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Some Weak Separation Axioms via $(1,2)S_\beta$-Open Sets in Bitopological Spaces

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ABSTRACT: In topology, separation axioms establish the fundamental criteria for the degree of separation between different subsets. Weak separation axioms, an expansion of separation axioms offer a finer level of distinction between subsets. In 1963, Levine and Kelly initiated the notions of semi-open sets and bitopological spaces respectively. After that, many papers have been published to extend topological concepts to bitopological spaces. In this paper, we define a new class of semi-open sets namely (1, 2)S_β - open sets by using (1, 2)semi - open sets in bitopological spaces. Also, we define and study (1, 2)S_β-separation axioms in biopology. We present some characterizations of weak separation axioms in bitopological spaces and the invariance properties by using $(1,2)S_\beta$- open sets. We compare some weak separation axioms among themselves and establish that the two pairs (1, 2)S_β-R_T space and (1, 2)S_β-T_{YS} are dependent on each other in general topology and independent on each other in bitopological spaces.

Keywords: (1, 2)semi-open set, (1, 2)S_β-open set, (1, 2)S_β-kernel, (1, 2)S_β-R_D, (1, 2)S_β-R_T, (1, 2)S_β-R_{YS}, (1, 2)S_β-T_{YS}, (1, 2)S_β-R_Y.

Contents

1. Introduction

Open sets are one of the most important keys to topology. Many researchers have developed different versions of open sets including their weaker and stronger versions. The first development was done by Levine [12] in 1963, where he presented the notions of semi-open set, semi-closed set and semi-continuity of functions in topological spaces. In the same year, Kelly [8] initiated the systematic study of bitopology which is a triple (X,τ,σ), where X is a non-empty set together with two distinct topologies τ, σ on X. Separation axioms are fundamental tools in the study of topology, providing a framework to distinguish between points and sets with a topological space. These axioms of function denoted by the symbols T_0, T_1, T_2, T_3, T_4, T_5, T_6 describe various levels of 'separability' and form the basis for understanding many important topological properties such as continuity, compactness and convergence.

In recent decades, the concept of bitopological spaces has emerged as a powerful extension of classical topology. In 1914, Hausdorff introduced T_2 axiom in his work, which required that two distinct points in a space can be separated by disjoint sets. In 1920s - 1930s, Kuratowski, Alexandrov, Frechet and others were developed the types of weak separation axioms in topological spaces. In 1943, Shanin [16] offered a new weak separation axiom called R_0. A topological space is R_0 of every open set contains the closure of each of its singleton. In 1982, Mashour et al. introduced the concept of separation axioms in bitopological spaces. In 1983, Monsef [1] introduced the notion of β-open sets in topological spaces. Kelly [8] initiated the study of separation properties in bitopological spaces. In 2012, Swidi [17] and Mohammed [17] introduced separation axiom via kernel set in topological spaces. A kernel of a set is the intersection of all open subsets of A. In 2013, Khalaf and Ahmed [3] introduced S_β-open sets in topological spaces. A semi-open subset A of a topological space $(X,\, \tau)$ is said to be S_β-open if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$. In 2009, Raja Rajeswari et al. [15] introduced the concept of ultra-separation axioms by using α-open sets in bitopological spaces. The results obtained

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by the authors in the paper motivate us to define and study the weak separation axioms by using (1, 2)S_β-open sets in bitopological spaces. The motive for drafting this paper is to anticipate a new category of open sets in the frame of bitopological spaces and we simplify the path for many future articles on this topic.

This paper is structured as follows: In the introduction, we give a birds eye view about the development of various concepts relevant to the study, introduced by several researchers in topology and bitopology over a period of time. In Section 2, we recall the fundamental definitions, propositions and theorems which are used in the subsequent sequel. In Section 3, we define the concept of (1, 2)S_β-separation axioms and present the main properties of this class and establish its relationship with other weak separation axioms. In addition, we study the weak separation axioms such as (1, 2)S_β-R_D, (1, 2)S_β-R_T, (1, 2)S_β-R_{YS}, (1, 2)S_β-T_{YS}, (1, 2)S_β-R_Y by using (1, 2)S_β-open set and establish the relationship among themselves. In Section 4, we give a brief conclusion and further scope for future research.

