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The Numerical Scheme and Convergence Analysis of the Stochastic Schnakenberg Model

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ABSTRACT: This article discusses coupled nonlinear stochastic partial differential equations, specifically the stochastic Schnakenberg model which is a reaction-diffusion system. The numerical approximation of the model is achieved using the proposed stochastic forward Euler (SFE) scheme, which is consistent with the given system of equations. The article also includes discussions on linear stability analysis, which shows that the proposed SFE scheme is conditionally stable. Additionally, the convergence of the schemes is discussed in the mean square sense. The numerical solution is obtained through simulations using the Python package for various parameter values. The effects of randomness are also discussed. In terms of graphical behavior, the stochastic Schnakenberg model exhibits self-replicating behavior.

Keywords: Stochastic Schnakenberg model, SFE scheme, finite difference method, analysis of schemes, graphical behavior.

Contents

1. Introduction

The importance of reaction-diffusion systems has grown in various fields, such as chemistry and biochemistry. Several noteworthy examples of these systems include the lattice Boltzmann model ([1]), Brusselator model ([2]), Lengyel-Epstein model ([3]), glycolysis model ([5]), and the Schnakenberg model ([4]), among others. Notably, the Schnakenberg model is commonly used to simulate the spread of morphogens, like calcium, in the hair cells of the whorl of Acetabularia. The mathematical representation of this model involves the following equations:

$$
\begin{aligned}
\frac{\partial C_a}{\partial t} &= D_1 \nabla^2 C_a + \kappa \left(a - C_a + C_a^2 C_l\right) \\
\frac{\partial C_l}{\partial t} &= D_2 \nabla^2 C_l + \kappa \left(b - C_a^2 C_l\right)
\end{aligned}
\tag{1.1}
$$

Here, the concentrations of activator and inhibitor are represented by C_a and C_l, respectively. The diffusion coefficients are denoted by D_1 and D_2, while the rate constants of the biochemical reactions are κ, a, and b. This model has a wide range of applications, including the study of morphogenesis processes, pattern formation in biological tissues, and spatial organization of cells in biology. It is also used in chemistry to investigate oscillatory chemical reactions, and in physics to explore self-organization phenomena in physical systems, such as liquids and crystals. Moreover, this model is employed in engineering to analyze fluid dynamics in chemical reactors and optimize industrial production conditions.

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In this analysis, we examine the 2D coupled Schnakenberg model that is subject to multiplicative time noise, as expressed below:

$$\begin{aligned}\frac{\partial C_a}{\partial t} &= D_1\nabla^2 C_a + \kappa\left(a - C_a + C_a^2 C_l\right) + v_1 C_l \dot{W}_1(t)\\ \frac{\partial C_l}{\partial t} &= D_2\nabla^2 C_l + \kappa\left(b - C_a^2 C_l\right) + v_2 C_a \dot{W}_2(t)\end{aligned} \tag{1.2}$$

The initial conditions are given by

$$C_a(x,y,0) = f(x,y) \quad C_l(x,y,0) = g(x,y) \tag{1.3}$$

The boundary conditions are either Dirichlet or homogeneous Neumann conditions, which are expressed as

$$C_a(x,y,t) = \alpha_1, \quad C_l(x,y,t) = \alpha_2,\ for\ (x,y) \in \partial D. \tag{1.4}$$

The noise strengths are determined by the Borel functions v_1 and v_2, and the time series of white noise are denoted by $\dot{W}_1(t)$ and $\dot{W}_2(t)$, respectively. The standard Brownian motions are represented by $W_1(t)$ and $W_2(t)$, and the state variables are independent of the state of the Brownian motion. In many physical systems at the micro-level, stochastic behavior is frequently observed. As a result, it is often beneficial to include some form of noise in classical models. It is true that when noise is present in the differential equation's solution, the resulting equations are known as stochastic differential equations (SDEs). Solving SDEs numerically can be a challenging task, especially when dealing with nonlinear stochastic partial differential equations (SPDEs). However, there are numerical methods available for solving such equations, which aim to provide solutions that are consistent with the given SPDEs in the mean square sense.

When dealing with stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs), it is crucial to analyze the stability of the numerical schemes used to solve them. For this reason, the von-Neumann technique is often employed to determine the stability of numerical schemes. This technique is particularly useful in assessing numerical truncation and dispersion errors in the solutions of SDEs and SPDEs. So, stability analysis is an essential step in the process of numerically solving SDEs and SPDEs.

Numerical solutions of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs) face a significant challenge: the random selection of the numerical scheme at each time step can lead to abrupt behavior of the solution. However, this issue can be controlled by ensuring stability and consistency of the numerical scheme.

In many real-world problems, physical systems are often affected by stochastic factors and production fluctuations. The stochastic effect can be either intrinsic or extrinsic. When partial differential equations are used to model such effects as a source term, they are called SPDEs. The source term can exist in equations or conditions, and SPDEs are thus considered for better representation of physical systems. Analytical solutions to SPDEs are often difficult or even impossible to obtain, so numerical approximation is carried out for such problems. Classical numerical techniques often fail to provide an approximation; therefore, various new techniques have been proposed by different researchers.

Iqbal et al. investigated the stochastic Newell-Whitehead-Segel equation and proposed two numerical schemes for its solutions. They discussed the consistency and stability of the schemes and derived the smoothness of the solution ([6]). Du and Zhang studied the numerical solutions of linear SPDEs with special additive noise and analyzed the error and convergence of finite difference and finite element methods ([7]). Pettersson and Signahl developed a numerical scheme for Ito-type SPDEs and discussed the uniform convergence of the method ([8]). Additional research on solutions of differential equations under the influence of noise can be found in ([9]- [16]). Baccouch and co-authors proposed a discontinuous Galerkin method for SDEs and proved its convergence in the mean-square sense ([17], [18]).

The paper is structured as follows. In Section 2, a stochastic forward Euler scheme is proposed to solve the problem. In Section 3, the consistency of the scheme is proven in the mean square sense. The stability analysis of the proposed scheme is provided using the Von-Neumann method in Section 4. The convergence in the mean square sense is then discussed in Section 5. Finally, the graph and discussion are presented in Section 6 and conclusion in 7.

