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Hybrid Block Method to Solving First Order Initial Value Problem in Ordinary Differential Equations

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ABSTRACT: This study examines the hybrid block method (HBM) derivation for solving first order initial value problems (FOIVP) in ordinary differential equations (ODEs). Using the collocation and interpolation procedure at equally spaced locations in the interval of consideration, a continuous formula is generated from the estimated answer, which is assumed to be in the form of power series. The accuracy of the procedure is significantly influenced by the advantage of using data off-step points. The efficacy of the suggested approach is demonstrated by a few numerical examples that show how close the answers are to the precise solutions. All types of FOIVP involving ODEs can be solved utilizing this method.

Key Words: Hybrid block method, collocation, interpolation, data off-step points.

Contents

1. INTRODUCTION

ODEs play a crucial role in applied sciences and engineering, as they provide a fundamental framework for modeling the dynamic behavior of real-world systems. These equations are extensively used across various disciplines such as chemical, electrical, and mechanical engineering, as well as in physics, chemistry, and mathematical biology. In these fields, ODEs help describe how physical quantities evolve over time or space—capturing processes like chemical reaction kinetics, electrical circuit behavior, mechanical vibrations, heat conduction, population dynamics, and the spread of diseases. By formulating and solving ODEs, researchers and engineers can analyze, predict, and optimize the performance of complex systems under different conditions, leading to improved design, control, and understanding of natural and engineered processes.
The main focus is to construct an accurate solution of an ODEs involving a FOIVP.
The FOIVP for an ODEs as follows,

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a, b]. \tag{1.1}$$

Where the function $f(x, y)$ is continuous and also follows the Lipschitz condition. Because of this, the existence and uniqueness of the solution for equation (1) is guaranteed by the related theorem.
Due to the limitations of the predictor-corrector approaches for solving ODEs of form (1) put out by [5],

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HBM have to be introduced. Odejide and Adeniran [1] developed ninth order hybrid linear multistep approach to obtained FOIVP, which is consistent, zero-stable, convergent, and accurate when applied to block form. According to Dahlquist (1956), hybrid approaches were first developed to get over the zero-stability barrier that existed in HBM. Utilizing data off-step locations enhances the accuracy of the approach and is another advantage of HBM, in addition to the ability to modify step size by [6]. Nathaniel et al. [2] developed a simple and accurate method to solve first-order ODEs using shifted Legendre polynomials. Kashkari and Syam [3] developed an optimized one-step HBM to solve math problems, which is efficient and gives accurate results. Ortiz [4] explained Lanczos tau method in a systematic way, linking it to polynomial solutions The classic collocation approach was introduced by [7] and is discrete in nature. A continous multistep collocation technique, improving approximation accuracy at all interior points and reducing absolute error is developed by [8]. The concept of a multistep collocation technique against discrete systems was used to accomplish this. Soomro et al. [9] stated that the linear multi step approach offers several advantages over the discrete method. These include improved error estimation, easier approximation of solutions at interior points within the integration interval, and a simplified form of coefficients that facilitates further analytical work at various points. Techniques utilizing Taylor series expansion for initial values in the predictor-corrector approach have been proposed by [10, 11, 12,13] using the Adam-type method by [16], although high implementation costs due to specialized subroutines are a notable drawback. In order to overcome this obstacle, a technique that combines the advantages of the HBM with the predictor-corrector approach must be proposed. The Adomain decomposition technique, variational iteration approach, Chebyshev's wavelet technique, fourth order Runge Kutta technique, and homotopy perturbation technique have limitations, such as small convergence regions and inaccuracy in [14, 15]. Fotta et al. [17] developed a one-step , HBMs to solve math problems, which gave good and accurate results

Therefore, a self-starting continuous two-step HBM is propoesd that has a higher accuracy and a quick rate of convergence when applied for the integration of FOIVP in ODEs. This resulted in an equal selection of collocation spots through the consideration period. The HBM have the benefit of using data off step points which plays an important role in the accuracy of the result.