2. Preliminaries

In this section, we give some preliminary definitions which are necessary to obtain the main results.

Definition 2.1 [12] Let A be a subset of a bitopologial space (X, τ_1, τ_2). Then A is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))) where τ_1-Int(A) is the interior of A with respect to the topology τ_1 and $\tau_1\tau_2$-Cl(A) is the intersection of all $\tau_1\tau_2$-closed sets containing A.

(iv) (1, 2)β-Int(A) is the union of all (1, 2)β-open sets contained in A.

(v) (1, 2)β-Cl(A) is the intersection of all (1, 2)β-closed sets containing A.

Definition 2.2 [12] A subset A of X is said to be

(i) (1, 2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1, 2)regular-open if $A = \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1, 2)β-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int($\tau_1\tau_2$-Cl(A))).

The set of all (1, 2)semi-open, (1, 2)regular-open, (1, 2)β-open are denoted by (1, 2)SO(X, τ_1, τ_2), (1, 2)RO(X, τ_1, τ_2), (1, 2)βO(X, τ_1, τ_2) or simply, (1, 2)SO(X), (1, 2)RO(X), (1, 2)β-O(X) respectively.

Definition 2.3 [12] A subset A of X is said to be

(i) (1, 2)semi-closed if $\tau_1\tau_2$-Int(τ_1-Cl(A)) $\subseteq A$.

(ii) (1, 2)regular-closed if $A = \tau_1$-Cl($\tau_1\tau_2$-Int(A)).

(iii) (1, 2)β- closed if $\tau_1\tau_2$- Int(τ_1-Cl($\tau_1\tau_2$-Int(A))) $\subseteq A$.

The set of all (1, 2)semi-closed, (1, 2)regular-closed, (1, 2)β-closed are denoted by (1, 2)SCL(X, τ_1, τ_2), (1, 2)RCL(X, τ_1, τ_2), (1, 2)βCL(X, τ_1, τ_2) or simply, (1, 2)SCL(X), (1, 2)RCL(X), (1, 2)β-CL(X) respectively.

Remark 2.4 [12] For any subset A of X,

(i) τ_1-Int(A) $\subseteq \tau_1\tau_2$-Int(A) and τ_2-Int(A) $\subseteq \tau_1\tau_2$-Int(A).

(ii) $\tau_1\tau_2$-Cl(A) $\subseteq \tau_1$-Cl(A) and $\tau_1\tau_2$-Cl(A) $\subseteq \tau_2$-Cl(A).

(iii) $\tau_1\tau_2$-Cl($A \cap B$) $\subseteq \tau_1\tau_2$-Cl(A) $\cap$ $\tau_1\tau_2$-Cl(B).

(iv) $\tau_1\tau_2$-Int(A) $\cup$ $\tau_1\tau_2$-Int(B) $\subseteq \tau_1\tau_2$-Int($A \cup B$).

Definition 2.5 [3] A semi-open subset A of a topological space (X, τ) is said to be S_β-open if for each $x \in A$, then there exists a β-closed set F such that $x \in F \subseteq A$. A subset B of a topological space (X, τ) is S_β-closed if $X - B$ is S_β-open.

3. Main Results

We begin this section with the definition of (1, 2)S_β-open set and its characterizations.

Definition 3.1 Let A be a non-empty subset of a bitopological space (X, τ_1, τ_2). Then $(1,2)S_\beta$-kernel of A, denoted by $(1,2)S_\beta$-ker(A) and it is defined as $(1,2)S_\beta\text{-ker}(A) = \cap\,\{G \in (1,2)S_\beta\text{-O}(X)/\ A \subseteq G\}$.

Example 3.2 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d}} and τ_2 = {ϕ, X}. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, { b, c, d}}. Take A = {a, b}, then (1, 2)S_β-ker({a, b}) = {a, b}. Therefore, (1, 2)S_β-ker(X) = {ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, { b, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d}, { b, c, d}}.

