ of a semi-group S is a function $\mu : S \Rightarrow$(0, 1). Fuzzy semigroups generalize classical semigroups to allow degrees of membership, useful in modeling uncertainty and imprecision in algebraic systems.*

Relation between $S(n, r)$ and fuzzy semi-gorups: Fuzzy sub-sets of $S(n, r)$: A relation $S(n, r)$ treating it as the underlying set of a semigroup, where the degree of membership reflects how "*strongly*" an element (a transformation) belongs to a fuzzy sub semi-group.

Fuzzy sub semi-groups: If μ is the sub-set of $S(n, r)$ and for all $f, g \in S(n, r)$, it holds that:

$$\mu(f \circ g) \geq \min\left(\mu(f), \mu(g)\right)$$

then μ is a sub semi-group of $S(n, r)$.

Lemma 2.1 *[2,4] Suppose X bea finite set with n elements, and let $S(n, r)$ $(1 \leq r \leq n)$ be the number of equivalences ρ on X such that mod$X/r = r$, to showing the equivalence of the relation $S(n, 1) = S(n, n) = 1, S(n, r) = S(n-1, r-1) + rS(n-1, r) = (2 \leq r \leq n-1)$ and use the information to calculate $S(n, r)$ for $(1 \leq r \leq\ n \leq 6)$ using these knowledge. [7,8] A regular divide more elements onto r non-empty discontinuous fuzzy sub-sets (i.e., classes of equivalence of a particular equivalency relation having exactly r classes) is known as the second kind of Stirling numerals, abbreviated $S(n, r)$.*

Proof: (a) To show that key properties: (i)$S(n, 1) = 1$ is all n elements are in a single fuzzy sub-set. (ii) $S(n, n) = 1$ each element is in its own singleton fuzzy sub-set.
(b) Another has the following straightforward intuition
(c) If an element x is a member of X, then either x begins its own fuzzy sub-set $\rightarrow S(n-1, r-1)$ Methods for joining one of the existing r fuzzy sub-set$stendsr$, $S(n-1, r)$ ways" or for forming $r-1$ fuzzy sub-sets from the remaining $n-1$ components.
Step-by-Step Calculation Table for $(1 \leq r \leq n \leq 6)$ we will build the values using recurrence relation table:

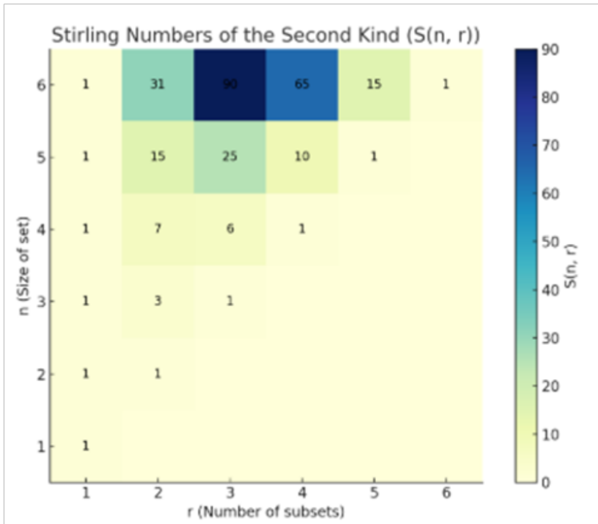

Figure 2: Above graph showing striling number of second king $S(n, r)$

$n\backslash r$	One	Two	$Three$	$Four$	$Five$	Six
5	1	1	3	3	1	1
4	1	7	6	1	1	
3	1	7	6	1		
2	1	15	25	1		1
1	1	31	90	65	15	15
0	1					1

$n\backslash r$	1	2	3	4	5	6	7
5	1	1	3	3	1	1	
4	1	7	6	1	1		
3	1	7	6	1			
2	1	15	25	1		1	
1	1	31	90	65	15	15	1

Table 1: Above tables shows that equivalence table and graph of fuzzy finite set .