2. Stochastic Forword Euler Scheme

Equations (1.1), and (1.2) are nonlinear SPDEs. The variable $\frac{\partial C_a}{\partial t}$, $\frac{\partial C_l}{\partial t}$, $\frac{\partial^2 C_a}{\partial x^2}$ and $\frac{\partial^2 C_l}{\partial y^2}$ are approximated as given below:

$$\frac{\partial C_a}{\partial t} = \frac{C^{n+1}_{a,i,j} - C^n_{a,i,j}}{\Delta t},$$

$$\frac{\partial C_l}{\partial t} = \frac{C^{n+1}_{l,i,j} - C^n_{l,i,j}}{\Delta t},$$

$$\frac{\partial^2 C_a}{\partial x^2} = \frac{C^n_{a,i+1,j} - 2C^n_{a,i,j} + C^n_{a,i-1,j}}{\Delta x^2},$$

$$\frac{\partial^2 C_a}{\partial y^2} = \frac{C^n_{a,i,j+1} - 2C^n_{a,i,j} + C^n_{a,i,j-1}}{\Delta y^2},$$

$$\frac{\partial^2 C_l}{\partial x^2} = \frac{C^n_{l,i+1,j} - 2C^n_{l,i,j} + C^n_{l,i-1,j}}{\Delta x^2},$$

$$\frac{\partial^2 C_l}{\partial y^2} = \frac{C^n_{l,i,j+1} - 2C^n_{l,i,j} + C^n_{l,i,j-1}}{\Delta y^2}.$$

Assuming that we divide the spatial and temporal coordinates, let Δx and Δy be the space step sizes, and Δt be the time step size:

$$x_i = i\Delta x, i = 0, 1, \ldots, M,$$

$$y_j = j\Delta y, j = 0, 1, \ldots, M,$$

$$t_n = n\Delta t, n = 0, 1, \ldots, N.$$

We can define $r_1 = \frac{\Delta t D_1}{\Delta x^2}$ and $r_2 = \frac{\Delta t D_2}{\Delta x^2}$ with $\Delta x = \Delta y$, and by substituting these values in Eqs 1.2, we obtain the following simplified equations:

$$\begin{aligned} C^{n+1}_{a,i,j} &= (1 - 4r_1 - \Delta t k)C^n_{a,i,j} + r_1(C^n_{a,i+1,j} + C^n_{a,i-1,j} + C^n_{a,i,j+1} + C^n_{a,i,j-1}) \\ &\quad + \Delta t k a + \Delta t k (C^n_{a,i,j})^2 C^n_{l,i,j} + v_1 C^n_{a,i,j}(W_1^{n+1} - W_1^n). \end{aligned} \tag{2.1}$$

$$\begin{aligned} C^{n+1}_{l,i,j} &= (1 - 4r_2)C^n_{l,i,j} + r_2(C^n_{l,i+1,j} + C^n_{l,i-1,j} + C^n_{l,i,j+1} + C^n_{l,i,j-1}) \\ &\quad + \Delta t k b + \Delta t k (C^n_{a,i,j})^2 C^n_{l,i,j} + v_2 C^n_{l,i,j}(W_2^{n+1} - W_2^n). \end{aligned} \tag{2.2}$$

So, Eqs 2.1 and 2.2 constitute the required proposed SFE scheme for Eqs 1.2.

3. Consistency of Proposed Scheme

The consistency of the scheme is proved in the mean square sense as describe in the following definition:

Definition 3.1 *[[19]–[21]]A stochastic finite difference scheme $L^n_{i,j}(U^n_{i,j}) = G^n_{i,j}$ is consistent with the SPDE $L(U) = G$ at a point (x, y, t); if there is any continuously differentiable function $\Psi = \Psi(x, y, t)$ then*

$$E||(L(\Psi) - G)^n_{i,j} - [L^n_{i,j}(\Psi(x_i, y_j, t_n)) - G^n_{i,j}]||^2 \to 0$$

as $(\Delta x, \Delta y, \Delta t) \to (0, 0, 0)$ and $(x_i, y_j, t_{n+1}) \to (x, y, t)$

Theorem 3.1 *The proposed SFE scheme for C_a and C_l in Eqs (2.1) and (2.2) is consistent with Eqs (1.2) in the mean square sense.*

Proof:

Suppose that $C_a(x,y,t)$ and $C_l(x,y,t)$ are smooth functions and applying $L(f) = \int_{t^n}^{t^{n+1}} f dt$ to the first Eq of (1.2), we get

$$L(C_a)_{i,j}^n = C_a(x_i,y_j,t_{n+1}) - C_a(x_i,y_j,t_n) - D_1\Big(\int_{t_n}^{t_{n+1}} \frac{\partial^2 C_a}{\partial x^2}(x_i,y_j,t)dt + \int_{t_n}^{t_{n+1}} \frac{\partial^2 C_a}{\partial y^2}(x_i,y_j,t)dt\Big)$$
$$+ k\int_{t_n}^{t_{n+1}} \Big(a - C_a(x_i,y_j,t) + C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\Big)dt + v_1\int_{t_n}^{t_{n+1}} C_a(x_i,y_j,t)dW_1(t)$$

By applying the proposed SFE scheme to Eq (1.2), we get

$$L_{i,j}^n(C_a) = C_a(x_i,y_j,t_{n+1}) - C_a(x_i,y_j,t_n) - D_1\Delta t\Big(\frac{C_a(x_{i+1},y_j,t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_{i-1},y_j,t_n)}{\Delta x^2}$$
$$+\frac{C_a(x_i,y_{j+1},t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_i,y_{j-1},t_n)}{\Delta y^2}\Big) - \Delta t k\Big(a - C_a(x_i,y_j,t_n) + C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\Big)$$
$$-v_1 C_a(x_i,y_j,t_n)\big(W_1^{n+1} - W_1^n\big)$$

than we have

$$L(C_a)_{i,j}^n - L_{i,j}^n(C_a) = -D_1\Big[\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial x^2}(x_i,y_j,t) - \frac{C_a(x_{i+1},y_j,t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_{i-1},y_j,t_n)}{\Delta x^2})dt$$
$$+\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial y^2}(x_i,y_j,t) - \frac{C_a(x_i,y_{j+1},t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big]$$
$$+k\Big[\int_{t_n}^{t_{n+1}}\Big(-\big[a - C_a(x_i,y_j,t) + C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big] + \big[a - C_a(x_i,y_j,t_n) + C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big]$$
$$+v_1\Big[\int_{t_n}^{t_{n+1}}(C_a(x_i,y_j,t) - C_a(x_i,y_j,t_n))dW_1(t)\Big]$$