In this study presents a novel application and refinement of the HBM by incorporating collocation and interpolation techniques at equally spaced points within the interval, which enhances the accuracy and stability of the solution. Unlike previous articles that primarily focused on theoretical development or specific types of differential equations, this study emphasizes the effective use of data off-step points, which significantly improves approximation accuracy across a broad class of first order FOIVPs. Additionally, the proposed method demonstrates high convergence properties and zero stability, making it a reliable, versatile tool for solving complex ODEs. Regarding practical applications, the HBM developed in this study can be effectively employed in various scientific and engineering fields such as chemical kinetics, electrical circuit analysis, mechanical vibrations, and population dynamics. These areas often involve solving complex, nonlinear first order differential equations in real-world models, where the accuracy, stability, and efficiency of the method are crucial for reliable predictions and system analysis.

2. DERIVATION OF THE METHOD

Let the solution of (1) is given in the form of a power series. The numerical results in the interval $[x_n, x_{n+1}]$ with step length h $= x_{n+1} - x_n$,

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \tag{2.1}$$

$$\Rightarrow \quad y(x) \approx \sum_{j=0}^{5} a_j x^j$$

differentiating (2) with respect to x

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \tag{2.2}$$

Let us consider from equations (2) and (3) as

$$y_{n+j} = \sum_{j=0}^{5} a_j x_{n+j}^{j} \tag{2.3}$$

$$f_{n+j} = \sum_{j=0}^{5} j a_j x_{n+j}^{j-1} \tag{2.4}$$

where $y_{n+j} = y_n + jhf_n$ and $f_n = f(x_n, y_n)$

Equation (2) is interpolated at $x = x_n$ and

Equation (3) is collocated at $x = 0, 1/2, 1, 3/2, 2$ that leads to the system of equations

$$y_n = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 \tag{2.5}$$

$$f_n = a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 \tag{2.6}$$

$$f_{n+1/2} = a_1 + 2a_2 x_{n+1/2} + 3a_3 x_{n+1/2}^2 + 4a_4 x_{n+1/2}^3 + 5a_5 x_{n+1/2}^4 \tag{2.7}$$

$$f_{n+1} = a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 \tag{2.8}$$

$$f_{n+3/2} = a_1 + 2a_2 x_{n+3/2} + 3a_3 x_{n+3/2}^2 + 4a_4 x_{n+3/2}^3 + 5a_5 x_{n+3/2}^4 \tag{2.9}$$

$$f_{n+2} = a_1 + a_2 x_{n+2} + 3a_{n+2}^2 + 4x_{n+2}^3 + 5x_{n+2}^4 \tag{2.10}$$

The six equations (6), (7), (8), (9), (10), (11) are written in a matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\times\frac{1}{2} & 3\times\frac{1}{4} & 4\times\frac{1}{8} & 5\times\frac{1}{16} \\ 0 & 1 & 2\times 1 & 3\times 1 & 4\times 1 & 5\times 1 \\ 0 & 1 & 2\times\frac{3}{2} & 3\times\frac{9}{4} & 4\times\frac{27}{8} & 5\times\frac{81}{16} \\ 0 & 1 & 2\times 2 & 3\times 4 & 4\times 8 & 5\times 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

which is simplified as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{2} & \frac{5}{16} & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & \frac{27}{4} & \frac{27}{2} & \frac{405}{16} \\ 0 & 1 & 4 & 12 & 32 & 80 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

Using Guass elimination method we get the values of $a_j's$ as follows,

$$a_0 = y_n, \tag{2.11}$$

$$a_1 = f_n, \tag{2.12}$$

$$a_2 = -\frac{25}{12} f_n + 4f_{n+\frac{1}{2}} - 3f_{n+1} + \frac{4}{3} f_{n+\frac{3}{2}} - \frac{1}{4} f_{n+2}, \tag{2.13}$$

$$a_3 = \frac{35}{18} f_n - \frac{52}{9} f_{n+\frac{1}{2}} + \frac{57}{9} f_{n+1} + \frac{4}{3} f_{n+\frac{3}{2}} - \frac{1}{4} f_{n+2}, \tag{2.14}$$

$$a_4 = -\frac{5}{6} f_n + 3f_{n+\frac{1}{2}} - 4f_{n+1} + \frac{1}{2} f_{n+\frac{3}{2}} - \frac{7}{4} f_{n+2}, \tag{2.15}$$

$$a_5 = \frac{2}{15}f_n - \frac{8}{15}f_{n+\frac{1}{2}} + \frac{4}{5}f_{n+1} - \frac{8}{15}f_{n+\frac{3}{2}} + \frac{2}{15}f_{n+2}, \tag{2.16}$$