Definition 3.3 Let $x \in X$. Then (1, 2)S_β-kernel of {x}, denoted by (1, 2)S_β-ker({x}) = $\cap$ {$G \in$ (1, 2)S_β-O(X)/ $x \in G$}.

Example 3.4 Let X = {a, b, c} with two topologies τ_1 = {ϕ, X, {a}, {b}, {a, b}, {b, c}} and τ_2 = {ϕ, X, {c}}. Also, (1, 2)S_β-O(X) = {ϕ, X, {a}, {b, c}}. Note that x = a, then (1, 2)S_β-ker({x}) ={a}, x = b implies (1, 2)S_β-ker({x}) ={b, c} and x = c implies (1, 2)S_β-ker({x}) = {b, c}.

Definition 3.5 In a bitopological space (X, τ_1, τ_2), a set A is said to be weakly-(1, 2)S_β-separated from set B if there exists a (1, 2)S_β-open set G such that $A \subset G$ and $G \cap B = \phi$ or $A \cap$ (1, 2)S_β-Cl(B) = ϕ.

Example 3.6 Let X = {a, b, c, d} with two topologies τ_1= {ϕ, X, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d}} and τ_2 = {ϕ, X}. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, { b, c, d}}. If A = {a, b}, then (1, 2)S_β-ker({a, b}) = {a, b}.

Lemma 3.7 Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are hold for $x, y \in X$.

(i) (1, 2)S_β-cl({x}) = {y : y is not weakly-(1, 2)S_β-separated from x} and

(ii) (1, 2)S_β-ker({x}) ={x is not weakly-(1, 2)S_β-separated from y}.

Definition 3.8 For any point x of a bitopological space (X, τ_1, τ_2),

(i) the derived set of x denoted by (1, 2)S_β-d({x}) and it is defined as (1, 2)S_β-d({x})= (1, 2)S_β-cl({x}) $-$ {x} = {y : $y \neq x$ and y is not weakly-(1, 2)S_β-separated from x}.

(ii) the shell of a singleton set {x} denoted by (1, 2)S_β-shl({x}) and it is defined as (1, 2)S_β-shl({x}) = (1, 2)S_β-ker({x}) $-$ {x} = {y : $y \neq x$ and x is not weakly- (1, 2)S_β-separated from y}.

Definition 3.9 A (1, 2)semi-open subset A of a bitopological space (X, τ_1, τ_2) is said to be (1, 2)S_β-open set if for each $x \in A$ there exists a (1, 2)β-closed set F such that $x \in F \subseteq A$.
The complement of (1, 2)S_β-open set is (1, 2)S_β-closed set and the family of all (1, 2)S_β-open ((1, 2)S_β-closed) subsets of X, is denoted by (1, 2)S_β-O(X) ((1, 2)S_β-CL(X)) respectively.

Proposition 3.10 A subset A of a bitopological space (X, τ_1, τ_2) is (1, 2)S_β-open set if and only if A is (1, 2)semi-open and it is the union of (1, 2)β-closed sets.

Proof: It is obvious from Definition 3.9. □

Definition 3.11 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-R_D space if for each $x \in X$, (1, 2)S_β-cl({x}) $\cap$ (1, 2)S_β-ker({x}) = {x} which implies that the (1, 2)S_β-derived set, (1, 2)S_β-d({x})= (1, 2)S_β-cl({x}) $-$ {x} is (1, 2)S_β-closed.

Example 3.12 Let X = {a, b, c, d} with two topologies $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a, b, d\}\}$. Then (1, 2)$S_\beta$-O($X$) = {$\phi$, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, X is a (1, 2)S_β-R_D space.

Definition 3.13 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-R_T if for each $x \in$ X, (1, 2)S_β-ker($\{x\}$) − (1, 2)S_β-cl($\{x\}$) and (1, 2)S_β-cl($\{x\}$) − (1, 2)S_β-ker($\{x\}$) are degenerate sets.

Example 3.14 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {b, d}, {a, b, c}, {a, b, d}} and $\tau_2 = \{\phi, X\}$. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, X is a (1, 2)S_β-R_T space.