In the mean square sense, the above equation can be written as

$$E\big|L(C_a)_{i,j}^n - L_{i,j}^n(C_a)\big|^2 \le 4D_1^2\Big[E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial x^2}(x_i,y_j,t) - \frac{C_a(x_{i+1},y_j,t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_{i-1},y_j,t_n)}{\Delta x^2})dt\Big|^2$$
$$+E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial y^2}(x_i,y_j,t) - \frac{C_a(x_i,y_{j+1},t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big|^2\Big]$$
$$+4k^2E\Big|\int_{t_n}^{t_{n+1}}\Big(-\big[a - C_a(x_i,y_j,t) + C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big] + \big[a - C_a(x_i,y_j,t_n) + C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big|^2$$
$$+4v_1^2E\Big|\int_{t_n}^{t_{n+1}}(C_a(x_i,y_j,t) - C_a(x_i,y_j,t_n))dW_1(t)\Big|^2$$

By using the Ito's integral square property, we get

$$E\big|L(C_a)_{i,j}^n - L_{i,j}^n(C_a)\big|^2 \le 4D_1^2\Big[E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial x^2}(x_i,y_j,t) - \frac{C_a(x_{i+1},y_j,t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_{i-1},y_j,t_n)}{\Delta x^2})dt\Big|^2$$
$$+E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2 C_a}{\partial y^2}(x_i,y_j,t) - \frac{C_a(x_i,y_{j+1},t_n) - 2C_a(x_i,y_j,t_n) + C_a(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big|^2\Big]$$
$$+4k^2E\Big|\int_{t_n}^{t_{n+1}}\Big(-\big[a - C_a(x_i,y_j,t) + C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big] + \big[a - C_a(x_i,y_j,t_n) + C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big|^2$$
$$+4v_1^2\int_{t_n}^{t_{n+1}} E\big|(C_a(x_i,y_j,t) - C_a(x_i,y_j,t_n))\big|^2 dt$$

than $E\big|L(C_a)_{i,j}^n - L_{i,j}^n(C_a)\big|^2 \longmapsto 0$ as $(i,j,n) \longmapsto \infty$, so the proposed scheme for C_a is consistent with the first equation of (1.2).

Similarly, we use the same step to show the consistency of Eq (2.2).

$$L(C_l)_{i,j}^n = C_l(x_i,y_j,t_{n+1}) - C_l(x_i,y_j,t_n) - D_2\Big(\int_{t_n}^{t_{n+1}} \frac{\partial^2 C_l}{\partial x^2}(x_i,y_j,t)dt + \int_{t_n}^{t_{n+1}} \frac{\partial^2 C_l}{\partial y^2}(x_i,y_j,t)dt\Big)$$

$$+k\int_{t_n}^{t_{n+1}}\Big(b-C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\Big)dt+v_2\int_{t_n}^{t_{n+1}}C_l(x_i,y_j,t)dW_2(t)$$

By applying the proposed SFE scheme to the second Eq of (1.2), we get

$$L_{i,j}^n(C_l)=C_l(x_i,y_j,t_{n+1})-C_l(x_i,y_j,t_n)-D_2\Delta t\Big(\frac{C_l(x_{i+1},y_j,t_n)-2C_l(x_i,y_j,t_n)+C_l(x_{i-1},y_j,t_n)}{\Delta x^2}$$

$$+\frac{C_l(x_i,y_{j+1},t_n)-2C_l(x_i,y_j,t_n)+C_l(x_i,y_{j-1},t_n)}{\Delta y^2}\Big)-\Delta tk\Big(b-C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\Big)$$

$$-v_2C_l(x_i,y_j,t_n)\big(W_2^{n+1}-W_2^n\big)$$

than we have

$$L(C_l)_{i,j}^n-L_{i,j}^n(C_l)=-D_2\Big[\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial x^2}(x_i,y_j,t)-\frac{C_l(x_{i+1},y_j,t_n)-2C_l(x_i,y_j,t_n)+C_l(x_{i-1},y_j,t_n)}{\Delta x^2})dt$$

$$+\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial y^2}(x_i,y_j,t)-\frac{C_l(x_i,y_{j+1},t_n)-2C_l(x_i,y_j,t_n)+C_l(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big]$$

$$+k\Big[\int_{t_n}^{t_{n+1}}\Big(-\big[b-C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big]+\big[b-C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big]$$

$$+v_2\Big[\int_{t_n}^{t_{n+1}}(C_l(x_i,y_j,t)-C_l(x_i,y_j,t_n))dW_2(t)\Big]$$

In the mean square sense, the above equation can be written as

$$E\big|L(C_l)_{i,j}^n-L_{i,j}^n(C_l)\big|^2\le 4D_2^2\Big[E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial x^2}(x_i,y_j,t)-\frac{C_l(x_{i+1},y_j,t_n)-2C_l(x_i,y_j,t_n)+C_l(x_{i-1},y_j,t_n)}{\Delta x^2})dt\Big|^2$$

$$+E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial y^2}(x_i,y_j,t)-\frac{C_l(x_i,y_{j+1},t_n)-2C_l(x_i,y_j,t_n)+C_l(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big|^2\Big]$$

$$+4k^2E\Big|\int_{t_n}^{t_{n+1}}\Big(-\big[b-C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big]+\big[b-C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big|^2$$

$$+4v_2^2E\Big|\int_{t_n}^{t_{n+1}}(C_l(x_i,y_j,t)-C_l(x_i,y_j,t_n))dW_2(t)\Big|^2$$

