



## Topological Rings through the Lens of $h$ -Open Sets

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**ABSTRACT:** This paper centers around the introduction of two novel concepts:  $h$ -topological ring and  $h$ -irresolute topological ring defined through the lens of  $h$ -open sets. A  $h$ -topological ring may not necessarily qualify as a topological ring. We also produced several examples of  $h$ -topological rings within infinite topological spaces characterized by co-countable and co-finite topology. Additionally, several essential properties of  $h$ -irresolute topological rings and  $h$ -topological rings are analyzed. This paper examines the translation of open and closed sets in an  $h$ -topological ring and an  $h$ -irresolute topological ring via an invertible element. A subring of a  $h$ -topological ring (or  $h$ -irresolute topological ring) exhibits closure ( $h$ -closure), which is also an  $h$ -topological ring (or  $h$ -irresolute topological ring).

**Keywords:**  $h$ -open sets,  $h$ -topological rings,  $h$ -irresolute topological ring, topological ring.

### Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>2</b>
<b>3</b>	<b><math>h</math>- topological ring</b>	<b>2</b>
<b>4</b>	<b><math>h</math>-irresolute topological ring</b>	<b>7</b>

### 1. Introduction

In modern mathematics, the study of algebraic structures endowed with topological properties has played a crucial role in various fields including functional analysis, algebraic geometry and number theory. Among such structures, topological rings serve as a fundamental concept that unifies algebraic and topological perspectives. A topological ring is a ring equipped with a topology that makes ring operations continuous. This additional structure allows for a more refined analysis of ring-theoretic properties in conjunction with topological constraints. The study of topological rings extends the classical theory of rings by incorporating continuity, thereby enabling the application of topological methods to algebraic problems. Notable examples of topological rings include topological fields, Banach algebras, and rings of continuous functions. These structures play an essential role in areas such as harmonic analysis,  $p$ -adic number theory and algebraic topology. Certain classical literature including [2,4,5,11] provide a thorough analysis of topological rings. Topological rings have been the subject of intense research since the 1930s, and they have undergone substantial developments recently. By developing a deeper understanding of topological rings, we gain valuable tools for bridging algebra and topology, leading to broader applications in modern mathematical research.

In this paper, we investigate the extension of topological rings via  $h$ -open sets, focusing on the conditions under which this extension preserves topological and algebraic properties. We explore fundamental concepts such as continuous ring homomorphisms and ideals that contribute to the construction and classification of these extensions. Many researchers have explored the concept of topological rings and provided some extensions. Some of the extensions of topological rings can be seen in [3,6,9,10]. The motivation for studying extensions of topological rings stems from their applications in functional analysis, algebraic topology, and arithmetic geometry, where topological structures influence algebraic behavior. Our approach includes both theoretical analysis and examples illustrating how extensions function in different contexts.