Definition 3.15 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-R_{YS} if for each $x, y \in X$, (1, 2)S_β-cl($\{x\}$) $\neq$ (1, 2)S_β-cl($\{y\}$) which implies (1, 2)S_β-cl($\{x\}$) $\cap$ (1, 2)S_β-cl($\{y\} = \phi$ or $\{x\}$ or $\{y\}$.

Example 3.16 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}} and $\tau_2 = \{\phi, X\}$. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}}. Here, X is a (1, 2)S_β-R_{YS} space.

Definition 3.17 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-T_{YS} if for each $x, y \in X$ and $x \neq y$, which implies (1, 2)S_β-cl($\{x\}$) $\cap$ (1,2)S_β-cl($\{y\} = \phi$ or $\{x\}$ or $\{y\}$.

Example 3.18 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d}} and τ_2 = {ϕ, X, {a, b, d}}. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, X is a (1, 2)S_β-T_{YS} space.

Definition 3.19 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-R_Y if for all $x, y \in X$, (1, 2)S_β-cl($\{x\}$) $\cap$ (1, 2)S_β-cl($\{y\}$ is a degenerate set.

Example 3.20 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {d}, {a, d}} and τ_2 = {ϕ, X, {a, c, d}}. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, X is a (1, 2)S_β-R_Y space.

Definition 3.21 A bitopological space (X, τ_1, τ_2) is called (1, 2)S_β-R_0 space if for each (1, 2)S_β-open set U and $x \in U$ implies (1, 2)S_β-cl($\{x\}$) $\subset U$.

Remark 3.22 Obviously, (1, 2)S_β-R_0 space implies (1, 2)S_β-R_T space. But the converse is not always true as it is shown in the following example.

Example 3.23 Let X = {a, b, c, d} with the topologies τ_1 = {ϕ, X, {b}, {a, c}, {a, b, c}, {a, c, d}} and $\tau_2 = \{\phi, X\}$. Then (1,2)S_β-O(X) = {ϕ, X, {b}, {a, c}, {a, b, c}, {a, c, d}} and (1,2)S_β-CL(X) = {ϕ, X, {b}, {d}, {b, d}, {a, c, d}}. Here, X is a (1,2)S_β-R_T space but not a (1, 2)S_β-R_0 space.

Theorem 3.24 If every bitopological space (X, τ_1, τ_2) is a(1,2)S_β-R_T space, then X is a (1, 2)S_β-R_D space.

Proof: Let (X, τ_1, τ_2) be a (1, 2)S_β-R_T space. Then by Definition 3.11, both (1, 2)S_β-ker($\{x\}$) − (1, 2)S_β-cl($\{x\}$) and (1, 2)S_β-cl($\{x\}$) − (1, 2)S_β-ker($\{x\}$) are degenerate sets. Also, let $< x >$ = (1, 2)S_β-cl($\{x\}$) $\cap$ (1, 2)S_β-ker($\{x\}$). Then (1, 2)S_β-ker($\{x\}$) = $< x > \cup D$ and (1,2)S_β-cl($\{x\}$) = $< x > \cup E$, where D is not a subset of (1, 2)S_β-cl($\{x\}$) and E is not a subset of (1, 2)S_β-ker($\{x\}$) which implies that D and E are degenerate sets. If $< x > = \{x\}$, then (1, 2)S_β-cl($\{x\}$) = $E \cup \{x\}$ and (1, 2)S_β-ker($\{x\}$) = $D \cup \{x\}$. Again, let U be a (1, 2)S_β-open set containing (1, 2)S_β-ker($\{x\}$). Then ($X - U$) is (1, 2)S_β-closed set. Hence, ($X - U$) $\cap$ (1, 2)S_β-cl($\{x\}$) = E or ϕ.
Case(i):
If ($X - U$) $\cap$ (1, 2)S_β-cl($\{x\}$) = E, then it is (1, 2)S_β-closed, where E is the intersection of two (1, 2)S_β-closed sets.

