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Characterization of Roughness in Gamma Near Algebras

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ABSTRACT: Rough sets offer a way to deal with uncertain or imprecise data. Gamma near algebra (GNA) is a broadening of near algebra (NA) and gamma near ring (GNR). In this paper we introduce the notions of rough sub gamma near algebras (RSGNAs) and rough ideals (RIs) in a GNA and derive their key characteristics. We further investigate the relationship between upper and lower approximations of these rough frameworks and their images under homomorphism.

Key Words: Rough set, gamma near algebra, rough ideal.

Contents

1. Introduction

In situations where information is limited, fuzzy set theory (FST), pioneered by L.A. Zadeh [35], offers a robust framework for decision-making and problem-solving under uncertainty. Rough set theory (RST), pioneered by Pawlak [17], offers a novel mathematical framework for managing uncertain information. It complements existing approaches like probability theory and FST, providing a powerful tool for analyzing data and processing under ambiguity [18]. RST has gained global recognition, attracting significant contributions from researchers [13, 15, 16, 19]. This has led to diverse applications, demonstrating its potential in tackling complex problems characterized by uncertainty and imprecision. RST offers a significant advantage in data analysis by operating without requiring additional information. It relies on a universal set and an equivalence relation, which captures indiscernibility and enables RST to handle uncertain and incomplete data sets. The application of RST to algebraic structures has been a significant area of research. Some studies (see [20, 21]) have focused on rough algebraic structures, while others (see [4, 9, 23, 29, 34]) have extended RST by using algebraic structures in place of the universal set, examining roughness in these contexts.

Extensive research has been conducted on applying RST to various algebraic structures. Biswas [1] explored rough groups (RGs) and rough subgroups (RSs). Kuroki [11] investigated structural properties of RSs and RGs. Kuroki [10] initiated the idea of rough ideals (RIs) in semigroups (SGs). Davvaz [3] applied RST to ring theory, defined rough subrings and RIs concerning an ideal in a ring. Selvan [24] developed the thought of RIs in a semiring.

A near-ring (NR) is an algebraic structure similar to a ring, but with relaxed conditions. Unlike a ring, a near-ring only requires one distributive law to hold, either the left or the right, and its additive group doesn't necessarily have to be commutative. Subha [33] utilized rough set theory to examine ideals in NRs. Mary Nirmala [12] explored RIs in a Rough NR. We recall that a gamma ring (GR) is a type of algebraic structure that generalizes the concept of a ring. It consists of an underlying set with two operations (addition and multiplication) and a set of operators (gamma set) that interact with these operations. Durgadevi [5, 6, 7] focused on rough fuzzy ideals in GRs, presented properties and theoretical results. Satyanarayana [26] presented the idea of Γ-near-ring (GNR), a unified generalization of the two

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concepts NR and the GR. Subha [29, 30, 31, 32] initiated the study of various RIs in GNR. A near algebra (NA) is defined as a NR that allows a field to act as a right operator domain. NAs (see [2, 8, 14, 25]) occur naturally when considering mappings between linear spaces. Rajani [22] pioneered the idea of RIs in a NA. The concept of gamma near algebras (see [28]) extends that of NAs and GNRs.

This study entails the replacement of universal set by a GNR. We present the ideas of rough sub gamma near algebras and RIs in GNAs, investigating their unique properties. Our analysis focuses on the interplay between upper and lower approximations (U&LAs) of these rough structures and their homomorphism images.

Now we introduce the notion of a GNA and related notions.

Definition 1.1 *[27] Let M denote a linear space (LS) on a field F and Γ be a set containing at least one element. Then M is referred to as a GNA on F if there exists a mapping $M \times \Gamma \times M \to M$ that meets the following criteria for all $\xi_1, \xi_2, \xi_3 \in M, \alpha_1, \alpha_2 \in \Gamma$ and $k \in F$:*
(i) $(\xi_1\alpha_1\xi_2)\alpha_2\xi_3 = \xi_1\alpha_1(\xi_2\alpha_2\xi_3)$,
(ii) $(\xi_1 + \xi_2)\alpha_1\xi_3 = \xi_1\alpha_1\xi_3 + \xi_2\alpha_1\xi_3$,
(iii) $(k\xi_1)\alpha_1\xi_2 = k(\xi_1\alpha_1\xi_2)$.

Throughout this manuscript, M, M' are GNAs on a field F.

Definition 1.2 *[27] A non-empty subset S of a GNA M is a sub GNA (SGNA) of M if for all $\xi_1, \xi_2 \in S, k \in F$ and $\alpha_1 \in \Gamma$:*
(i) $\xi_1 - \xi_2, k\xi_1 \in S$,
(ii) $\xi_1\alpha_1\xi_2 \in S$.

Definition 1.3 *[27] The linear subspace I of the LS M is termed as a GNA ideal if for all $\xi_1, \xi_2 \in M, i \in I$ and $\alpha_1 \in \Gamma$:*
(i) $\xi_2\alpha_1(\xi_1 + i) - \xi_2\alpha_1\xi_1 \in I$,
(ii) $i\alpha_1\xi_1 \in I$.