Finally By using the Ito's integral square property, we get

$$E\big|L(C_l)_{i,j}^n-L_{i,j}^n(C_l)\big|^2\le 4D_2^2\Big[E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial x^2}(x_i,y_j,t)-\frac{C_l(x_{i+1},y_j,t_n)-2C_l(x_i,y_j,t_n)+C_l(x_{i-1},y_j,t_n)}{\Delta x^2})dt\Big|^2$$

$$+E\Big|\int_{t_n}^{t_{n+1}}(\frac{\partial^2C_l}{\partial y^2}(x_i,y_j,t)-\frac{C_l(x_i,y_{j+1},t_n)-2C_l(x_i,y_j,t_n)+C_l(x_i,y_{j-1},t_n)}{\Delta y^2})dt\Big|^2\Big]$$

$$+4k^2E\Big|\int_{t_n}^{t_{n+1}}\Big(-\big[b-C_a^2(x_i,y_j,t)C_l(x_i,y_j,t)\big]+\big[b-C_a^2(x_i,y_j,t_n)C_l(x_i,y_j,t_n)\big]\Big)dt\Big|^2$$

$$+4v_2^2\int_{t_n}^{t_{n+1}}E\big|(C_l(x_i,y_j,t)-C_l(x_i,y_j,t_n))\big|^2dt$$

than $E\big|L(C_l)_{i,j}^n-L_{i,j}^n(C_l)\big|^2\longmapsto 0$ as $(i,j,n)\longmapsto\infty$, so the proposed scheme for C_l is consistent with with the second equation of (1.2).

□

4. Stability of the Proposed Scheme

Finite-difference computations require determinations of spatial and temporal sampling criteria. As pointed out by Kelly and Marfurt ([22]), spatial sampling is generally chosen to avoid numerical dispersion in solutions. Then, the temporal sampling is chosen to avoid numerical instability. The stability analysis for FD solutions of partial differential equations is handled using a method originally developed by Von Neumann ([23]).In this section, we will give the stability criteria of the current scheme using the von- Neumann Method .

For that, $C^n_{a,i,j}$ is replaced in the differentiel equation as given below:

$$C^n_{a,i,j} = f(t_n)e^{\iota(\lambda_x x_i+\lambda_y y_j)}, \quad \textit{with } \iota \textit{ is the complexe numbre } (\iota^2 = -1). \tag{4.1}$$

and by doing some basic calculation one gets the amplification as follows [39]:

$$E|\frac{f(t_n + \Delta t)}{f(t_n)}|^2 \leq 1 + \chi\Delta t \tag{4.2}$$

where χ is a constant. it is a necessary and sufficient condition of stability.

Theorem 4.1 *if* $|(1-4r_1-\Delta tk)+2r_1(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))| \leq 1$ *and* $|(1-4r_2)+2r_2(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))| \leq 1$*, then the proposed difference scheme is stable in mean sequence with* $(n+1)\Delta t = T$*.*

Proof: Here we suppose that $\Delta x = \Delta y$, as von-Neumann method for stability is applied to linear scheme , so non linear terms in Equations (2.1) is linearised ,so equation (2.1) can be written as

$$C^{n+1}_{a,i,j} = (1-4r_1-\Delta tk)C^n_{a,i,j} + r_1(C^n_{a,i+1,j}+C^n_{a,i-1,j}+C^n_{a,i,j+1}+C^n_{a,i,j-1}) + v_1C^n_{a,i,j}(W^{n+1}_1 - W^n_1). \tag{4.3}$$

by using equation 4.1, the above equation takes the following form:

$$f(t_n+\Delta t)e^{\iota(\lambda_x x_i+\lambda_y y_j)} = \Big[(1-4r_1-\Delta tk)+r_1(e^{\iota\lambda_x\Delta x}+e^{-\iota\lambda_x\Delta x}+e^{\iota\lambda_y\Delta x}+e^{-\iota\lambda_y\Delta x})+v_1(W^{n+1}_1-W^n_1)\Big]f(t_n)e^{\iota(\lambda_x x_i+\lambda_y y_j)} \tag{4.4}$$

than

$$f(t_n+\Delta t) = \Big[(1-4r_1-\Delta tk)+2r_1(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))+v_1(W^{n+1}_1-W^n_1)\Big]f(t_n) \tag{4.5}$$

Using the independence of brownian motion and amplification factor can be written as given below

$$E|\frac{f(t_n+\Delta t)}{f(t_n)}|^2 \leq \big|(1-4r_1-\Delta tk)+2r_1(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))\big|^2 + \big|v_1\big|^2\Delta t \tag{4.6}$$

if $\big|(1-4r_1-\Delta tk)+2r_1(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))\big| \leq 1$ then

$$E|\frac{f(t_n+\Delta t)}{f(t_n)}|^2 \leq 1+\chi\Delta t \tag{4.7}$$

where $|v_1|^2 = \chi$ is a constant, so the given scheme for C_a is stable. Repeating the same procedure for the stability of Eq (2.2),we get the following equation:

$$E|\frac{f(t_n+\Delta t)}{f(t_n)}|^2 \leq \big|(1-4r_2)+2r_2(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))\big|^2 + \big|v_2\big|^2\Delta t \tag{4.8}$$

if $\big|(1-4r_2)+2r_2(cos(\lambda_x\Delta x)+cos(\lambda_y\Delta x))\big| \leq 1$ then

$$E|\frac{f(t_n+\Delta t)}{f(t_n)}|^2 \leq 1+\chi\Delta t \tag{4.9}$$

where $|v_2|^2 = \chi$ is a constant, so the given scheme for C_l is stable. □

5. Convergence of the Proposed Scheme

The convergence of the scheme is discussed in the mean square sense.

Theorem 5.1 *The proposed scheme given by Eqs (2.1) and (2.2) is convergent in the mean square sense.*

Proof:

$$E\left|C_{a,i,j}^{n} - C_a\right|^2 = E\left|(L_{i,j}^{n})^{-1}(L_{i,j}^{n}(C_{a,i,j}^{n}) - L_{i,j}^{n}(C_a))\right|^2 \tag{5.1}$$

as the proposed scheme is consistent in the mean square sense i.e.,$(L_{i,j}^{n}(C_{a,i,j}^{n}) \rightarrow L_{i,j}^{n}(C_a))$ as $(\Delta x, \Delta y, \Delta t) \rightarrow (0, 0, 0)$:

$$E\left|(L_{i,j}^{n})^{-1}(L_{i,j}^{n}(C_{a,i,j}^{n}) - L_{i,j}^{n}(C_a))\right|^2 \rightarrow 0; \tag{5.2}$$

also, because the scheme is stable, this $(L_{i,j}^{n})^{-1}$ is bounded. So $E\left|C_{a,i,j}^{n} - C_a\right|^2 \rightarrow 0$.Hence the proposed scheme C_a is convergent in the mean square sense.Similiry we proof that C_l is convergent in the mean square sense. □