Case(ii):
If $(X - U) \cap (1, 2)S_\beta$-cl$(\{x\}) = \phi$, then $(1, 2)S_\beta$-cl$(\{x\}) \subset U$ that is $E \subset U$. Since E is not a subset of $(1, 2)S_\beta$-ker$(\{x\})$, there is a $(1, 2)S_\beta$-open set V such that $x \in V$ and E is not a subset of V. Then $(1, 2)S_\beta$-cl$(\{x\}) \cap (X - V) = E$ is a $(1, 2)S_\beta$-closed. Hence U is a $(1,2)S_\beta$-R_D space. □

Remark 3.25 The converse of Theorem 3.24, is not true and it is shown in the following example.

Example 3.26 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}} and τ_2 = {ϕ, X}. Then $(1, 2)S_\beta$-O(X) = {ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}}. $(1, 2)S_\beta$-CL(X) = {ϕ, X, {b}, {c}, {d}, {b, c}, {b, d}, {c, d}, {b, c, d}}. Here, X is a $(1, 2)S_\beta$-R_D space but not a $(1, 2)S_\beta$-R_T space.

Theorem 3.27 Every $(1, 2)S_\beta$-R_T space is $(1, 2)S_\beta$-R_{YS} space.

Proof: Let (X, τ_1, τ_2) be $(1, 2)S_\beta$-R_T space. Let $x, y \in X$. If $(1,2)S_\beta$-cl$(\{x\}) \neq (1, 2)S_\beta$-cl$(\{y\})$ and $(1, 2)S_\beta$-cl$(\{x\}) \cap (1, 2)S_\beta$-cl$(\{y\}) \neq \phi$, then there exists an element $a \in X$ such that $a \in (1,2)S_\beta$-cl$(\{x\})$ $\cap$ $(1, 2)S_\beta$-cl$(\{y\})$ where $a \neq x$, $a \neq y$ then $a \in (1,2)S_\beta$-cl$(\{x\})$ and $a \in (1,2)S_\beta$-cl$(\{y\})$. Also $x, y \in (1, 2)S_\beta$-ker$(\{a\})$. Since (X, τ_1, τ_2) is a $(1, 2)S_\beta$-R_T space, we have $(1, 2)S_\beta$-ker$(\{a\}) = < a > \cup E$ where E is a degenerate set and not a subset of $(1, 2)S_\beta$-cl$(\{a\})$. Therefore, there exist four possible cases if $x \in (1,2)S_\beta$-ker$(\{a\})$ and $y \in (1, 2)S_\beta$-ker$(\{a\})$.
Case(1):
Let $x \in < a >$ and $y \in < a >$. Then $x \in (1, 2)S_\beta$-cl$(\{a\})$ and $y \in (1, 2)S_\beta$-cl$(\{a\})$. Hence $(1, 2)S_\beta$-cl$(\{x\})$ $= (1,2)S_\beta$-cl$(\{a\}) = (1, 2)S_\beta$-cl$(\{y\})$, a contradiction.
Case(2):
Let $\{x\} = E$ and $y \in < a >$. So $\{x\} \notin (1, 2)S_\beta$-cl$(\{a\})$ and $y \in (1, 2)S_\beta$-cl$(\{a\})$, then $(1, 2)S_\beta$-cl$(\{y\}) = (1, 2)S_\beta$-cl$(\{a\})$. Here also we have two subcases:
Subcase(i) Let $y \in (1, 2)S_\beta$-cl$(\{x\})$. By assumption $x \notin (1, 2)S_\beta$-cl$(\{a\})$. So $x \in X - (1, 2)S_\beta$-cl$(\{a\})$ where $(1, 2)S_\beta$-cl$(\{a\})$ is a $(1,2)S_\beta$-open set containing the point x. Hence $(1, 2)S_\beta$-ker$(\{x\}) \subset X-$ $(1, 2)S_\beta$-cl$(\{a\})$. Then $(1, 2)S_\beta$-cl$(\{x\}) - (1, 2)S_\beta$-ker$(\{x\}) \supseteq (1, 2)S_\beta$-cl$(\{a\}) \supseteq \{y, a\}$, which implies that $(1, 2)S_\beta$-cl$(\{x\}) - (1, 2)S_\beta$-ker$(\{x\})$ is not a degenerate set, which is a contradiction.
Subcase(ii) Let $y \notin (1, 2)S_\beta$-cl$(\{x\})$. Since $y \in (1, 2)S_\beta$-cl$(\{y\})$ and $a \in (1, 2)S_\beta$-cl$(\{x\})$, therefore $y \in$ $(1, 2)S_\beta$-cl$(\{x\})$, which is a contradiction.
Case(3):
Let $x = < a >$ and $\{y\} = E$. Then it is similar to Case (2).
Case(4):
Let $\{x\} = \{y\} = E$. Then $(1, 2)S_\beta$-cl$(\{x\}) = (1, 2)S_\beta$-cl$(\{y\})$, which is a contradiction. If $(1, 2)S_\beta$-cl$(\{x\}) \neq (1, 2)S_\beta$-cl$(\{y\})$, then $(1,2)S_\beta$-cl$(\{x\}) \cap (1, 2)S_\beta$-cl$(\{y\}) = \phi$ or $\{x\}$ or $\{y\}$. Hence, X is a $(1,2)S_\beta$-R_{YS} space. □