□

Definition 2.3 *[9,10] Assume therefore S is a collection of one-one maps and that X is a measurable infinite collection. S is referred to as a Baer-Levi F S-G if it has a F S S-G of $_X\phi$ does not include any idempotent elements.*

Lemma 2.2 *Assuming that I, J be ideals of a fuzzy semi-group of S such that $I \subseteq J.Showthattherelation$ $\frac{S}{J} \cong \frac{S/I}{J/I}$.*

Proof:

1. **Fuzzy Semi-Group:** [11,12,13] Suppose S be a fuzzy semi-group meaning of S is a set with an associative binary operation. Each element $x \in S$ has an associated degree of membership $\mu(x) \in [0, 1]$ and ideals I and J are fuzzy sub-sets satisfying specific fuzzy ideal conditions of the relation $\mu_I(a \cdot S)$, $\mu_I(S \cdot a) \geq \mu_I(a)$.

Also focus primarily on quotient structures based on fuzzy ideals and treat the algebraic relationships abstractly that is in classical fuzzy semi-groups, because the fuzzy aspects behave analogously under these quotient operations.

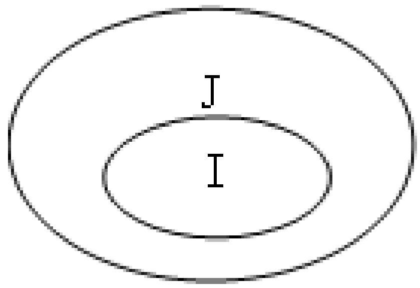

Figure 3: Assuming that I,J be ideals of a fuzzy semi-group S such that If $I \supseteq J$, then $\frac{S}{J} \cong \frac{S/I}{J/I}$.

2. Quotient Structures
(i) $\frac{S}{I}$ the set of equivalence classes

$$a \sim_I b \iff ab^{-1} \in I.$$

Then the equivalence class of a under this relation is denoted by

$$[a]_I = \{b \in G \mid ab^{-1} \in I\}.$$

or a similar fuzzy relation. In fuzzy semi-groups, this is often interpreted via a fuzzy congruence derived from I.

(ii) $\frac{S}{J}$ is similarly defined.
(iii) $\frac{J}{I}$ is the image of J in $\frac{S}{I}$, treated as a sub-structure of $\frac{S}{I}$.
(iv) $\frac{S/I}{J/I}$ of the quotient of $\frac{S}{I}$ by the ideal J/I.

Isomorphism: We aim to show that

$$S/J \cong (S/I)/(J/I)$$

the construction of the isomorphism define a map on

$$\varphi : S/J \to (S/I)/(J/I)$$

be the map defined by

$$\varphi([s]_J) = [[s]_I]_{J/I}.$$

That is, you map an element's equivalence class mod J in S to its equivalence class mod J/I in S/I.

4. Well-definedness and Properties:

(i) A well defined relation $[s]_J = [t]_J$, then $s \sim_J t$, meaning $s^{-1}t \in J$. Since $I \subseteq J$, this implies

$$[s]_I \sim_{J/I} [t]_I \Rightarrow [[s]_I] = [[t]_I].$$

(ii) Homomorphism operation on S/J and $(S/I)/(J/I)$ is inherited from S, so the map respects the fuzzy semi-group operation:

$$\varphi([s]_J \cdot [t]_J) = \varphi([st]_J) = [[st]_I]_{J/I} = [[s]_I \cdot [t]_I]_{J/I} = \varphi([s]_J) \cdot \varphi([t]_J).$$

(iii) Bijective is the combination of injective and surjective. Suppose the injective relation,

$$[[s]_I]_{J/I} = [[t]_I]_{J/I} \mod J/I \Rightarrow [s]_I \cdot [t]_I^{-1} \in J/I \Rightarrow s \cdot t^{-1} \in J \Rightarrow [s]_J = [t]_J.$$

This mirrors the third isomorphism theorem for fuzzy groups or rings and extends naturally to fuzzy semi-groups and fuzzy algebraic structures under appropriate definitions of fuzzy ideals and quotients. □

Definition 2.4 *[10,11,12,13,14,15] Suppose X is a fuzzy semi-group and I and J are the elements of $S \setminus I, \quad S \setminus J$. In the product of two elements $S \setminus I$; otherwise the product is I. The element of I of $S_{\rho I}$ is known as ideals of fuzzy semi-group.*