6. Graph and Discussion

We take stochastic Schnakenberg equation as follow,

$$\begin{aligned} \frac{\partial C_a}{\partial t} &= D_1 \nabla^2 C_a + \kappa \left(a - C_a + C_a^2 C_l\right) + v_1 C_l \dot{W}_1(t) \\ \frac{\partial C_l}{\partial t} &= D_2 \nabla^2 C_l + \kappa \left(b - C_a^2 C_l\right) + v_2 C_a \dot{W}_2(t) \end{aligned} \tag{6.1}$$

with the initial conditions

$$\begin{aligned} &C_a(x, y, 0) = a + b + 10^{-3} exp(-10((x - 1/2)^2 + (y - 1/2)^2)). \\ &C_l(x, y, 0) = a/(a + b)^2. \end{aligned} \tag{6.2}$$

and the boundary conditions are taken as homogeneous Neumann

$$C_a(x, y, t) = \alpha_1, \quad C_l(x, y, t) = \alpha_2, \ for\ (x, y) \in \partial D. \tag{6.3}$$

Here, we have taken the problems for the numerical solution of the stochastic Schnakenberg model perturbed by time series with white noise. The physical behavior of Problem is depicted through Figures 1–13 respectively. We have compared the numerical results of the stochastic Schnakenberg model with the classical Schnakenberg model when the noise strength $v_1 = v_2 \rightarrow 0$ and these results are the same. When the values of the noise strength increased, the fluctuation in the plots increased. The noisy strength disturbed the pattern formed in Schnakenberg models and it can be controlled with the parameter v_1 and v_2. The proposed SFE finite difference scheme is conditionally stable and conditions are given in the corresponding stability theorem of the scheme

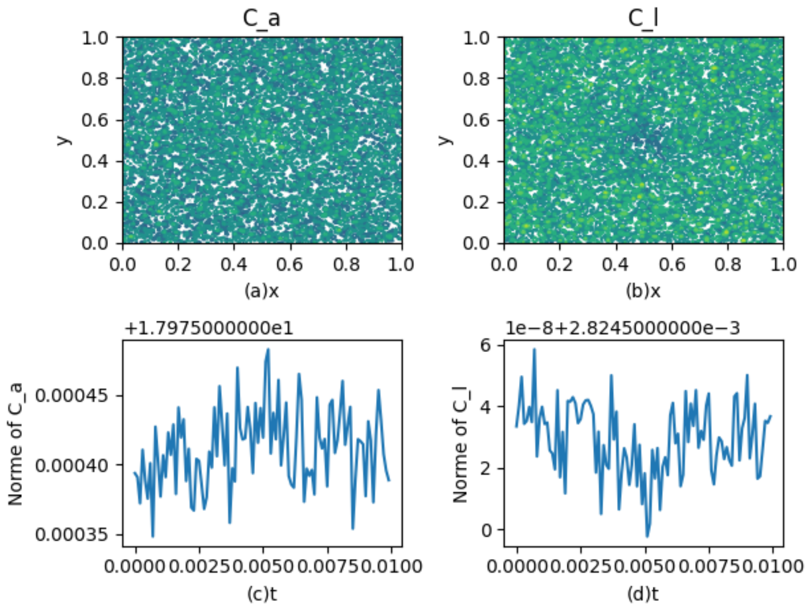

Figure 1: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 0.001, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 0.1$

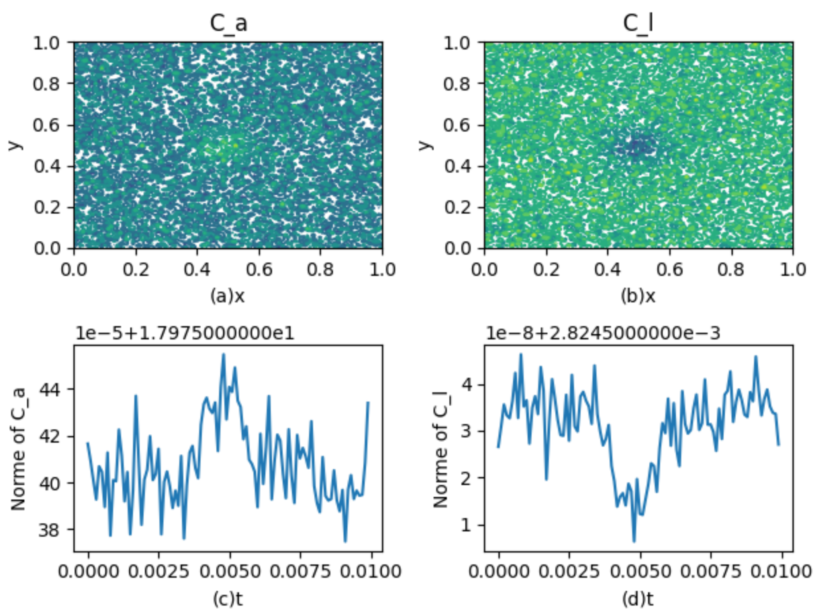

Figure 2: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 0.001, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 0.001$.

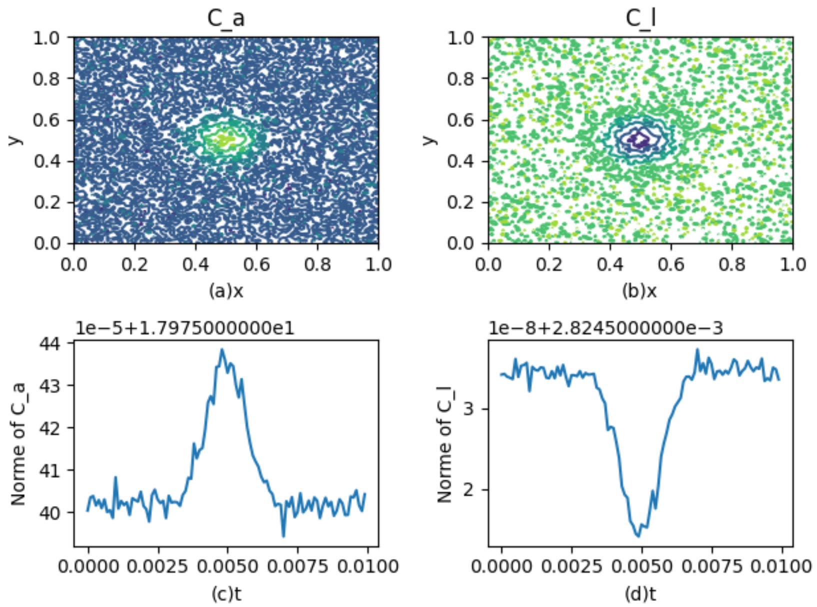

Figure 3: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, ,$D_1 = D_2 = 0.001, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 0.00001$.