Remark 3.28 The converse of Theorem 3.27 is not true and it is shown in the following example.

Example 3.29 Let X = {a, b, c, d} with two topologies τ_1 ={ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}} and τ_2 = {ϕ, X}. Then $(1, 2)S_\beta$-O(X) = {ϕ, X, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, b, d}, {a, c, d}}. $(1, 2)S_\beta$-CL(X) = {ϕ, X, {b}, {c}, {d}, {b, c}, {b, d}, {c, d}, {b, c, d}}. Here, X is a $(1, 2)S_\beta$-R_D space but not a $(1, 2)S_\beta$-R_T space.

Theorem 3.30 Every $(1, 2)S_\beta$-R_{YS} space is $(1, 2)S_\beta$-R_D space but the converse is not true.

Proof: Let (X, τ_1, τ_2) be $(1, 2)S_\beta$-R_{YS} space. Here we have three cases.
Case(i):
Let $(1, 2)S_\beta$-cl$(\{x\}) \cap (1, 2)S_\beta$-cl$(\{y\}) = \{x\}$. Then $(1, 2)S_\beta$-$d(\{x\}) = \phi$ and also $(1, 2)S_\beta$-closed set.
Case(ii):
Let $(1, 2)S_\beta$-cl$(\{x\}) \cap (1, 2)S_\beta$-cl$(\{y\}) = \{y\}$. Then $(1, 2)S_\beta$-$d(\{y\}) = \phi$.

Case(iii):
Let (1, 2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) = ϕ. Since X is a (1, 2)S_β-R_{YS} space, (1,2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-ker($\{x\}$) = $\{x\}$, which is a contradiction. Therefore, X is a (1, 2)S_β-R_D space. □

Remark 3.31 Obviously every (1, 2)S_β-R_{YS} space is (1, 2)S_β-R_Y space.

Remark 3.32 Every (1, 2)S_β-T_{YS} space is (1, 2)S_β-R_{YS} space but the converse is not true.

Example 3.33 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {a}, {a, b}} and τ_2 = {ϕ, X, {b, c}}. Then (1, 2)S_β-O(X) = {ϕ, X, {a}, {a, b}, {a, b, c}, {a, b, d}}, (1, 2)S_β-CL(X) = {ϕ, X, {b, c, d}, {c, d}, {d}, {c}}. Here, X is a (1, 2)S_β-R_{YS} space but not (1, 2)S_β-T_{YS} space.

Theorem 3.34 For a bitopological space (X, τ_1, τ_2), the following statements are equivalent.

(i) X is (1, 2)S_β-T_{YS}.

(ii) For any two distinct points $x, y \in X$, either of the points has an empty (1, 2)S_β-derived set or (1, 2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) = ϕ.

(iii) The (1, 2)S_β-closure of (1, 2)S_β-derived sets of any two distinct points are disjoint.

(iv) The (1, 2)S_β-derived sets of any two distinct points are (1, 2)S_β-separated sets.