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## 2. Preliminaries

This section covers some fundamental definitions that will be utilized in the next sections.  $\mathcal{X}$  and  $\mathcal{Y}$  will represent two topological spaces with topologies  $\tau$  and  $\sigma$  respectively, on which no separation axioms are imposed. Notations  $Int(\mathcal{A})$  and  $Cl(\mathcal{A})$  stand for the interior and closure of a subset  $\mathcal{A}$  of topological space  $\mathcal{X}$ , respectively. In 2021, F. Abbas introduced the idea of  $h$ -open sets. A subset  $\mathcal{A}$  of a topological space  $\mathcal{X}$  is said to be  **$h$ -open** [1] if  $\mathcal{A} \subseteq Int(\mathcal{A} \cup \mathcal{U})$  for every non-empty open set  $\mathcal{U}$ , where  $\mathcal{U} \neq \mathcal{X}$ . The definition of  $h$ -open sets makes it clear that every open set is  $h$ -open set but converse need not be true. Notation  $\tau_h$  will be used for the family of  $h$ -open sets in  $\mathcal{X}$ . Now let us recall some definitions from [1] that will be used frequently.  **$h$ -interior** of a subset  $\mathcal{A}$  of  $\mathcal{X}$  is denoted by  $Int_h(\mathcal{A})$  and is defined as the union of all  $h$ -open sets in  $\mathcal{X}$  contained in  $\mathcal{A}$ . A point  $x \in Int_h(\mathcal{A})$  if there exists an  $h$ -open set  $\mathcal{U}$  containing  $x$  such that  $x \in \mathcal{U} \subseteq \mathcal{A}$ . Furthermore,  **$h$ -closure** of  $\mathcal{A} \subseteq \mathcal{X}$  is denoted by  $Cl_h(\mathcal{A})$  and is defined as the intersection of  $h$ -open sets in  $\mathcal{X}$  containing  $\mathcal{A}$ . A point  $x \in Cl_h(\mathcal{A})$  if every  $h$ -open set containing  $x$  intersects  $\mathcal{A}$ . A mapping  $g : \mathcal{X} \rightarrow \mathcal{Y}$  is said to be  **$h$ -continuous**( **$h$ -irresolute**) if inverse image of every open set( $h$ -open set respectively) in  $\mathcal{Y}$  is  $h$ -open in  $\mathcal{X}$ . Further,  $g$  is said to be  **$h$ -totally continuous** if inverse image of every  $h$ -open set in  $\mathcal{Y}$  is a clopen set in  $\mathcal{X}$ . Also,  $g$  is said to be  **$h$ -open** if image of every open set in  $\mathcal{X}$  is  $h$ -open in  $\mathcal{Y}$ . Furthermore,  $g$  is  **$h$ -homeomorphism** if it is bijective,  $h$ -continuous and  $h$ -open. For additional results regarding  $h$ -open sets, one can see [1,7,8,10].

## 3. $h$ - topological ring

This section delves into the interplay between topological rings and  $h$ -open sets, focusing on their impact on continuity, ideal structures, and homomorphisms. We explore fundamental properties, provide illustrative examples, and establish key results that highlight the significance of  $h$ -open sets in characterizing and refining the topological rings.

**Definition 3.1** A ring  $(\mathcal{R}, +, \cdot)$  with the topology  $\tau$  is said to be an  **$h$ -topological ring** if ring operations are  $h$ -continuous. Equivalently, if following holds:

- (1)  $(\mathcal{R}, +)$  is an  $h$ -topological group [7].
- (2) For every  $r_1, r_2 \in \mathcal{R}$  and open set  $\mathcal{U}$  in  $\mathcal{R}$  containing  $r_1, r_2$ , there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ .

We shall usually denote an  $h$ -topological ring by  $(\mathcal{R}, +, \cdot, \tau)$ .

**Note 1** Throughout this section,  $\mathcal{R}$  refers to an  $h$ -topological ring endowed with the topology  $\tau$  and binary operations  $+$  and  $\cdot$ .

It is evident from the definition that every topological ring is an  $h$ -topological ring but converse need not be true.

**Example 3.1** (1) Any ring with discrete or indiscrete topology are trivial examples of  $h$ -topological ring.

- (2)  $\mathbb{R}$ , the ring of real numbers with the usual topology is another example of an  $h$ -topological ring.
- (3) Consider  $\mathcal{R} = (\mathbb{Z}_2[i], \oplus, \odot)$ , the ring of Gaussian integers modulo 2 with  $\tau = \{\emptyset, \{0, 1, i\}, \mathcal{R}\}$ . Then  $\tau_h = \mathcal{P}(\mathcal{R})$ , where  $\mathcal{P}(\mathcal{R})$  denotes the power set of  $\mathcal{R}$ . Clearly  $(\mathcal{R}, \tau)$  is an  $h$ -topological ring. Moreover, it does not form a topological ring because  $\{0, 1, i\}$  is an open neighborhood of 0 in  $\mathcal{R}$  and  $(1+i) \oplus (1+i) = 0$ . Here  $\mathcal{R}$  is the only possible neighborhood of  $1+i$  such that  $\mathcal{R} \oplus \mathcal{R}$  is not contained in  $\{0, 1, i\}$ .