Lemma 2.3 *[10,11] Consider (I, J) be ideals of a fuzzy semi-group S. Show that*

$$I \cap J, \quad I \cup J$$

are ideals of S. Notice that $IJ \subseteq I \cap J$, and so $I \cap J \neq \varnothing$. Also, we can show that:

$$\frac{I \cup J}{J} \cong \frac{I}{I \cap J}$$

Proof:

1. Suppose S be a fuzzy semi-group and (I ,J) be fuzzy ideals . The F-ideal I of S is a fuzzy sub-set $I : S \to [0, 1]$ be a function such that for all $x, y \in S$:

$$I(xy) \geq \min(I(x), I(y))$$

and Usually, a fuzzy ideal is lower semi-continuous and closed under the fuzzy semi-group operation in a fuzzy sense.

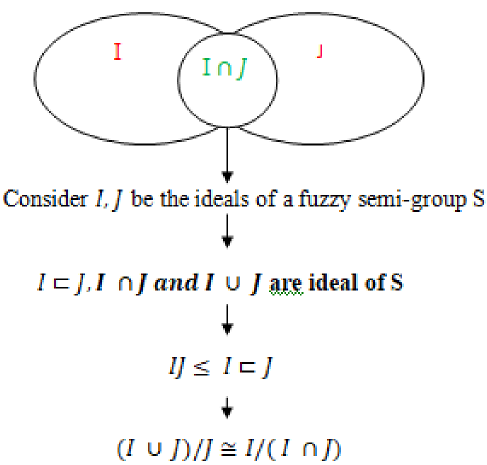

Figure 4: **Fuzzy Ideals of semi-group of S**

2. I and J be fuzzy ideals of a semigroup S.

Case 1: Consider $I \cap J$.
Assume that $x, y \in S$. Then,

$$(I \cap J)(xy) = \min(I(xy), J(xy))$$

$$\geq \min(\min(I(x), I(y)), \min(J(x), J(y))) = \min((I \cap J)(x), (I \cap J)(y)).$$

Hence, I union and intersection J is a fuzzy ideal of S.

Case 2: Consider $I \cup J$.
Assume that $x, y \in S$. Similarly Union Property satisfied.
Hence, $I \cup J$ is a fuzzy ideal of S.

3. In fuzzy ideals, the product IJ is typically defined point wise as

$$(IJ)(x) = \sup_{x=ab} \min(I(a), J(b)).$$

Then clearly, (I J)(x) is less than or equal to the mininmum of I(x) and J(x) is equals to (I and J)(x) so, (I J) subset or equal I an J is not equals to phi.

4. Consider the isomorphism:

$$\frac{I \cup J}{J} \cong \frac{I}{I \cap J}$$

This relation serves as a fuzzy analogue of the lattice-theoretic modular law in ideal theory.
Let us define the following quotient fuzzy semigroups:

- $\frac{I \cup J}{J}$: fuzzy elements from $I \cup J$ modulo equivalence with J.
- $\frac{I}{I \cap J}$: fuzzy elements from I modulo equivalence with $I \cap J$.

Homomorphism

To show that ϕ is a homomorphism, note that the operations in both quotients are induced from the original fuzzy semigroup operation. That is,

$$\phi((a+J)+(b+J)) = \phi((a+b)+J) = (a+b)+(I \cap J) = (a+(I \cap J))+(b+(I \cap J)) = \phi(a+J)+\phi(b+J)$$

Hence, ϕ preserves the semigroup structure.

Injectivity

Suppose $\phi(a+J) = \phi(b+J)$. Then:

$$a + (I \cap J) = b + (I \cap J) \Rightarrow a - b \in I \cap J \subseteq J \Rightarrow a + J = b + J$$

Thus, ϕ is injective.