Proof: (i)⇒(ii)
Let (X, τ_1, τ_2) be a (1, 2)S_β-T_{YS} space and if for any $x, y \in X$, (1,2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) = ϕ then there is nothing to prove. If not (1, 2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) = $\{x\}$, then (1, 2)S_β-d($\{x\}$) = ϕ.
(ii)⇒(iii)
Let (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) ∩ (1, 2)S_β-cl((1, 2)S_β-d($\{y\}$)) = ϕ. If any one of the derived sets is empty, then there is nothing to prove. Now (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) ⊂ (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) then (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) ∩ (1, 2)S_β-cl((1, 2)S_β-d($\{y\}$)) ⊂ (1, 2)S_β-cl($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) ϕ. Hence, the result.
(iii)⇒(iv) obvious.
(iv)⇒(i)
Let $x, y \in X$ and $x \neq y$ such that (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) ∩ ((1,2)S_β-d($\{y\}$)) = ϕ. Here there are two possibilities:
(1, 2)S_β-cl((1, 2)S_β-d($\{x\}$))= (1, 2)S_β-d($\{x\}$) or
(1, 2)S_β-cl((1, 2)S_β-d($\{x\}$))= (1, 2)S_β-cl($\{x\}$).
Case(i):
Let (1, 2)S_β-d($\{x\}$) be a (1, 2)S_β-closed for each $x \in X$. Then (1,2)S_β-d($\{x\}$) ∩ (1, 2)S_β-d($\{y\}$) = ϕ or (1, 2)S_β-d($\{x\}$) ∩ (1, 2)S_β-cl($\{y\}$) = ϕ or y. If it is y, then y is (1, 2)S_β-closed and hence (1, 2)S_β-cl($\{x\}$)) ∩ (1, 2)S_β-cl($\{y\}$) = $\{y\}$. If it is ϕ, then (1, 2)S_β-cl($\{x\}$)) ∩ (1, 2)S_β-cl($\{y\}$) = ϕ or $\{x\}$.
Case(ii):
Let (1, 2)S_β-cl((1, 2)S_β-d($\{x\}$)) = (1,2)S_β-cl($\{x\}$) then (1,2)S_β-cl($\{x\}$) ∩ (1,2)S_β-d($\{y\}$) = ϕ. Hence, (1,2)S_β-cl($\{x\}$)) ∩ (1,2)S_β-cl($\{y\}$) = ϕ or $\{y\}$. □

Remark 3.35 The (1, 2)S_β-R_T space and (1, 2)S_β-T_{YS} space are independent on each other.

Example 3.36 Let X = {a, b, c, d} with two topologies τ_1 = {ϕ, X, {b}, {a, c}, {a, b, c}, {a, c, d}} and τ_2 ={ϕ, X}. Then (1,2)S_β-O(X) = {ϕ, X, {b}, {a, c}, {a, b, c}, {a, c, d}}, (1, 2)S_β-CL(X) = {ϕ, X, {b}, {d}, {b, d}, {a, c, d}}. Here, X is a (1, 2)S_β- T_{YS} space but not a (1, 2)S_β-R_T space.

From the above discussion, we have the following diagram.

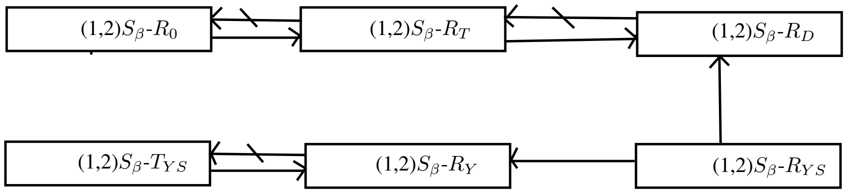

Figure 1:

4. Conclusion

In this work, we define and study the weak separation axioms of (1, 2)S_β-open sets in bitopological spaces. By comparative analysis, we identify the similarities and differences among various axioms and highlighted the logical implications that typically hold in general topology but failed in bitopological spaces. Also, this work will lead to study various types of separation axioms and their properties by using the corresponding set. The future researchers can investigate alternative concepts such as covering characteristics and separation axioms via proposed class of (1, 2)S_β-open and (1, 2)β-closed sets. Also, we propose to introduce $T_{\frac{1}{2}}$, D_0, D_1 and D_2 separation axioms by using (1, 2)S_β-open set, (1, 2)S_β-closed sets and study their properties.

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