**Theorem 3.1** Consider a ring  $(\mathcal{R}, +, \cdot)$  endowed with topology  $\tau$ . Suppose  $\tau_h$  is discrete. Then  $(\mathcal{R}, +, \cdot, \tau)$  is an  $h$ -topological ring.

**Proof:** Proof of this theorem can be obtained utilizing Theorem 3.1 [7]. □

We provided few results in [10] that generates  $h$ -open sets in infinite topological spaces.

**Theorem 3.2** [10] Consider a non-empty set  $\mathcal{X}$  endowed with co-finite topology. Then  $\{x\}$  is  $h$ -open for all  $x \in \mathcal{X}$ .

**Corollary 3.1** [10] Consider a non-empty set  $\mathcal{X}$  endowed with co-finite topology. Then  $\tau_h = \mathcal{P}(\mathcal{X})$ .

**Theorem 3.3** [10] Consider a non-empty set  $\mathcal{X}$  endowed with co-countable topology. Then  $\tau_h = \mathcal{P}(\mathcal{X})$ .

**Example 3.2** [10]  $\mathbb{R}$ , the set of real numbers with the topology  $\tau = \{\mathcal{A} \subseteq \mathbb{R} : \mathcal{A} = \phi \text{ or } \mathcal{X} \setminus \mathcal{A} \text{ is bounded}\}$ . Then  $\tau_h = \mathcal{P}(\mathcal{X})$ .

Now we can generate few more examples of  $h$ -topological ring.

1.  $(\mathbb{R}, +, \cdot, \tau)$  endowed with co-finite topology is an  $h$ -topological ring (see Theorem 3.1 and Corollary 3.1).
2.  $(\mathbb{R}, +, \cdot, \tau)$  endowed with co-countable topology is an  $h$ -topological ring (see Theorem 3.1 and Theorem 3.3).
3. In Example 3.2,  $\mathbb{R}$ , the set of real numbers is an  $h$ -topological ring (see Theorem 3.1).

In the following section, we demonstrate results on ring multiplication alone that also hold for ring addition in order to avoid duplicate proofs.

**Note 2**  $\mathcal{I}(\mathcal{R})$  denotes the set of all invertible elements in an  $h$ -topological ring  $\mathcal{R}$ .

**Theorem 3.4** Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{I}(\mathcal{R})$ . If  $\mathcal{A} \in \tau$ , then

- (1)  $r \cdot \mathcal{A} \in \tau_h$ ;
- (2)  $\mathcal{A} \cdot r \in \tau_h$

**Proof:**

- (1) Let  $p \in r \cdot \mathcal{A}$ . Then  $a = r^{-1} \cdot p$  for some  $a \in \mathcal{A}$ . Since ring multiplication is  $h$ -continuous, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $p$ , respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{A}$ . This implies that  $p \in \mathcal{U}_2 \subseteq r \cdot \mathcal{A}$ . Thus,  $r \cdot \mathcal{A} \in \tau_h$ .
- (2) Proof for second part follows along similar lines.

□

**Corollary 3.2** Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{R}$ . If  $\mathcal{A} \in \tau$ , then

- (1)  $r + \mathcal{A} \in \tau_h$ ;
- (2)  $\mathcal{A} + r \in \tau_h$

**Theorem 3.5** Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{I}(\mathcal{R})$ . Let  $\mathcal{K}$  be a closed set in  $\mathcal{R}$ . Then

- (1)  $r \cdot \mathcal{K}$  is  $h$ -closed;
- (2)  $\mathcal{K} \cdot r$  is  $h$ -closed

**Proof:**

- (1) Let  $p \in Cl_h(r \cdot \mathcal{K})$  and an open set  $\mathcal{U}$  in  $\mathcal{R}$  containing  $q$ , where  $q = r^{-1} \cdot p$ . Then by given hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $p$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Clearly,  $\mathcal{U}_2$  intersects  $r \cdot \mathcal{K}$  and suppose  $s \in \mathcal{U}_2 \cap r \cdot \mathcal{K}$ . Then  $r^{-1} \cdot s \in \mathcal{K} \cap \mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{K} \cap \mathcal{U}$ . This implies that  $q \in \mathcal{K}$  which further implies that  $p \in r \cdot \mathcal{K}$ . Thus  $Cl_h(r \cdot \mathcal{K}) \subseteq r \cdot \mathcal{K}$ .