Definition 1.4 *[27] A mapping $\zeta : M \to M'$ is called a GNA homomorphism if the subsequent criteria are fulfilled for all $\xi_1, \xi_2 \in M, k \in F, \alpha_1 \in \Gamma$:*
(i) $\zeta(\xi_1 + \xi_2) = \zeta(\xi_1) + \zeta(\xi_2)$,
(ii) $\zeta(k\xi_1) = k\zeta(\xi_1)$,
(iii) $\zeta(\xi_1\alpha_1\xi_2) = \zeta(\xi_1)\alpha_1\zeta(\xi_2)$.

Remark 1.1 *For basic definitions of rough sets, refer to [17, 18].*

2. Rough Sub Gamma Near Algebras

Here we outline and investigate rough sub gamma near algebras, explore their unique characteristics. We examine the interconnections among upper and lower rough sub gamma near algebras and the U&LAs of their homomorphism images.

Definition 2.1 *A binary relation θ on M is an equivalence relation (ER) if it fulfills reflexivity, symmetry, and transitivity. For θ being an ER on M, the equivalence class of $\xi_1 \in M$ consists of the set $\{\xi_2 \in M \mid (\xi_1, \xi_2) \in \theta\}$. We express it as $[\xi_1]_\theta$.*

Definition 2.2 *An ER θ on M is termed as a full congruence relation (FCR) if $(\xi_1, \xi_2) \in \theta$ implies $(\xi_1 + \xi_3, \xi_2 + \xi_3), (k\xi_1, k\xi_2), (\xi_1\alpha_1\xi_3, \xi_2\alpha_1\xi_3)$ and $(\xi_3\alpha_1\xi_1, \xi_3\alpha_1\xi_2) \in \theta$ for all $k \in F, \alpha_1 \in \Gamma$ and $\xi_3 \in M$.*

Proposition 2.1 *Let θ be a FCR on M. Then $(\xi_1, \xi_2) \in \theta$ and $(\xi_1', \xi_2') \in \theta$ implies $(\xi_1 + \xi_1', \xi_2 + \xi_2') \in \theta$ and $(\xi_1\alpha_1\xi_1', \xi_2\alpha_1\xi_2') \in \theta$ for every $\xi_1, \xi_1', \xi_2, \xi_2' \in M, \alpha_1 \in \Gamma$.*

Theorem 2.1 *Let θ be a FCR on M. If $\xi_1, \xi_2 \in M, k \in F$ and $\alpha_1 \in \Gamma$, then*
(i) $[\xi_1]_\theta + [\xi_2]_\theta = [\xi_1 + \xi_2]_\theta$,
(ii) $[k\xi_1]_\theta = k[\xi_1]_\theta$,
(iii) $[\xi_1]_\theta\alpha_1[\xi_2]_\theta \subseteq [\xi_1\alpha_1\xi_2]_\theta$.

Proof: The proofs of (i) and (ii) follow directly.
(iii) Let $\xi_3 = \xi_1'\alpha_1\xi_2' \in [\xi_1]_\theta\alpha_1[\xi_2]_\theta$. Then $\xi_1' \in [\xi_1]_\theta$ and $\xi_2' \in [\xi_2]_\theta$. This implies $(\xi_1', \xi_1) \in \theta$ and $(\xi_2', \xi_2) \in \theta$. Since θ is a FCR, $(\xi_1'\alpha_1\xi_2', \xi_1\alpha_1\xi_2) \in \theta$. Thus $\xi_1'\alpha_1\xi_2' \in [\xi_1\alpha_1\xi_2]_\theta$, proving $[\xi_1]_\theta\alpha_1[\xi_2]_\theta \subseteq [\xi_1\alpha_1\xi_2]_\theta$. □

Definition 2.3 *A FCR θ on M is said to be complete if $[\xi_1]_\theta\alpha_1[\xi_2]_\theta = [\xi_1\alpha_1\xi_2]_\theta$ for all $\xi_1, \xi_2 \in M$.*

Definition 2.4 *Let θ be a FCR on M and $S \subseteq M$. Consequently, the sets, $LApr_\theta(S) = \{\xi_1 \in M \mid [\xi_1]_\theta \subseteq S\}$ and $UApr_\theta(S) = \{\xi_1 \in M \mid [\xi_1]_\theta \cap S \neq \phi\}$ are designated in the same order as the θ-lower and θ-upper approximations of S.*

Remark 2.1 *For any set S containing at least one element of M, we have $LApr_\theta(S) \subseteq S \subseteq UApr_\theta(S)$.*

Definition 2.5 *For any non-empty subset S of M, $Apr_\theta(S) = (LApr_\theta(S), UApr_\theta(S))$ is referred to as a rough set (RS) with reference to θ if $LApr_\theta(S) \neq UApr_\theta(S)$.*