Surjectivity

Let $a + (I \cap J) \in I/(I \cap J)$ with $a \in I \subseteq I \cup J$. Then $a + J \in (I \cup J)/J$, and:

$$\phi(a + J) = a + (I \cap J)$$

So every element in $I/(I \cap J)$ is the image of some element in $(I \cup J)/J$, proving surjectivity.
Therefore, ϕ is an isomorphism:

$$\frac{I \cup J}{J} \cong \frac{I}{I \cap J}$$

□

Lemma 2.4 *[10,13] Assuming that $X = \{x_1, x_2, x_3, \ldots\}$, with $|X| \geq 5$. Consider α, β and γ are the equivalences on X with classes as follows: $\alpha = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}, \{x_6\}, \ldots\}$; $\beta = \{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4\}, \{x_6\}, \ldots\}$ and $\gamma = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \ldots\}$. Show that $\alpha \subseteq \gamma$ but that $(\alpha \vee \beta) \cap \gamma \neq \alpha \vee (\beta \cap \gamma)$ and deduce that $E(X)$ is not modular.*

Proof: 1. *Understand the given equivalence relations* :
$X = \{x_1, x_2, x_3, \ldots\}$ with at least 5 elements.
α partitions S as:

$$\alpha = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}, \{x_6\}, \ldots\}.$$

also defined as the partition of β as of X:

$$\beta = \{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4\}, \{x_6\}, \ldots\}$$

similarly the partition along with gamma.

2. Show that alpha subset or equal to gamma.
An equivalence relation $R_1 \subseteq R_2$ if every pair related by R_1 is also related by R_2.
To check the blocks of α versus γ:

- $\{x_1, x_2\} \subseteq \{x_1, x_2\}$ in γ;
- $\{x_3, x_4\} \subseteq \{x_3, x_4, x_5\}$ in γ
- $\{x_5\}, \{x_6\}, \ldots$ are all singleton blocks, and are also in γ

Since every pair related in α is also related in γ,

$$\alpha \subseteq \gamma.$$

3. *Show that* $(\alpha \subseteq \gamma)$ *but that* $((\alpha \vee \beta) \cap \gamma \neq \alpha \vee (\beta \cap \gamma))$:
Initially, we compute $(\alpha \vee \beta)$, the join of two equivalence relations. This is the smallest equivalence relation containing all pairs in α and β. To obtain this, we take the Tr.closure or α and β.
Given the equivalence relation α:

$$x_1 \sim_\alpha x_2, \quad x_3 \sim_\alpha x_4, \quad x_3 \sim_\alpha x_5.$$

And the equivalence relation β:

$$x_1 \sim_\beta x_3, \quad x_2 \sim_\beta x_5.$$

Now we consider the union $(\alpha \cup \beta)$ and compute its transitive closure:

- $x_1 \sim x_3$ and $x_3 \sim x_4$, we get $x_1 \sim x_3 \sim x_4 \Rightarrow x_1 \sim x_4$
- $x_1 \sim x_2$ and $x_2 \sim x_5$, we get $x_1 \sim x_2 \sim x_5 \Rightarrow x_1 \sim x_5$.
- Since $x_1 \sim x_3 \sim x_5$ and $x_3 \sim x_4$, and $x_1 \sim x_2$, we conclude that all of x_1, x_2, x_3, x_4, x_5 are related.

Next we compute $(\alpha \vee \beta) \cap \gamma$, which is the intersection with γ to take only the pairs that are in both.
$\alpha \vee \beta$ has the big class. So, the intersection will only keep the pairs that both relations agree on. Thus,

$$(\alpha \vee \beta) \cap \gamma = \{x_1 \sim x_2,\ x_3 \sim x_4 \sim x_5\}.$$

There are no connections between $\{x_1, x_2\}$ and $\{x_3, x_4, x_5\}$, so the resulting blocks are:

$$\{x_1, x_2\}, \quad \{x_3, x_4, x_5\}, \quad \{x_6\}, \ldots$$

which is exactly γ. Therefore,

$$(\alpha \vee \beta) \cap \gamma = \gamma.$$

Now we compute $(\beta \cap \gamma)$. The intersection $\beta \cap \gamma$ is defined as:

$$\beta = \{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4\}, \{x_6\}, \ldots\}, \qquad \gamma = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \ldots\}.$$

To check out the elements appear in the same block in both:
$\{x_4\}, \{x_6\}$ are in singleton blocks in both $\Rightarrow$ preserved.