(2) Proof for  $\mathcal{K} \cdot r$  follows in the similar manner as above. □

**Corollary 3.3** *Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{R}$ . Let  $\mathcal{K}$  be a closed set in  $\mathcal{R}$ . Then*

(1)  $r + \mathcal{K}$  is  $h$ -closed;

(2)  $\mathcal{K} + r$  is  $h$ -closed

**Definition 3.2** [10] *Let  $(\mathcal{X}, \tau)$  be a topological space. A collection  $\{\mathcal{H}_i : i \in I\}$  of  $h$ -open sets in  $(\mathcal{X}, \tau)$  is called an  $h$ -open cover if  $\mathcal{X} \subseteq \bigcup_{i \in I} \mathcal{H}_i$ .*

**Definition 3.3** [10] *A topological space  $(\mathcal{X}, \tau)$  is called  $h$ -compact if every  $h$ -open cover has a finite subcover.*

**Theorem 3.6** *Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{I}(\mathcal{R})$ . If  $\mathcal{A}$  is  $h$ -compact subset of  $\mathcal{R}$ , then*

(1)  $r \cdot \mathcal{A}$  is compact;

(2)  $\mathcal{A} \cdot r$  is compact.

**Proof:**

(1) Let  $\{\mathcal{H}_i : i \in I\}$  be an open cover of  $r \cdot \mathcal{A}$ . Then  $r \cdot \mathcal{A} \subseteq \bigcup_{i \in I} \mathcal{H}_i$  implies  $\mathcal{A} \subseteq \bigcup_{i \in I} (r^{-1} \cdot \mathcal{H}_i)$ . Since each  $\mathcal{H}_i$  is open subset of  $\mathcal{R}$ , by Theorem 3.4,  $r^{-1} \cdot \mathcal{H}_i$  is  $h$ -open. Also,  $\mathcal{A}$  is  $h$ -compact implies  $\mathcal{A} \subseteq \bigcup_{i=1}^n (r^{-1} \cdot \mathcal{H}_i)$ . Thus  $r \cdot \mathcal{A} \subseteq \bigcup_{i=1}^n (\mathcal{H}_i)$ . This implies that  $r \cdot \mathcal{A}$  is compact.

(2) Analogous to (1). □

**Corollary 3.4** *Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{R}$ . If  $\mathcal{A}$  is  $h$ -compact subset of  $\mathcal{R}$ , then*

(1)  $r + \mathcal{A}$  is compact;

(2)  $\mathcal{A} + r$  is compact.

**Theorem 3.7** *Consider an  $h$ -topological ring  $\mathcal{R}$  and a subset  $\mathcal{A}$  of  $\mathcal{R}$ . Then for each  $r \in \mathcal{R}$ , we have*

(1)  $Int(r \cdot \mathcal{A}) \subseteq r \cdot Int_h(\mathcal{A})$ ;

(2)  $r \cdot Cl_h(\mathcal{A}) \subseteq Cl(r \cdot \mathcal{A})$

**Proof:**

(1) Let  $p \in Int(r \cdot \mathcal{A})$ . Then  $p = r \cdot a$ , for some  $a \in \mathcal{A}$ . By given hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r$  and  $a$  respectively, such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq Int(r \cdot \mathcal{A})$ . This implies that  $r \cdot \mathcal{U}_2 \subseteq \mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq Int(r \cdot \mathcal{A}) \subseteq r \cdot \mathcal{A}$ . Consequently,  $r \cdot \mathcal{U}_2 \subseteq r \cdot Int_h(\mathcal{A})$ . This results in  $p \in r \cdot Int_h(\mathcal{A})$ .