Definition 2.6 *A non-empty subset S of M is termed as an upper rough sub gamma near algebra (URSGNA) of M if $UApr_\theta(S)$ is a SGNA of M and a lower rough sub gamma near algebra (LRSGNA) of M if $LApr_\theta(S)$ is a SGNA of M.*

Definition 2.7 *Let S be a subset of M and $(LApr_\theta(S), UApr_\theta(S))$, a RS. If $(LApr_\theta(S), UApr_\theta(S))$ are SGNAs of M, then we call $(LApr_\theta(S), UApr_\theta(S))$ a rough sub gamma near algebra(RSGNA).*

Example 2.1 *Consider the linear space $M = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\upsilon_1, \upsilon_2) \mid \upsilon_1, \upsilon_2 \in \mathbb{Z}_2\}$ on the field $\mathbb{Z}_2$. Let $\Gamma = \{\alpha, \beta\}$. Construct a mapping $M \times \Gamma \times M \to M$ as follows:*

α	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

β	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

It is clear that M is a GNA over the field $\mathbb{Z}_2$. Define θ on $\mathbb{Z}_2 \times \mathbb{Z}_2$ as $(\upsilon_1, \upsilon_2)\theta(\upsilon_1', \upsilon_2')$ if and only if $\upsilon_1 + \upsilon_2 = \upsilon_1' + \upsilon_2'(mod2)$. Then θ is a FCR on $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the congruence classes $C_1 = \{(0,0), (1,1)\}$ and $C_2 = \{(0,1), (1,0)\}$. Let $S = \{(0,0), (0,1), (1,1)\}$. Then $UApr_\theta(S) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $LApr_\theta(S) = \{(0,0), (1,1)\}$ are SGNAs of M. Thus $(LApr_\theta(S), UApr_\theta(S))$ is a RSGNA.

Theorem 2.2 *Let θ be a FCR on M. If S is a SGNA of M, then $UApr_\theta(S)$ is a SGNA of M.*

Proof: Hypothesize that $\xi_1, \xi_2 \in UApr_\theta(S), \alpha_1 \in \Gamma$ and $k \in F$. Consequently $[\xi_1]_\theta \cap S \neq \phi$ and $[\xi_2]_\theta \cap S \neq \phi$. Hence we can find $\xi_1' \in [\xi_1]_\theta \cap S$ and $\xi_2' \in [\xi_2]_\theta \cap S$.Thus $\xi_1' \in [\xi_1]_\theta, \xi_1' \in S$ and $\xi_2' \in [\xi_2]_\theta, \xi_2' \in S$. Due to the fact that S is a SGNA of $M, \xi_1' - \xi_2' \in S, k\xi_1' \in S, \xi_1'\alpha_1\xi_2' \in S$. Now, $\xi_1' - \xi_2' \in [\xi_1]_\theta - [\xi_2]_\theta = [\xi_1 - \xi_2]_\theta$. Hence $\xi_1' - \xi_2' \in [\xi_1 - \xi_2]_\theta \cap S$, which implies $[\xi_1 - \xi_2]_\theta \cap S \neq \phi$ or $\xi_1 - \xi_2 \in UApr_\theta(S)$.
Also $k\xi_1' \in k[\xi_1]_\theta = [k\xi_1]_\theta$. Therefore $k\xi_1' \in [k\xi_1]_\theta \cap S$ and hence $[k\xi_1]_\theta \cap S \neq \phi$ or $k\xi_1 \in UApr_\theta(S)$. We have $(\xi_1', \xi_1) \in \theta$ and $(\xi_2', \xi_2) \in \theta$. Because θ is a FCR, $(\xi_1'\alpha_1\xi_2', \xi_1\alpha_1\xi_2) \in \theta$ and hence $\xi_1'\alpha_1\xi_2' \in [\xi_1\alpha_1\xi_2]_\theta$. Thus $\xi_1'\alpha_1\xi_2' \in [\xi_1\alpha_1\xi_2]_\theta \cap S$ or $[\xi_1\alpha_1\xi_2]_\theta \cap S \neq \phi$. So $\xi_1\alpha_1\xi_2 \in UApr_\theta(S)$. Hence $UApr_\theta(S)$ is a SGNA of M. □

Theorem 2.3 *Let θ be a complete congruence relation (CCR) on M. If $LApr_\theta(S)$ is a set consisting of at least one element and S is a SGNA of M, then $LApr_\theta(S)$ is a SGNA of M.*