Substituting the values of $a_0, a_1, a_2, a_3, a_4, a_5$ in equation (4) by replacing the variable with $x = x_n + th$ gives the continuous formulation written in the following form as,

$$y(x) = \alpha_0 y_n + h[\beta_0 f_n + \beta_{1/2} f_{n+1/2} + \beta_1 f_{n+1} + \beta_{3/2} f_{n+3/2} + \beta_2 f_{n+2}], \tag{2.17}$$

where from the equations (12), (13), (14), (15), (16), (17).

$$\alpha_0 = \text{coefficient of } y_n\,, \quad \beta_0 = \text{coefficient of } f_n, \quad \beta_{\frac{1}{2}} = \text{coefficient of } f_{n+\frac{1}{2}}$$

$$\beta_1 = \text{coefficient of } f_{n+1}, \quad \beta_{\frac{3}{2}} = \text{coefficient of } f_{n+\frac{3}{2}}, \quad \beta_2 = \text{coefficient of } f_{n+2}.$$

Therefore,

$$\alpha_0(t) = 1$$

$$\beta_0(t) = -\frac{25}{12}t^2 + \frac{35}{18}t^3 - \frac{5}{6}t^4 + \frac{2}{15}t^5$$

$$\beta_{\frac{1}{2}}(t) = 4t^2 - \frac{52}{9}t^3 + 3t^4 - \frac{8}{15}t^5$$

$$\beta_1(t) = -3t^2 + \frac{57}{9}t^3 - 4t^4 + \frac{4}{5}t^5$$

$$\beta_{\frac{3}{2}}(t) = \frac{4}{3}t^2 - \frac{28}{9}t^3 + \frac{7}{3}t^4 - \frac{8}{15}t^5$$

$$\beta_2(t) = -\frac{1}{4}t^2 + \frac{11}{18}t^3 - \frac{7}{4}t^4 + \frac{2}{15}t^5$$

Substituting the values $t = 0, 1/2, 1, 3/2, 2$.

$$\alpha_0(0) = 1,\ \beta_0(0) = 0,\ \beta_{\frac{1}{2}}(0) = 0,\ \beta_1(0) = 0,\ \beta_{\frac{3}{2}}(0) = 0,\ \beta_2(0) = 0$$

$$\alpha_0\left(\frac{1}{2}\right) = 1,\ \beta_0\left(\frac{1}{2}\right) = \frac{251}{1440},\ \beta_{\frac{1}{2}}\left(\frac{1}{2}\right) = \frac{323}{720},\ \beta_1\left(\frac{1}{2}\right) = -\frac{11}{60},\ \beta_{\frac{3}{2}}\left(\frac{1}{2}\right) = \frac{53}{720},\ \beta_2\left(\frac{1}{2}\right) = -\frac{19}{1440}$$

$$\alpha_0(1) = 1,\ \beta_0(1) = \frac{29}{180},\ \beta_{\frac{1}{2}}(1) = \frac{31}{45},\ \beta_1(1) = \frac{2}{15},\ \beta_{\frac{3}{2}}(1) = \frac{1}{45},\ \beta_2(1) = -\frac{1}{180}$$

$$\alpha_0\left(\frac{3}{2}\right) = 1,\ \beta_0\left(\frac{3}{2}\right) = \frac{27}{160},\ \beta_{\frac{1}{2}}\left(\frac{3}{2}\right) = \frac{51}{80},\ \beta_1\left(\frac{3}{2}\right) = \frac{9}{20},\ \beta_{\frac{3}{2}}\left(\frac{3}{2}\right) = \frac{21}{80},\ \beta_2\left(\frac{3}{2}\right) = -\frac{3}{160}$$

$$\alpha_0(2) = 1,\ \beta_0(2) = \frac{7}{45},\ \beta_{\frac{1}{2}}(2) = \frac{32}{45},\ \beta_1(2) = \frac{4}{15},\ \beta_{\frac{3}{2}}(2) = \frac{32}{45},\ \beta_2(2) = \frac{7}{45}$$