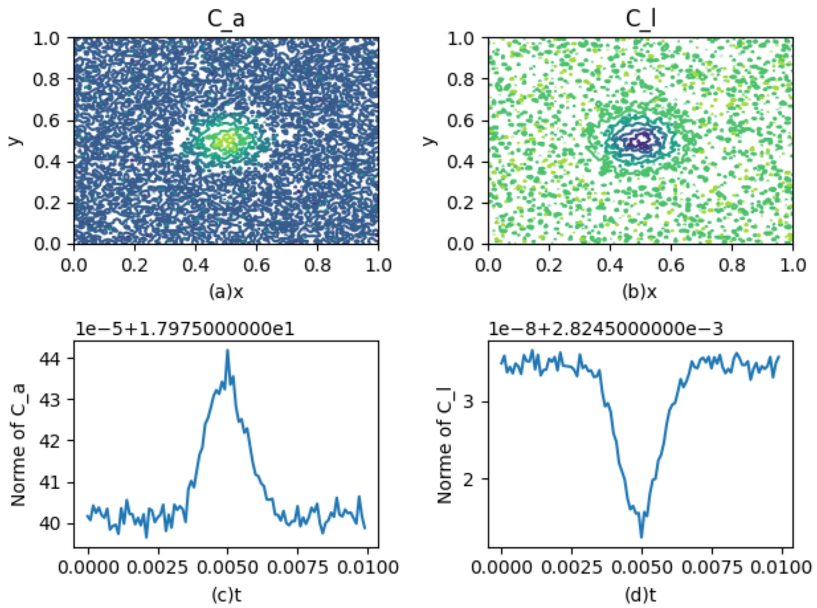

Figure 4: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 0.001, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 0.0000001$

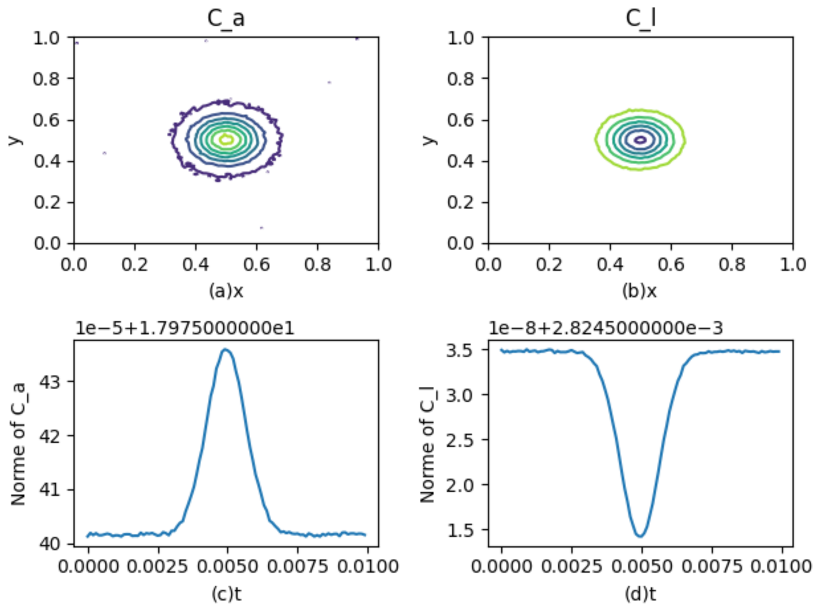

Figure 5: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 0.001, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-9}$

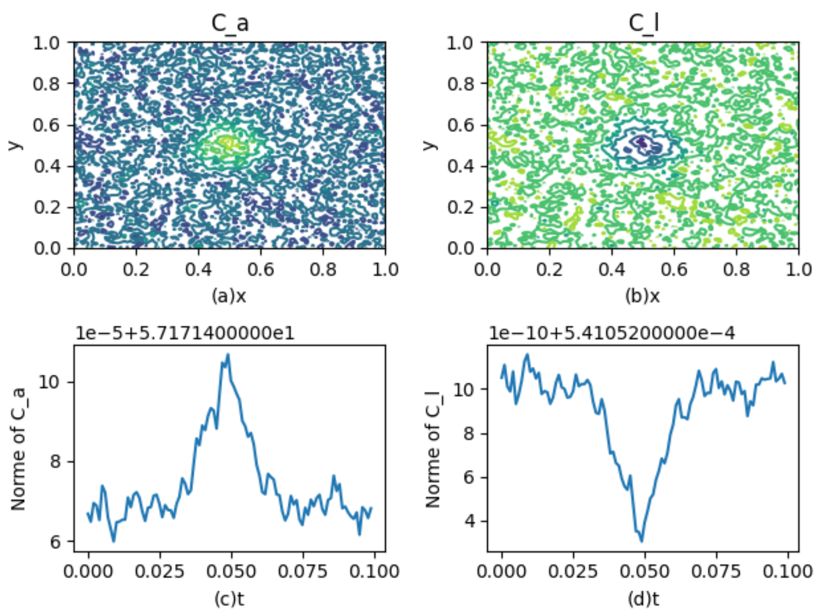

Figure 6: (a,c) and (b,d) are the numerical solutions(a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 1, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$