Proof: Let $\xi_1, \xi_2 \in LApr_\theta(S), \alpha_1 \in \Gamma$ and $k \in F$. Then $[\xi_1]_\theta \subseteq S$ and $[\xi_2]_\theta \subseteq S$. Now, $[\xi_1 - \xi_2]_\theta = [\xi_1]_\theta - [\xi_2]_\theta \subseteq S$. This implies $\xi_1 - \xi_2 \in LApr_\theta(S)$.
Also, $[k\xi_1]_\theta = k[\xi_1]_\theta \subseteq kS \subset S$ and hence $k\xi_1 \in LApr_\theta(S)$.
Since θ is a CCR, $[\xi_1]_\theta\alpha_1[\xi_2]_\theta = [\xi_1\alpha_1\xi_2]_\theta \subseteq S$. Thus $\xi_1\alpha_1\xi_2 \in LApr_\theta(S)$ and therefore $LApr_\theta(S)$ is a SGNA of M. □

Theorem 2.4 *Let $\zeta : M \to M'$ be an onto homomorphism. Let ρ be a FCR on M' and S be contained in M. Then*
(i) $\theta = \{(\xi_1, \xi_2) \in M \times M' \mid (\zeta(\xi_1), \zeta(\xi_2)) \in \rho\}$ *is a FCR on M.*
(ii) If ζ is injective and ρ is complete, then θ is complete.
(iii) $\zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$.
(iv) $\zeta(LApr_\theta(S)) \subseteq LApr_\rho(\zeta(S))$. *If ζ is one-one, then* $\zeta(LApr_\theta(S)) = LApr_\rho(\zeta(S))$.