Therefore HBM is constructed as follows

$$y_{n+\frac{1}{2}} = y_n + h\left[\frac{251}{1440}f_n + \frac{323}{720}f_{n+\frac{1}{2}} - \frac{11}{60}f_{n+1} + \frac{53}{720}f_{n+\frac{3}{2}} - \frac{19}{1440}f_{n+2}\right] \tag{2.18}$$

$$y_{n+1} = y_n + h\left[\frac{29}{180}f_n + \frac{31}{45}f_{n+\frac{1}{2}} + \frac{2}{15}f_{n+1} + \frac{1}{45}f_{n+\frac{3}{2}} - \frac{1}{180}f_{n+2}\right] \tag{2.19}$$

$$y_{n+\frac{3}{2}} = y_n + h\left[\frac{27}{160}f_n + \frac{51}{80}f_{n+\frac{1}{2}} + \frac{9}{20}f_{n+1} + \frac{21}{80}f_{n+\frac{3}{2}} - \frac{3}{160}f_{n+2}\right] \tag{2.20}$$

$$y_{n+2} = y_n + h\left[\frac{7}{45}f_n + \frac{32}{45}f_{n+\frac{1}{2}} + \frac{4}{15}f_{n+1} + \frac{32}{45}f_{n+\frac{3}{2}} + \frac{7}{45}f_{n+2}\right] \tag{2.21}$$

The points $n+\frac{1}{2}$ and $n+\frac{3}{2}$ are called data off step points which are helpful in the accurate of the solution at $n+1$ and $n+2$.

3. ANALYSIS OF THE METHOD

The fundamental properties such as order, zero stability, and consistency are discussed.

3.1. ORDER:

Operating with linear operator L on both sides of (18) we get,

$$L[y(x_n);h] = \alpha_0 y(x_n+h) + h\left[\beta_0 y'(x_n) + \beta_{\frac{1}{2}}y'\left(x_n+\frac{h}{2}\right) + \beta_1 y'(x_n+h) + \beta_{\frac{3}{2}}y'\left(x_n+\frac{3h}{2}\right) + \beta_2 y'(x_n+2h)\right], \tag{3.1}$$

where $y(x)$ is an arbitrary function that is continuously differentiable on $[a,b]$.
Expanding about x_n in $y(x_n+h), y'(x_n), y'\left(x_n+\frac{h}{2}\right), y'(x_n+h), y'\left(x_n+\frac{3h}{2}\right), y'(x_n+2h)$ by Taylor's series and collecting the coefficients.
$L[y(x_n);h] = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + c_3 h^3 y^{(3)}(x_n) + \cdots + c_q h^q y^{(q)}(x_n)$

$$\text{Evaluating, } y\left(x = x_n + \tfrac{1}{2}h\right) \text{ we get } C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \text{ and } C_6 = \frac{3h^6}{10240}$$

$$y(x = x_{n+1}) \text{ we get } C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \text{ and } C_6 = \frac{h^6}{5760}$$

$$y\left(x = x_n + \tfrac{3}{2}h\right) \text{ we get } C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \text{ and } C_6 = \frac{3h^6}{10240}$$

$$y(x = x_{n+2}) \text{ we get } C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0 \text{ and } C_7 = \frac{-h^7}{15120}$$

Therefore the error constants are $\left(\frac{3}{10240}, \frac{1}{5760}, \frac{3}{10240}, \frac{-1}{15120}\right)^T$ and the orders are $(5,5,5,6)^T$

3.2. CONSISTENCY

The HBM is consistent [18], as all the orders are greater than 1.