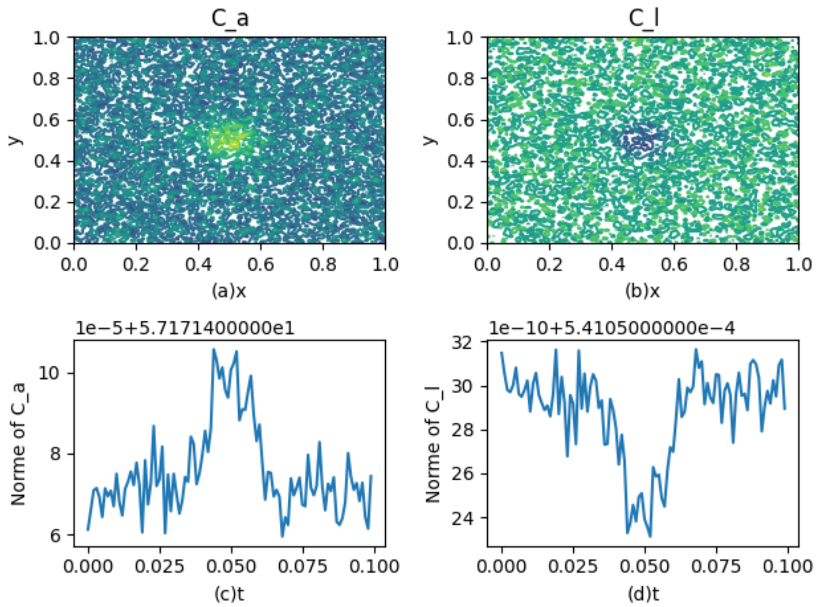

Figure 7: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 0.1, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$

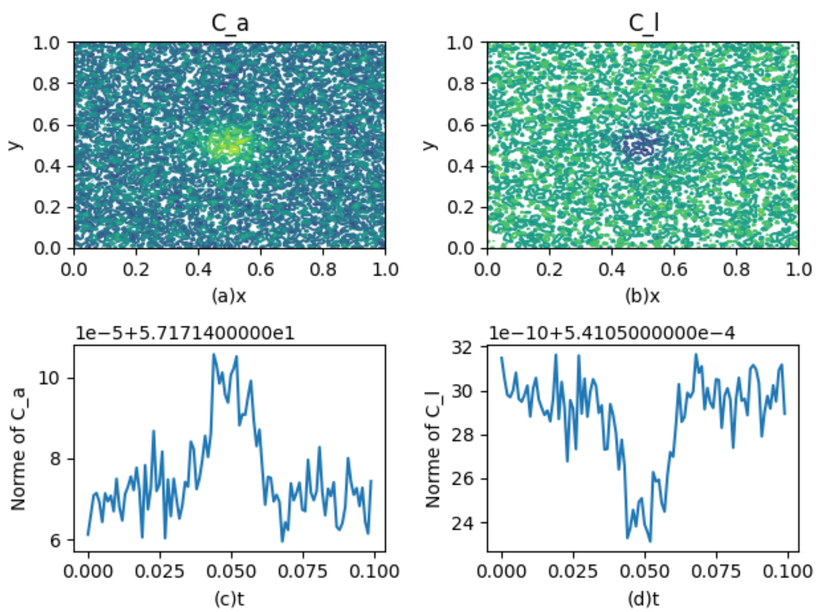

Figure 8: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 0.01, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$.

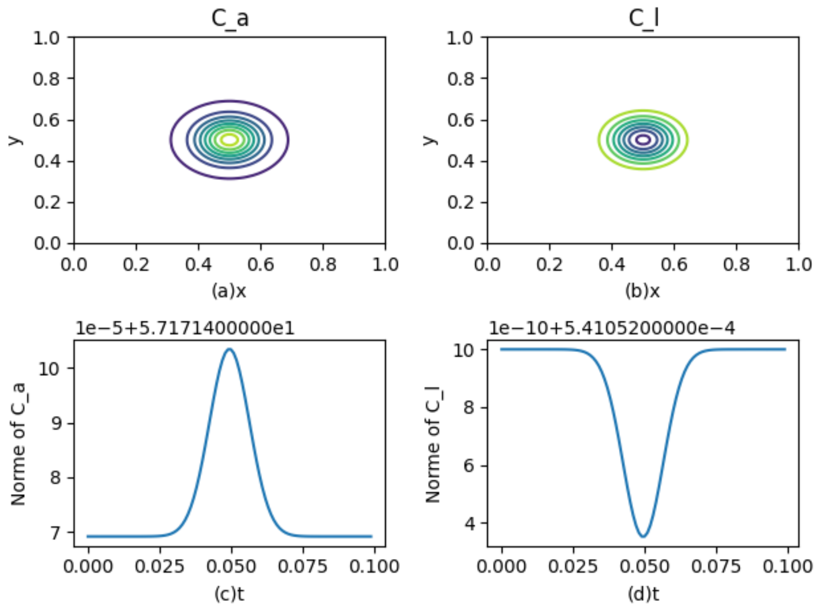

Figure 9: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e,$D_1 = D_2 = 0.01, k = 100, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-10}$.

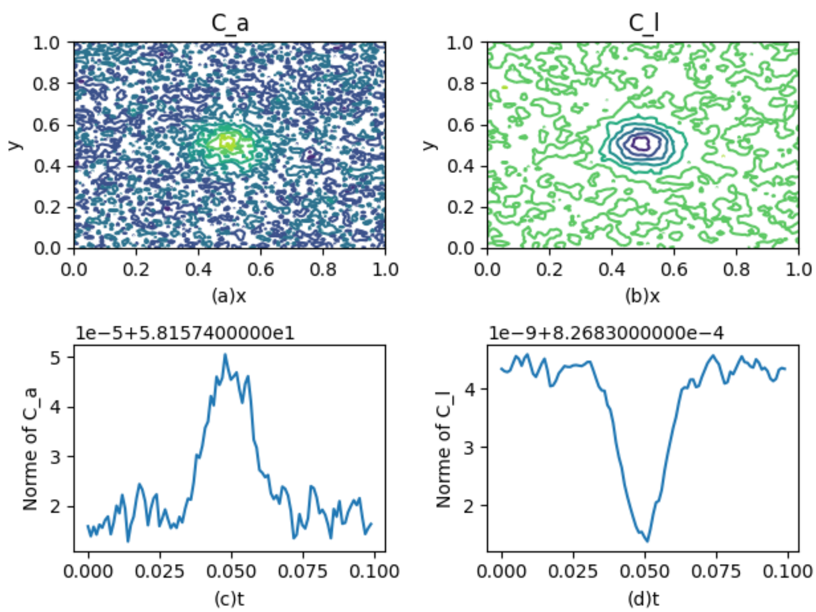

Figure 10: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 1, k = 10, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$.