(2) Let  $p \in r \cdot Cl_h(\mathcal{A})$ . Then  $p = r \cdot a$ , for some  $a \in Cl_h(\mathcal{A})$ . Now consider an open set  $\mathcal{U}$  in  $\mathcal{R}$  containing  $p$ . By given hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r$  and  $a$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . As  $a \in Cl_h(\mathcal{A})$ , we have  $\mathcal{A}$  intersects  $\mathcal{U}_2$  and suppose  $q \in \mathcal{A} \cap \mathcal{U}_2$ . Thus  $r \cdot q \in (r \cdot \mathcal{A}) \cap (\mathcal{U}_1 \cdot \mathcal{U}_2) \subseteq (r \cdot \mathcal{A}) \cap \mathcal{U}$  and we have  $(r \cdot \mathcal{A}) \cap \mathcal{U}$  non-empty. Clearly,  $p \in Cl(r \cdot \mathcal{A})$ . □

**Theorem 3.8** Consider an  $h$ -topological ring  $\mathcal{R}$  with unity and a subset  $\mathcal{A}$  of  $\mathcal{R}$ . Then for each  $r \in \mathcal{I}(\mathcal{R})$ , we have following:

- (1)  $r \cdot \text{Int}(\mathcal{A}) \subseteq \text{Int}_h(r \cdot \mathcal{A})$ ;
- (2)  $r \cdot \text{Int}(\mathcal{A}) \subseteq \text{Int}_h(r \cdot \text{Int}(\mathcal{A}))$ ;
- (3)  $\text{Cl}_h(r \cdot \mathcal{A}) \subseteq r \cdot \text{Cl}(\mathcal{A})$ ;
- (4)  $\text{Cl}_h(r \cdot \text{Cl}(\mathcal{A})) \subseteq r \cdot \text{Cl}(\mathcal{A})$ .

**Proof:**

- (1) Let  $p \in r \cdot \text{Int}(\mathcal{A})$ . Then  $p = r \cdot a$  for some  $a \in \text{Int}(\mathcal{A})$ . Then by given hypothesis, we have  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $a$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \text{Int}(\mathcal{A})$ . This implies that  $r^{-1} \cdot \mathcal{U}_2 \subseteq \mathcal{A}$  and hence  $\mathcal{U}_2 \subseteq r \cdot \mathcal{A}$ . Thus,  $p \in \text{Int}_h(r \cdot \mathcal{A})$ .
- (2) Let  $p \in r \cdot \text{Int}(\mathcal{A})$ . Then  $p = r \cdot a$  for some  $a \in \text{Int}(\mathcal{A})$ . Then by given hypothesis, we have  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $a$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \text{Int}(\mathcal{A})$ . Now  $r^{-1} \cdot \mathcal{U}_2 \subseteq \text{Int}_h(\mathcal{A})$  and thus  $\mathcal{U}_2 \subseteq r \cdot \text{Int}_h(\mathcal{A})$ . Clearly,  $p \in \text{Int}_h(r \cdot \text{Int}(\mathcal{A}))$ .
- (3) Let  $p \in \text{Cl}_h(r \cdot \mathcal{A})$  and  $\mathcal{U}$  be an open set containing  $q = r^{-1} \cdot p$ . Since  $\mathcal{R}$  is given to be an  $h$ -topological ring, we have  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $q$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Since  $p \in \text{Cl}_h(r \cdot \mathcal{A})$ ,  $\mathcal{U}_2$  intersects  $r \cdot \mathcal{A}$  and suppose  $s \in (r \cdot \mathcal{A}) \cap \mathcal{U}_2$ . This implies that  $r^{-1} \cdot s \in \mathcal{A} \cap (\mathcal{U}_1 \cdot \mathcal{U}_2) \subseteq \mathcal{A} \cap \mathcal{U}$  and thus  $\mathcal{A}$  intersects  $\mathcal{U}$ . Therefore, every open set containing  $q$  intersects  $\mathcal{A}$  and consequently  $q \in \text{Cl}(\mathcal{A})$ . Hence,  $p = r \cdot q \in r \cdot \text{Cl}(\mathcal{A})$ .
- (4) Let  $p \in \text{Cl}_h(r \cdot \text{Cl}(\mathcal{A}))$  and  $\mathcal{U}$  be an open set containing  $q = r^{-1} \cdot p$ . By given hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $q$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . In particular,  $\mathcal{U}_2 \subseteq r \cdot \mathcal{U}$ . It is evident from  $p \in \text{Cl}_h(r \cdot \text{Cl}(\mathcal{A}))$  that  $r \cdot \mathcal{U}$  intersects  $r \cdot \text{Cl}(\mathcal{A})$ . This implies  $\mathcal{U}$  intersects  $\text{Cl}(\mathcal{A})$  and thus  $\mathcal{U}$  intersects  $\mathcal{A}$ . Hence,  $q \in \text{Cl}(\mathcal{A})$  and so  $p = r \cdot q \in r \cdot \text{Cl}(\mathcal{A})$ .