3.3. ZERO STABILITY:

The equations (19), (20), (21), (22) are put in the form of

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix} = \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+2}\\y_{n+\frac{3}{2}}\\y_{n+1}\\y_n\end{bmatrix} + h\begin{bmatrix}\frac{323}{720}&-\frac{11}{60}&\frac{53}{720}&-\frac{19}{1440}\\\frac{31}{45}&\frac{2}{15}&\frac{1}{45}&-\frac{1}{180}\\\frac{51}{80}&\frac{9}{20}&\frac{21}{80}&-\frac{3}{160}\\\frac{32}{45}&\frac{4}{15}&\frac{32}{45}&\frac{7}{45}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix} + h\begin{bmatrix}0&0&\frac{7}{45}\\0&0&\frac{45}{27}\\0&0&\frac{160}{29}\\0&0&\frac{251}{1440}\end{bmatrix}\begin{bmatrix}f_{n+2}\\f_{n+\frac{3}{2}}\\f_{n+1}\\f_{n+\frac{1}{2}}\end{bmatrix} \tag{17}$$

The characteristic polynomial of the HBMs is given by

$$\text{P}(\mathbf{R}) = \left|\mathbf{R}A^0 - A^1\right| \quad \text{where } A^1 = \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$$

$$\text{Therefore P}(\mathbf{R}) = \mathbf{R}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix} = \begin{bmatrix}R&0&0&-1\\0&R&0&-1\\0&0&R&-1\\0&0&0&R-1\end{bmatrix}$$

$$\text{Implies P}(\mathbf{R}) = R^4 - R^3 \rightarrow \rho(\mathbf{R}) = 0 \rightarrow R^4 - R^3 = 0$$

Therefore the roots of the first characteristic polynomial are $R_1 = R_2 = R_3 = 0$ and $R_4 = 1$

Since $|R| \leq 1$ and $|R| = 1$ HBMs is zero stable.

3.4. CONVERGENCE:

The HBM is convergent because it satisfies the conditions of consistency and zero stability.

4. IMPLEMENTATION OF THE METHOD

The effectiveness and validity of the HBM is tested by some non-linear first ODEs. The error formula as follows, $E(x) = |y(x) - y(x_n)|$ where $y(x)$ is the exact solution and $y(x_n)$ is the approximate solution obtained by the HBM.

Steps for the proposed method:

- Find the values of $x_n, x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}$
- Find the corresponding values of $y_n, y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}$ using

$$y_{n+j} = y_n + jhf_n \quad \text{and} \quad f_n = f(x_n, y_n)$$

- Substitute the values in equations (19), (20), (21), (22) to get the most accurate solutions.

4.1. NUMERICAL EXPERIMENTS

4.1.1 *Example 1:*

The following non-linear ODEs:

$$y' = 2x^4y, \quad y(0) = 1 \text{ with } h = 0.1$$

The exact solution is $y = e^{\frac{2}{5}x^5}$

Table 1: Table 1: Comparison of HBM Solutions (Proposed Method) with exact solutions and corresponding errors at Various X values.