Proof: (i) Let $(\xi_1, \xi_2) \in \theta, \xi_3 \in M, \alpha_1 \in \Gamma$ and $k \in F$. Then $(\zeta(\xi_1), \zeta(\xi_2)) \in \rho$ and $\zeta(\xi_3) \in M'$. Since ρ is a FCR on M', we get $(\zeta(\xi_1)+\zeta(\xi_3), \zeta(\xi_2)+\zeta(\xi_3)) \in \rho, (k\zeta(\xi_1), k\zeta(\xi_2)) \in \rho, (\zeta(\xi_1)\alpha_1\zeta(\xi_3), \zeta(\xi_2)\alpha_1\zeta(\xi_3)) \in \rho$ and $(\zeta(\xi_3)\alpha_1\zeta(\xi_1), \zeta(\xi_3)\alpha_1\zeta(\xi_2)) \in \rho$. Then $(\zeta(\xi_1 + \xi_3), \zeta(\xi_2 + \xi_3)) \in \rho, (\zeta(k\xi_1), \zeta(k\xi_2)) \in \rho, (\zeta(\xi_1\alpha_1\xi_3),$ $\zeta(\xi_2\alpha_1\xi_3)) \in \rho$ and $(\zeta(\xi_3\alpha_1\xi_1), \zeta(\xi_3\alpha_1\xi_2)) \in \rho$. Hence $(\xi_1+\xi_3, \xi_2+\xi_3) \in \theta, (k\xi_1, k\xi_2) \in \theta, (\xi_1\alpha_1\xi_3, \xi_2\alpha_1\xi_3) \in$ θ and $(\xi_3\alpha_1\xi_1, \xi_3\alpha_1\xi_2) \in \theta$. Consequently θ is a FCR on M.
(ii) We have $[\xi_1]_\theta\alpha_1[\xi_2]_\theta \subseteq [\xi_1\alpha_1\xi_2]_\theta$ and we show that $[\xi_1\alpha_1\xi_2]_\theta \subseteq [\xi_1]_\theta\alpha_1[\xi_2]_\theta$ to prove θ is complete. Let $\xi_3 \in [\xi_1\alpha_1\xi_2]_\theta$. Then $(\xi_1\alpha_1\xi_2, \xi_3) \in \theta$. By the definition, $(\zeta(\xi_1\alpha_1\xi_2), \zeta(\xi_3)) \in \rho$. Thus $\zeta(\xi_3) \in [\zeta(\xi_1\alpha_1\xi_2)]_\rho = [\zeta(\xi_1)\alpha_1\zeta(\xi_2)]_\rho = [\zeta(\xi_1)]_\rho\alpha_1[\zeta(\xi_2)]_\rho$. Hence we get $\xi_1', \xi_2' \in M$ so that $\zeta(\xi_3) =$ $\zeta(\xi_1')\alpha_1\zeta(\xi_2') = \zeta(\xi_1'\alpha_1\xi_2')$. Since ζ is one-one, we have $\xi_3 = \xi_1'\alpha_1\xi_2'$ where $\xi_1' \in [\xi_1]_\theta$ and $\xi_2' \in [\xi_2]_\theta$. Thus $z \in [\xi_1]_\theta\alpha_1[\xi_2]_\theta$. Hence θ is complete.
(iii) Let $\xi_2 \in \zeta(UApr_\theta(S))$. Then we can find $\xi_1 \in UApr_\theta(S)$ so that $\zeta(\xi_1) = \xi_2$. Now $\xi_1 \in UApr_\theta(S)$, implies $[\xi_1]_\theta \cap S \neq \phi$. Let $\xi_1' \in [\xi_1]_\theta \cap S$. Then $(\xi_1', \xi_1) \in \theta$ and $\xi_1' \in S$. By the definition, $(\zeta(\xi_1'), \zeta(\xi_1)) \in \rho$ and $\zeta(\xi_1') \in \zeta(S)$. Consequently, $\zeta(\xi_1') \in [\zeta(\xi_1)]_\rho$ and hence $\zeta(\xi_1') \in [\zeta(\xi_1)]_\rho \cap \zeta(S)$. Thus $[\zeta(\xi_1)]_\rho \cap \zeta(S) \neq \phi$. Hence $\xi_2 = \zeta(\xi_1) \in UApr_\rho(\zeta(S))$ and therefore $\zeta(UApr_\theta(S)) \subseteq UApr_\rho(\zeta(S))$.
Conversely, let $\xi_2 \in UApr_\rho(\zeta(S))$. Then $[\xi_2]_\rho \cap \zeta(S) \neq \phi$. Also there is $\xi_1 \in M$ satisfying $\zeta(\xi_1) = \xi_2$. Therefore $[\zeta(\xi_1)]_\rho \cap \zeta(S) \neq \phi$. Let $\zeta(\xi_1') \in [\zeta(\xi_1)]_\rho \cap \zeta(S)$ for certain $\xi_1' \in S$. Then $(\zeta(\xi_1'), \zeta(\xi_1)) \in \rho$. This follows that $(\xi_1', \xi_1) \in \theta$ and hence $\xi_1' \in [\xi_1]_\theta$. Thus $[\xi_1]_\theta \cap S \neq \phi$. Hence $\xi_1 \in UApr_\theta(S)$ and so $\xi_2 = \zeta(\xi_1) \in \zeta(UApr_\theta(S))$. Thus $UApr_\rho(\zeta(S)) \subseteq \zeta(UApr_\theta(S))$ and $\zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$.
(iv) Let $\xi_2 \in \zeta(LApr_\theta(S))$. Then we can find $\xi_1 \in LApr_\theta(S)$ satisfying $\zeta(\xi_1) = \xi_2$. Now $\xi_1 \in LApr_\theta(S)$, implies $[\xi_1]_\theta \subseteq S$. Let $\xi_2' \in [\xi_2]_\theta$. Then there is $\xi_1' \in M$ so that $\zeta(\xi_1') = \xi_2$ and $\zeta(\xi_1') \in [\zeta(\xi_1)]_\rho$. Thus $\xi_1' \in [\xi_1]_\theta \subseteq S$ and therefore $\xi_2' = \zeta(\xi_1') \in \zeta(S)$. Thus $[\xi_2]_\theta \subseteq \zeta(S)$ or $\xi_2 \in LApr_\rho(\zeta(S))$. So $\zeta(LApr_\theta(S)) \subseteq LApr_\rho(\zeta(S))$.
Suppose ζ is one-one. Let $\xi_2 \in LApr_\rho(\zeta(S))$. Then we can find $\xi_1 \in M$ so that $\zeta(\xi_1) = \xi_2$ and $[\zeta(\xi_1)]_\rho \subseteq \zeta(S)$. Let $\xi_1' \in [\xi_1]_\theta$. Then $\zeta(\xi_1') \in [\zeta(\xi_1]_\rho \subseteq \zeta(S)$ and hence $\xi_1' \in S$. Thus $[\xi_1]_\theta \subseteq S$ implies, $\xi_1 \in LApr_\theta(S)$. Then $\xi_2 = \zeta(\xi_1) \in \zeta(LApr_\theta(S))$ and accordingly $LApr_\rho(\zeta(S)) \subseteq \zeta(LApr_\theta(S))$.Thus $\zeta(LApr_\theta(S)) = LApr_\rho(\zeta(S))$. □

Remark 2.2 *From here onwards, θ and ρ are FCRs on M and M' respectively, and S and I are subsets of M containing at least one element.*

Theorem 2.5 *Let $\zeta : M \to M'$ be an epimorphism. Then $UApr_\theta(S)$ is a SGNA of M if and only if $UApr_\rho(\zeta(S))$ is a SGNA of M'.*