$$\beta \cap \gamma = \{\{x_4\}, \{x_6\}, \ldots\}.$$

which is essentially a trivial relation.
Therefore,

$$\bigvee(\beta \cap \gamma) = \alpha \vee (\text{trivial relation}) = \alpha.$$

we conclude that

$$(\alpha \vee \beta) \cap \gamma \neq \alpha \vee (\beta \cap \gamma).$$

4. E(X) is not modular:
A lattice L is **modular** if for all $a \leq c$, we have:

$$a \vee (b \wedge c) = (a \vee b) \wedge c.$$

We found $\alpha \subseteq \gamma$, but

$$(\alpha \vee \beta) \wedge \gamma \neq \alpha \vee (\beta \wedge \gamma)$$

$\Rightarrow$ Modular law fails in $E(X)$.
Thus, $E(X)$, the lattice of equivalence relations on X, is not modular.

□

3. Conclusion

In this paper, we study the number of equivalence relations on a finite set X such that $|X/\rho| = r$, where $1 \leq r \leq n$ and $n = |X|$. The number of such equivalence relations is denoted by $S(n, r)$, known as the Stirling numbers of the second kind. We examine their recurrence relations and compute explicit values for $1 \leq r \leq n \leq 6$. The numbers $S(n, r)$ are the strirling number is another type.

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References

1. A. Dawar. The nature and power of fixed point logic with counting. *ACM SIGLOG News*, pages 8–21, 2015.
2. C. J. Gittelson, R. Andrev, and C. Schwab. Optimally adaptive Galerkin methods for random partial differential equations. *Journal of Computational and Applied Mathematics*, 263:189–201, 2014.
3. E. Lughofer, E. Weigl, W. Heidl, C. Eitzinger, and T. Radauer. Integrating new classes on the fly in evolving fuzzy classifier designs and their application in visual inspection. *Applications of Soft Computing*, 35:558–582, 2015.
4. F. Abu Zaid, E. Grädel, M. Grohe, and W. Pakusa. Choiceless polynomial time on structures with small abelian color classes. In *Mathematical Foundations of Computer Science 2014*, pages 50–62. Springer, 2014.
5. F. Meng and X. Chen. A new method for a triangular fuzzy compare wise judgment matrix process based on consistency analysis. *International Journal of Fuzzy Systems*, pages 1–20, 2016.
6. H. T. Banks, J. A. Burns, and E. M. Cliff. Parameter estimation and identification for systems with delays. *SIAM Journal on Control and Optimization*, 19(6), November 1981.
7. H. T. Banks and K. Ito. A unified framework for approximation and inverse problems for distributed parameter systems. *Citric - Theory and Adv. Tech.*, 4:73–90, 1988.
8. H. T. Banks and C. Thompson. Least squares estimation of probability measures in the Prohorov metric frameworks. Technical report, CRSC-TR12-21, N. C. State University, Raleigh, NC, November 2012.
9. I. G. Rosen, S. E. Luczak, and J. Weiss. Blind deconvolution for distributed parameter systems with unbounded input and output and determining blood alcohol concentration from transdermal biosensor data. *Applied Mathematics and Computation*, 213:357–376, 2014.
10. S. Shelke and S. Apte. A fuzzy-based classification scheme for unconstrained handwritten Devanagari character recognition. In *2015 International Conference on Communication, Information & Computing Technology (ICCICT)*, 2015.
11. S. Kubler, J. Robert, W. Derigent, A. Voisin, and Y. Le Traon. A state-of-the-art survey & testbed of fuzzy AHP (F-AHP) applications. *Expert Systems with Applications*, 65:398–422, 2016.
12. P. Tufekci. Prediction of a full load electrical power output of a baseload operated combined cycle power plant using machine learning methods. *International Journal of Electrical Power & Energy Systems*, 60:126–140, 2014.
13. W. Pakusa. Linear equation systems and the search for a logical characterisation of polynomial time. Ph.D. thesis, RWTH Aachen University, 2016.
14. Y. L. He, X.-Z. Wang, and J. Z. Huang. Fuzzy nonlinear regression analysis using a random weight network. *Information Sciences*, 364:222–240, 2016.
15. Z. Dai, I. G. Rosen, C. Wang, N. J. Barnett, and S. E. Luczak. Identifying drinking diary based pharmacokinetic models to calibrate transdermal alcohol biosensor data analysis software. *Mathematical Biosciences and Engineering*, to appear, 2016.

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