X values	Initial values of y	Initial values of f	Values of y by proposed method	Exact solution	Error
$x_0 = 0$	$y_0 = 1$	$f_0 = 0$	$y_0 = 1$	$y_0 = 1$	0.0000000
$x_{\frac{1}{2}} = 0.05$	$y_{\frac{1}{2}} = 1$	$f_{\frac{1}{2}} = 0.00000125$	$y_{\frac{1}{2}} = 1.000000125$	$y_{\frac{1}{2}} = 1.000000125$	0.0000000
$x_1 = 0.1$	$y_1 = 1$	$f_1 = 0.0002$	$y_1 = 1.0000004$	$y_1 = 1.0000004$	0.0000000
$x_{\frac{3}{2}} = 0.15$	$y_{\frac{3}{2}} = 1$	$f_3 = 0.0010125$	$y_{\frac{3}{2}} = 1.000020375$	$y_{\frac{3}{2}} = 1.000030375$	0.00001
$x_2 = 0.2$	$y_2 = 1$	$f_2 = 0.0032$	$y_2 = 1.0000272$	$y_2 = 1.000128008$	0.000143992
$x_{\frac{5}{2}} = 0.25$	$y_{\frac{5}{2}} = 1.000432044$	$f_{\frac{5}{2}} = 0.0078158753$	$y_{\frac{5}{2}} = 1.0002982722$	$y_{\frac{5}{2}} = 1.000390701$	0.000092429
$x_3 = 0.3$	$y_3 = 1.000592087$	$f_3 = 0.0162095181$	$y_3 = 1.001116399$	$y_3 = 1.000972473$	0.000143926
$x_{\frac{7}{2}} = 0.35$	$y_{\frac{7}{2}} = 1.000752131$	$f_{\frac{7}{2}} = 0.0300350733$	$y_{\frac{7}{2}} = 1.00233252$	$y_{\frac{7}{2}} = 1.00213083$	0.000229437
$x_4 = 0.4$	$y_4 = 1.000912174$	$f_4 = 0.0512467033$	$y_4 = 1.003525387$	$y_4 = 1.0041044$	0.000057013
$x_{\frac{9}{2}} = 0.45$	$y_{\frac{9}{2}} = 1.006094412$	$f_{\frac{9}{2}} = 0.08251231796$	$y_{\frac{9}{2}} = 1.009395662$	$y_{\frac{9}{2}} = 1.007408433$	0.001987229
$x_5 = 0.5$	$y_5 = 1.008663437$	$f_5 = 0.1260830463$	$y_5 = 1.014552773$	$y_5 = 1.012578452$	0.005782565
$x_{\frac{11}{2}} = 0.55$	$y_{\frac{11}{2}} = 1.011232462$	$f_{\frac{11}{2}} = 0.1856818$	$y_{\frac{11}{2}} = 1.022260687$	$y_{\frac{11}{2}} = 1.020335378$	0.001925349
$x_6 = 0.6$	$y_6 = 1.013801487$	$f_6 = 0.2627773454$	$y_6 = 1.033371473$	$y_6 = 1.031592784$	0.001778689

Table 1 presents a comparison of numerical results obtained by a HBM comparision with exact solution to the Examople 1, likely a differential equation, at various X values, and it also shows the initial values used for the calculation. The table has different columns, including X values that represent the independent variable's values at which the solution is being evaluated, starting from $x_0 = 0$ and increasing in increments. The initial values of y are the starting or initial conditions for the dependent variable $'y'$ at each corresponding X value, often starting with 1.0 for initial X values and calculated from previous steps for later X values. The initial values of f is represent the initial values of a function f, possibly the derivative dy/dx or a related function, used in the numerical method, and these values increase as x increases. The values of y by HBM column shows the numerical solutions for $'y'$ calculated using the HBM at each corresponding X value. The exact solution column provides the true or analytical solution for $'y'$ at each X value, used as a benchmark to assess the accuracy of the HBM. The error column quantifies the difference between the Values of y by HBM and the exact solution for each X value, with a smaller error indicating higher accuracy of the HBM. The error values generally increase as $x$increases, which is a common characteristic of numerical methods where errors accumulate over steps.

Example 2: One of the most prevalent types of nonlinear ODEs for simulating real-world usage in a variety of domains is the Riccati ODEs.
Consider the Riccati's differential equation (RDE) of the form,

$$\frac{dv}{dt} = 1 - t^2 + v(t), \tag{4.1}$$

subject to $v(0) = 0$ with $h = 0.1$.
The exact solution of equation (24) is

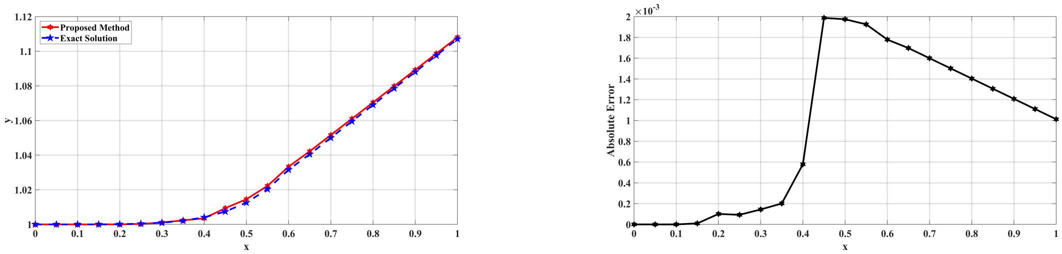
$$y = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + 0.5\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right).$$