Proof: Let $UApr_\theta(S)$ be a SGNA of M. Let $\xi_1', \xi_2' \in UApr_\rho(\zeta(S)), k \in F, \alpha_1 \in \Gamma$.Then $\xi_1', \xi_2' \in$ $\zeta(UApr_\theta(S))$. So we can find $\xi_1, \xi_2 \in UApr_\theta(S)$ satisfying $\xi_1' = \zeta(\xi_1), \xi_2' = \zeta(\xi_2)$. Since $UApr_\theta(S)$ is a SGNA of $M, \xi_1 - \xi_2, k\xi_1, \xi_1\alpha_1\xi_2 \in UApr_\theta(S)$.
Now, $\xi_1' - \xi_2' = \zeta(\xi_1) - \zeta(\xi_2) = \zeta(\xi_1 - \xi_2) \in \zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$.
$k\xi_1' = k\zeta(\xi_1) = \zeta(k\xi_1) \in \zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$ and $\xi_1'\alpha_1\xi_2' = \zeta(\xi_1)\alpha_1\zeta(\xi_2) = \zeta(\xi_1\alpha_1\xi_2) \in$ $\zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$.
Hence $UApr_\rho(\zeta(S))$ is a SGNA of M'.
In contrast, suppose that $UApr_\rho(\zeta(S))$ is a SGNA of M'. Let $\xi_1, \xi_2 \in UApr_\theta(S), k \in F, \alpha_1 \in \Gamma$. Thus $\xi_1' = \zeta(\xi_1), \xi_2' = \zeta(\xi_2) \in \zeta(UApr_\theta(S)) = UApr_\rho(\zeta(S))$.
Since $UApr_\rho(\zeta(S))$ is a SGNA of $M', \xi_1' - \xi_2', k\xi_1', \xi_1'\alpha_1\xi_2' \in UApr_\rho(\zeta(S))$.
Now, $\zeta(\xi_1 - \xi_2) = \zeta(\xi_1) - \zeta(\xi_2) = \xi_1' - \xi_2' \in UApr_\rho(\zeta(S)) = \zeta(UApr_\theta(S))$.
$\zeta(k\xi_1) = k\zeta(\xi_1) = k\xi_1' \in UApr_\rho(\zeta(S)) = \zeta(UApr_\theta(S))$.
$\zeta(\xi_1\alpha_1\xi_2) = \zeta(\xi_1)\alpha_1\zeta(\xi_2) = \xi_1'\alpha_1\xi_2' \in UApr_\rho(\zeta(S)) = \zeta(UApr_\theta(S))$. There exists $\upsilon_1, \upsilon_2, \upsilon_3 \in UApr_\theta(S)$

such that $\xi_1 - \xi_2 = \upsilon_1, k\xi_1 = \upsilon_2, \xi_1\alpha_1\xi_2 = \upsilon_3$.Thus we have $[\upsilon_1]_\theta \cap S \neq \phi$,$[\upsilon_2]_\theta \cap S \neq \phi$ and $[\upsilon_3]_\theta \cap S \neq \phi$. Also we have $\xi_1 - \xi_2 \in [\upsilon_1]_\theta, k\xi_1 \in [\upsilon_2]_\theta, \xi_1\alpha_1\xi_2 \in [\upsilon_3]_\theta$. Hence $[\xi_1 - \xi_2]_\theta \cap S \neq \phi, [k\xi_1]_\theta \cap S \neq \phi, [\xi_1\alpha_1\xi_2]_\theta \cap S \neq \phi$ and therefore $\xi_1 - \xi_2, k\xi_1, \xi_1\alpha_1\xi_2 \in UApr_\theta(S)$. Thus $UApr_\theta(S)$ is a SGNA of M. □

Theorem 2.6 *Let $\zeta : M \to M'$ be an isomorphism. Then $LApr_\theta(S)$ is a SGNA of M if and only if $LApr_\rho(\zeta(S))$ is a SGNA of M'.*

Proof: Analogous to the preceding theorem. □

3. Rough Ideals in Gamma Near Algebras

We present definitions of rough ideals, explore their features and focus on their homomorphism properties in this part.

Definition 3.1 *A non-empty subset I of M is known as an upper rough ideal (URI) of M if $UApr_\theta(I)$ is an ideal of M and a lower rough ideal (LRI) of M if $LApr_\theta(I)$ is an ideal of M.*

Definition 3.2 *Let I be a subset of M and $(LApr_\theta(I), UApr_\theta(I))$, a RS. If $(LApr_\theta(I), UApr_\theta(I))$ are ideals of M, then we designate $(LApr_\theta(I), UApr_\theta(I))$ as a rough ideal (RI).*

Example 3.1 *Consider the gamma near algebra from example 3.1. Let $I = \{(0,0),(0,1),(1,1)\}$. Then $UApr_\theta(I) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $LApr_\theta(I) = \{(0,0),(1,1)\}$ are ideals of M. Thus $(LApr_\theta(I), UApr_\theta(I))$ is a RI.*