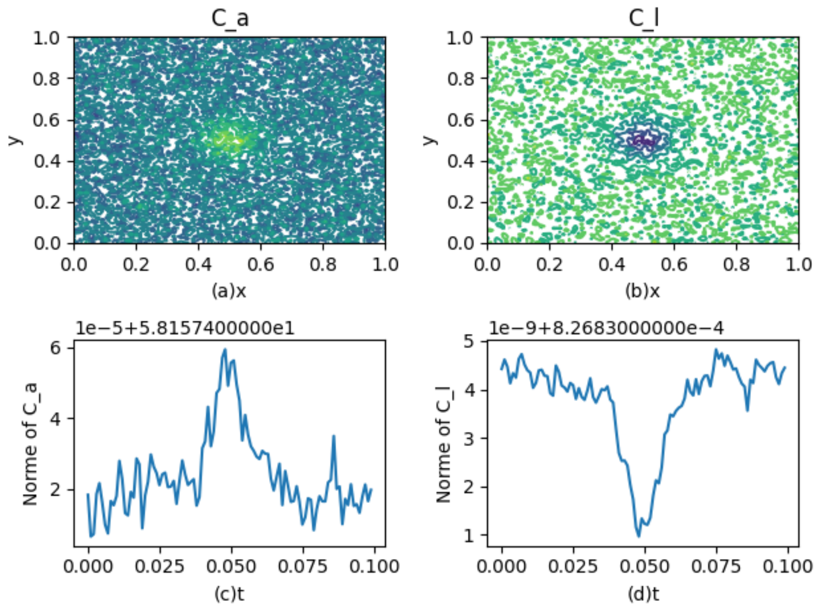

Figure 11: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 0.1, k = 10, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$.

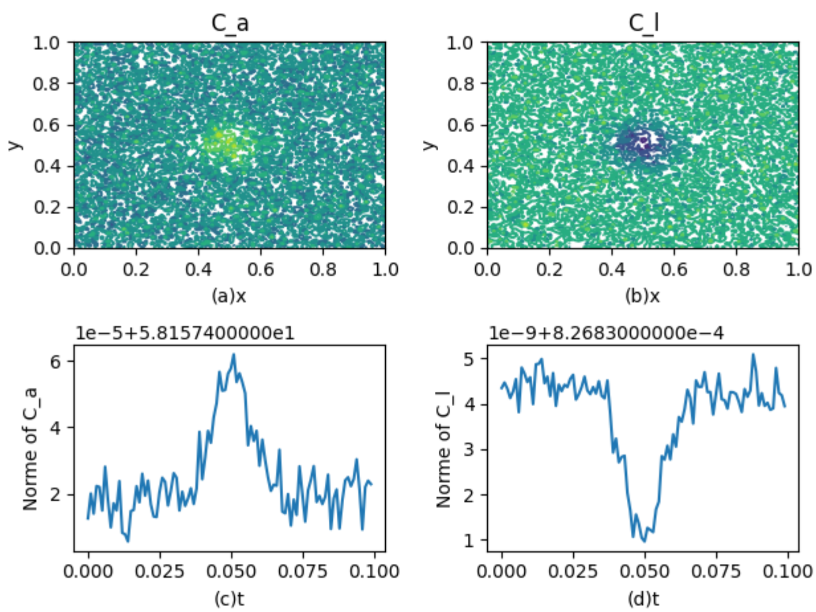

Figure 12: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 0.01, k = 10, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-3}$.

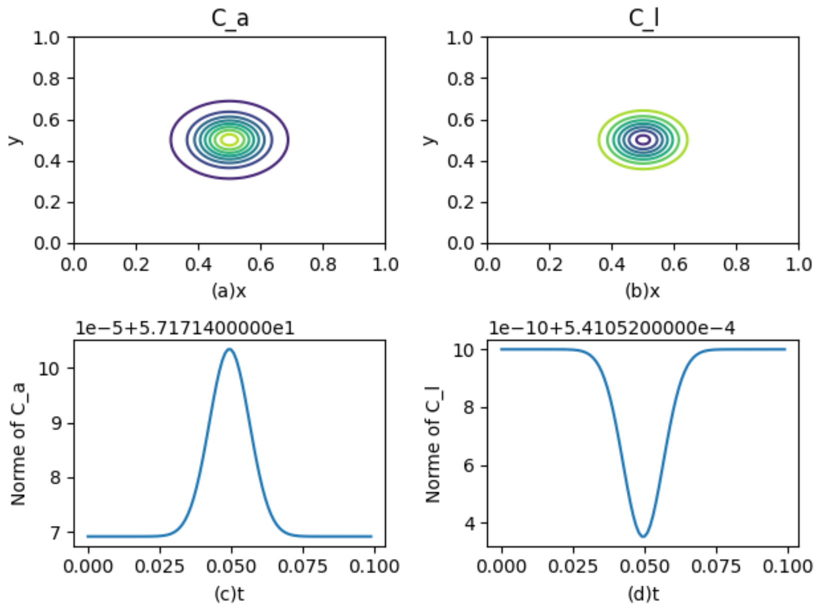

Figure 13: (a,c) and (b,d) are the numerical solutions of C_a and C_l in 2D, and their norm over time respectively were obtained using the proposed SFE finite difference scheme,for various parameter values i.e, $D_1 = D_2 = 0.01, k = 10, a = 0.13, b = 0.765, v_1 = v_2 = 10^{-10}$.

7. Conclusion

In this manuscript, the stochastic Schnakenberg model has been numerically discussed. The underlying model is an autocatalytic chemical reaction-diffusion system that produces a variety of patterns.The numerical solution has been obtained by using the proposed SFE scheme. The linear stability analysis of each scheme with the help of the von- Neumann criteria has been presented. The proposed SFE scheme is conditionally stable and the condition of stability is given in the corresponding stability theorem .The scheme is consistent in the mean square sense with a given system of equations. The convergence of the scheme is also proved. When $v_1 = v_2 = 10^{-10}$, the stochastic Schnakenberg model truly resembles classical Schnakenberg models as shown in Figures 5, 9 and 13 for the proposed scheme. When the value of the noise strength increased then the pattern deformed.The 2D graphical representations yielded the efficacy of the time-efficient scheme. The stochastic behavior of the underlying model is the novelty. This article will motivate and encourage the researchers, providing deep insight into the chemical reaction models under the influence of random processes.

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