The RDE is a nonlinear ODE with numerous applications across engineering and scientific domains. In control systems, it plays a crucial role in solving optimal control problems, such as the Linear Quadratic Regulator (LQR) and Linear Quadratic Gaussian (LQG) control, contributing to the stability and performance of dynamic systems. In fluid dynamics, it is used to model the transient settling velocity of non-spherical particles in viscous fluids, which inherently involves nonlinear behavior. Additionally, in mathematical biology, the RDE is employed in constructing models like the SIR model to describe the dynamics of infectious disease transmission over time

Table 2: Table 2: Evaluation of the proposed numerical method: Comparison of Calculated 'v' Values with exact solutions and absolute errors at various time t values.

t values	**Initial values of v**	**Initial values of f**	**Values of v by proposed method**	**Exact solution**	**Error**
$t_0 = 0$	$v_0 = 0$	$f_0 = 1$	$v_0 = 0$	$v_0 = 0$	0.000000
$t_{\frac{1}{2}} = 0.05$	$v_{\frac{1}{2}} = 0.05$	$f_{\frac{1}{2}} = 1.0975$	$v_{\frac{1}{2}} = 0.052458333$	$v_{\frac{1}{2}} = 0.05253943521$	0.00008
$t_1 = 0.1$	$v_1 = 0.1$	$f_1 = 1.19$	$v_1 = 0.109666667$	$v_1 = 0.1102951969$	0.0006
$t_{\frac{3}{2}} = 0.15$	$v_{\frac{3}{2}} = 0.15$	$f_{\frac{3}{2}} = 1.2775$	$v_{\frac{3}{2}} = 0.171375$	$v_{\frac{3}{2}} = 0.173419388$	0.00282
$t_2 = 0.2$	$v_2 = 0.2$	$f_2 = 1.36$	$v_2 = 0.237333333$	$v_2 = 0.2419767996$	0.004
$t_{\frac{5}{2}} = 0.25$	$v_{\frac{5}{2}} = 0.313246841$	$f_{\frac{5}{2}} = 1.528370099$	$v_{\frac{5}{2}} = 0.3158634013$	$v_{\frac{5}{2}} = 0.3159264087$	0.00006
$t_3 = 0.3$	$v_3 = 0.3845168824$	$f_3 = 1.621180532$	$v_3 = 0.3946444955$	$v_3 = 0.3951048487$	0.00005
$t_{\frac{7}{2}} = 0.35$	$v_{\frac{7}{2}} = 0.4557869238$	$f_7 = 1.703832128$	$v_{\frac{7}{2}} = 0.4778121405$	$v_{\frac{7}{2}} = 0.4792136573$	0.0014
$t_4 = 0.4$	$v_4 = 0.5270569652$	$f_4 = 1.776324883$	$v_4 = 0.5648583943$	$v_4 = 0.5678121663$	0.002954
$t_{\frac{9}{2}} = 0.45$	$v_{\frac{9}{2}} = 0.6382027746$	$f_9 = 1.435702775$	$v_{\frac{9}{2}} = 0.6389208725$	$v_{\frac{9}{2}} = 0.6389208425$	0.00004
$t_5 = 0.5$	$v_5 = 0.7085933829$	$f_5 = 1.458593383$	$v_5 = 0.7097348747$	$v_5 = 0.7097345547$	0.00003
$t_{\frac{11}{2}} = 0.55$	$v_{\frac{11}{2}} = 0.778989912$	$f_{11} = 1.476483991$	$v_{\frac{11}{2}} = 0.7846968781$	$v_{\frac{11}{2}} = 0.7846578781$	0.000256
$t_6 = 0.6$	$v_6 = 0.8493745995$	$f_6 = 1.4893746$	$v_6 = 0.8588641762$	$v_6 = 0.8588644762$	0.000003

Table 2 presents a detailed evaluation of a HBM for solving a problem, likely a ODEs or similar numerical task, by comparing its results against an exact solution. This table uses t values as the independent variable, suggesting a time-dependent problem, with values starting from $t_0 = 0$ and increasing