Theorem 3.1 *If I is an ideal of M, then I is a RI of M.*

Proof: Suppose that $\xi'_1, \xi'_2 \in UApr_\theta(I), k \in F, \alpha_1 \in \Gamma$ and $\xi_1, \xi_2 \in M$. This follows that $[\xi'_1]_\theta \cap I \neq \phi$ and $[\xi'_2]_\theta \cap I \neq \phi$. Hence we can find $i \in [\xi'_1]_\theta \cap I$ and $j \in [\xi'_2]_\theta \cap I$. So $i \in [\xi'_1]_\theta, i \in I$ and $j \in [\xi'_2]_\theta, j \in I$. As I is an ideal of $M, i - j \in I, ki \in I, i\alpha_1\xi_1 \in I$ and $\xi_2\alpha_1(\xi_1 + i) - \xi_2\alpha_1\xi_1 \in I$.
Now, $i - j \in [\xi'_1]_\theta - [\xi'_2]_\theta = [\xi'_1 - \xi'_2]_\theta$. Thus $i - j \in [\xi'_1 - \xi'_2]_\theta \cap I$, which implies $[\xi'_1 - \xi'_2]_\theta \cap I \neq \phi$ or $\xi'_1 - \xi'_2 \in UApr_\theta(I)$.
Also, $ki \in k[\xi'_1]_\theta = [k\xi'_1]_\theta$. Therefore $ki \in [k\xi'_1]_\theta \cap I$ and hence $[k\xi'_1]_\theta \cap I \neq \phi$ or $k\xi'_1 \in UApr_\theta(I)$.
Since $(i, \xi'_1) \in \theta$, then $(i\alpha_1\xi_1, \xi'_1\alpha_1\xi_1) \in \theta$ or $i\alpha_1\xi_1 \in [\xi'_1\alpha_1\xi_1]_\theta$. Hence $i\alpha_1\xi_1 \in [\xi'_1\alpha_1\xi_1]_\theta \cap I$ and therefore $[\xi'_1\alpha_1\xi_1]_\theta \cap I \neq \phi$ or $\xi'_1\alpha_1\xi_1 \in UApr_\theta(I)$.
Now $(i, \xi'_1) \in \theta$ implies $(i+\xi_1, \xi'_1+\xi_1) \in \theta$ which in turn implies $(\xi_2\alpha_1(i+\xi_1), \xi_2\alpha_1(\xi'_1+\xi_1)) \in \theta$ and hence $(\xi_2\alpha_1(i+\xi_1) - \xi_2\alpha_1\xi_1, \xi_2\alpha_1(\xi'_1+\xi_1) - \xi_2\alpha_1\xi_1) \in \theta$. Thus $\xi_2\alpha_1(i+\xi_1) - \xi_2\alpha_1\xi_1 \in [\xi_2\alpha_1(\xi'_1+\xi_1) - \xi_2\alpha_1\xi_1]_\theta$ and hence $[\xi_2\alpha_1(\xi'_1 + \xi_1) - \xi_2\alpha_1\xi_1]_\theta \cap I \neq \phi$. Therefore $\xi_2\alpha_1(\xi'_1 + \xi_1) - \xi_2\alpha_1\xi_1 \in UApr_\theta(I)$ and thus $UApr_\theta(I)$ is an ideal of M. □

Theorem 3.2 *Let I be an ideal of M. If $LApr_\theta(I)$ is a set containing at least one element, then it is equal to I.*

Proof: Since $LApr_\theta(I) \subseteq I$, we now demonstrate that $I \subseteq LApr_\theta(I)$. Let $\xi_1 \in LApr_\theta(I)$ and $\xi'_1 \in I$. Then $[0]_\theta = [\xi_1 - \xi_1]_\theta = [\xi_1]_\theta + [-\xi_1]_\theta \subseteq I + I = I$ and hence $a + [0]_\theta \subseteq a + I = I$. We have $\xi_1 \in \xi'_1 + [0]_\theta$ iff $\xi_1 - \xi'_1 \in [0]_\theta$ iff $(\xi_1 - \xi'_1, 0) \in \theta$ iff $(\xi_1, \xi'_1) \in \theta$ iff $\xi_1 \in [\xi'_1]_\theta$ and hence $[\xi'_1]_\theta \subseteq I$, which implies $\xi'_1 \in LApr_\theta(I)$. Hence $I \subseteq LApr_\theta(I)$ and thus $LApr_\theta(I) = I$. □

Theorem 3.3 *Let I be an ideal of M and $LApr_\theta(I)$ be a set that is non-empty. Then $(LApr_\theta(I), UApr_\theta(I))$ is a RI of M.*

Proof: It is straightforward. □

Theorem 3.4 *Let I_1 and I_2 be ideals of M and $LApr_\theta(I_1 \cap I_2)$is a set that is non-empty. Then $(LApr_\theta(I_1 \cap I_2), UApr_\theta(I_1 \cap I_2))$ is a RI of M.*

Proof: It follows directly. □

Theorem 3.5 *Let $\zeta : M \to M'$ be an epimorphism. Then $UApr_\theta(I)$ is an ideal of M if and only if $UApr_\rho(\zeta(I))$ is an ideal of M'.*