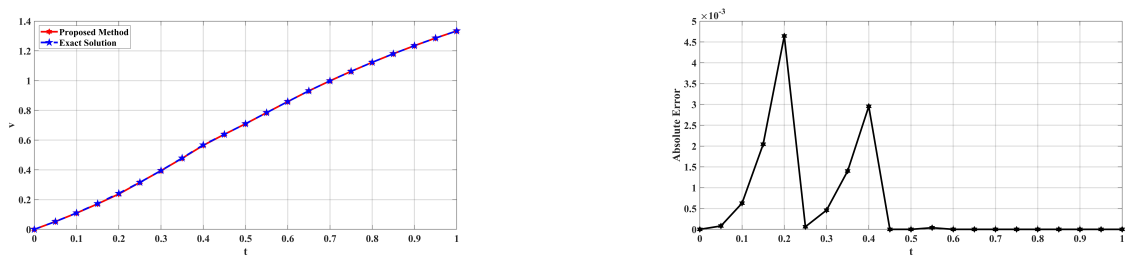


(a) Proposed Method vs Exact Solution $x = 0$ to 1. (b) Absolute Error between Proposed and Exact Solution.

Figure 1: Comparison of HBMs and exact solution of Example 1.

in increments, including intermediate entries for specific time points of interest. The table has several columns, including initial values of v, which are the starting conditions for the dependent variable $'v'$ at each corresponding value of $'t'$, with v_0 starting at 0. The initial values of f probably represent the initial values of a function f, possibly related to the derivative dv/dt, used in the calculation, and these values generally increase as $t$increases. The column of values of v shows the numerical solutions for v calculated using the HBM at each t value, while the column of exact solutions provides the true or analytical solution for v at each v value, serving as a reference to gauge the accuracy of the HBM. The error column quantifies the absolute difference between the HBM values and the exact solution for each t value, with smaller errors indicating higher accuracy. The error values show fluctuations, sometimes very small, suggesting that the HBM is quite accurate for this specific problem.

Figure 1 illustrates the comparison between the HBM and exact solution for Example Problem 1 over the interval $x = 0$ to $x = 1$, along with the corresponding absolute error. In subfigure (a), the approximate results obtained using the HBM are represented by red circles, while the exact solution is shown using blue stars. The two curves closely overlap across the entire domain, indicating an excellent agreement between the proposed method and the exact analytical solution. This visual match demonstrates the high accuracy and reliability of the proposed approach. Subfigure (b) presents the absolute error between the HBM and exact solutions, which remains very small throughout the interval. The maximum error observed is approximately 2×10^{-3} near $x = 0.45$, after which it gradually decreases. Overall, Figure 1 confirms that the proposed numerical method provides an accurate and effective approximation of the exact solution for the given problem.



(a) Proposed Method vs Exact Solution for $v(t)$ at $t \in [0, 1]$.

(b) Absolute Error between Proposed and Exact Solution.

Figure 2: Comparison of HBM and exact solution of Example 2.

Figure 2 shows the comparison between the HBM and exact solution for Example Problem 2 over the time interval $t = 0$ to $t = 1$, along with the absolute error. In part (a), the HBM solution is shown using red circles, while the exact solution is marked with blue stars. Both curves are almost overlapping, which means that the proposed method gives solutions that are very closed to the exact solution for the function $v(t)$. This indicates that the numerical approach is accurate and reliable. In part (b), the absolute error between the approximate and exact solutions is shown. The maximum error is around 4.5×10^{-3}, observed near $t = 0.2$ and again slightly near $t = 0.4$, but for the rest of the interval, the error remains very small and nearly zero. This confirms that the HBM gives a very good approximation of the exact solution for the governing equation.

5. CONCLUSION

This study describes a HBM that has been demonstrated to be consistent, zero stable, and convergent for solving FOIVP of ODEs. When compared to exact solutions, the numerical results of the approach when applied to a few nonlinear ODEs turned out to be accurate. The error between the HBM (proposed technique) and the exact solution are observed to be decreased as the values of the independent variable are increased. The HBM is applicable to all forms of FOIVP of ODEs, both linear and nonlinear, which appear in various kinds of physical environments and in mathematical models.

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