Proof: Let $UApr_\theta(I)$ be an ideal of M. Let $i', j' \in UApr_\rho(\zeta(I)), k \in F, \alpha_1 \in \Gamma$ and $\xi_1', \xi_2' \in M'$. Then $i', j' \in \zeta(UApr_\theta(I))$. So there exist $i, j \in UApr_\theta(I)$ so that $i' = \zeta(i), j' = \zeta(j)$. As ζ is onto, we get $\xi_1, \xi_2 \in M$ so that $\xi_1' = \zeta(\xi_1), \xi_2' = \zeta(\xi_2)$. As $UApr_\theta(I)$ is an ideal of $M, i-j, ki, i\alpha_1\xi_1, \xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1 \in UApr_\theta(I)$.
Now, $i' - j' = \zeta(i) - \zeta(j) = \zeta(i-j) \in \zeta(UApr_\theta(I)) = UApr_\rho(\zeta(I))$.
Also, $ki' = k\zeta(i) = \zeta(ki) \in \zeta(UApr_\theta(I)) = UApr_\rho(\zeta(I))$.
We have, $i'\alpha_1\xi_1' = \zeta(i)\alpha_1\zeta(\xi_1) = \zeta(i\alpha_1\xi_1) \in \zeta(UApr_\theta(I)) = UApr_\rho(\zeta(I))$.
Finally, $\xi_2'\alpha_1(\xi_1' + i') - \xi_2'\alpha_1\xi_1' = \zeta(\xi_2)\alpha_1(\zeta(\xi_1) + \zeta(i)) - \zeta(\xi_2)\alpha_1\zeta(\xi_1) = \zeta(\xi_2\alpha_1(\xi_1+i)) - \zeta(\xi_2\alpha_1\xi_1) = \zeta(\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1) \in \zeta(UApr_\theta(I)) = UApr_\rho(\zeta(I))$.
Hence $UApr_\rho(\zeta(I))$ is an ideal of M'.
Conversely, presume that $UApr_\rho(\zeta(I))$ is an ideal of M'. Let $i, j \in UApr_\theta(I), k \in F, \alpha_1 \in \Gamma$ and $\xi_1, \xi_2 \in M$. Then $i' = \zeta(i), j' = \zeta(j) \in \zeta(UApr_\theta(I)) = UApr_\rho(\zeta(I))$ and $\xi_1' = \zeta(\xi_1), \xi_2' = \zeta(\xi_2) \in M'$.
Since $UApr_\rho(\zeta(I))$ is an ideal of $M', i'-j', ki', i'\alpha_1\xi_1'$ and $\xi_2'\alpha_1(\xi_1'+i') - \xi_2'\alpha_1\xi_1' \in UApr_\rho(\zeta(I))$.
Now, $\zeta(i-j) = \zeta(i) - \zeta(j) = i' - j' \in UApr_\rho(\zeta(I)) = \zeta(UApr_\theta(I))$.
$\zeta(ki) = k\zeta(i) = ki' \in UApr_\rho(\zeta(I)) = \zeta(UApr_\theta(I))$.
$\zeta(i\alpha_1\xi_1) = \zeta(i)\alpha_1\zeta(\xi_1) = i'\alpha_1\xi_1' \in UApr_\rho(\zeta(I)) = \zeta(UApr_\theta(I))$.
Also, $\zeta(\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1) = \zeta(\xi_2)\alpha_1(\zeta(\xi_1) + \zeta(i)) - \zeta(\xi_2)\alpha_1\zeta(\xi_1) = \xi_2'\alpha_1(\xi_1'+i') - \xi_2'\alpha_1\xi_1' \in UApr_\rho(\zeta(I)) = \zeta(UApr_\theta(I))$.
Thus there exist $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in UApr_\theta(I)$ such that $i - j = \upsilon_1, ki = \upsilon_2, i\alpha_1\xi_1 = \upsilon_3$ and $\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1 = \upsilon_4$. Thus we have $[\upsilon_1]_\theta \cap I \neq \phi, [\upsilon_2]_\theta \cap I \neq \phi, [\upsilon_3]_\theta \cap I \neq \phi$ and $[\upsilon_4]_\theta \cap I \neq \phi$.
Also we have $i - j \in [\upsilon_1]_\theta, ki \in [\upsilon_2]_\theta, i\alpha_1\xi_1 \in [\upsilon_3]_\theta$ and $\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1 \in [\upsilon_4]_\theta$.
Hence $[i-j]_\theta \cap I \neq \phi, [ki]_\theta \cap I \neq \phi, [i\alpha_1\xi_1]_\theta \cap I \neq \phi$ and $[\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1]_\theta \cap I \neq \phi$ and therefore $i - j, ki, i\alpha_1\xi_1$ and $\xi_2\alpha_1(\xi_1+i) - \xi_2\alpha_1\xi_1 \in UApr_\theta(I)$.
Thus $UApr_\theta(I)$ is an ideal of M. □

Theorem 3.6 *Let $\zeta : M \to M'$ be an isomorphism. Then $LApr_\theta(I)$ is an ideal of M if and only if $LApr_\rho(\zeta(I))$ is an ideal of M'.*

Proof: Same as above theorem. □

4. Conclusion

This study initiated and examined the concepts of RSGNAs and RIs within GNAs. We studied their properties, illustrated our results with relevant examples, and explored the connections between U&LAs of these rough structures and their homomorphism images.

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