□

**Corollary 3.5** Let  $\mathcal{R}$  be an  $h$ -topological ring and  $\mathcal{A}$  be a subset of  $\mathcal{R}$ . Then for every  $r \in \mathcal{R}$ , we have following:

- (1)  $r + \text{Int}(\mathcal{A}) \subseteq \text{Int}_h(r + \mathcal{A})$ ;
- (2)  $r + \text{Int}(\mathcal{A}) \subseteq \text{Int}_h(r + \text{Int}(\mathcal{A}))$ ;
- (3)  $\text{Cl}_h(r + \mathcal{A}) \subseteq r + \text{Cl}(\mathcal{A})$ ;
- (4)  $\text{Cl}_h(r + \text{Cl}(\mathcal{A})) \subseteq r + \text{Cl}(\mathcal{A})$ .

**Theorem 3.9** Consider two subsets  $\mathcal{A}$  and  $\mathcal{B}$  of an  $h$ -topological ring  $\mathcal{R}$ . Then  $\text{Cl}_h(\mathcal{A}) \cdot \text{Cl}_h(\mathcal{B}) \subseteq \text{Cl}(\mathcal{A} \cdot \mathcal{B})$ .

**Proof:** Let  $p \in \text{Cl}_h(\mathcal{A}) \cdot \text{Cl}_h(\mathcal{B})$ . Then  $p = q \cdot r$ , for some  $q \in \text{Cl}_h(\mathcal{A})$  and  $r \in \text{Cl}_h(\mathcal{B})$ . Let  $\mathcal{U}$  be an open set in  $\mathcal{R}$  containing  $p$ . Further, since  $\mathcal{R}$  is given to be an  $h$ -topological ring, we have  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $q$  and  $r$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . As  $q \in \text{Cl}_h(\mathcal{A})$  and  $r \in \text{Cl}_h(\mathcal{B})$ ,  $\mathcal{U}_1$  intersects  $\mathcal{A}$  and  $\mathcal{U}_2$  intersects  $\mathcal{B}$ . Suppose  $s \in \mathcal{U}_1 \cap \mathcal{A}$  and  $t \in \mathcal{U}_2 \cap \mathcal{B}$ . This implies that  $s \cdot t \in (\mathcal{A} \cdot \mathcal{B}) \cap \mathcal{U}$  and thus  $\mathcal{U}$  intersects  $\mathcal{A} \cdot \mathcal{B}$ . This further implies that  $p \in \text{Cl}(\mathcal{A} \cdot \mathcal{B})$ . □

**Corollary 3.6** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two subsets of an  $h$ -topological ring  $\mathcal{R}$ . Then  $\text{Cl}_h(\mathcal{A}) + \text{Cl}_h(\mathcal{B}) \subseteq \text{Cl}(\mathcal{A} + \mathcal{B})$ .

**Theorem 3.10** Consider an  $h$ -topological ring  $\mathcal{R}$  and  $r \in \mathcal{R}$ . Let  $\mathcal{A}$  and  $\mathcal{B}$  be any two subsets of  $\mathcal{R}$ . Then the following statements are true:

- (1) The mappings  $\sum_r : \mathcal{R} \rightarrow \mathcal{R} ; x \mapsto x \cdot r$  and  $\sum'_r : \mathcal{R} \rightarrow \mathcal{R} ; x \mapsto r \cdot x$ , for  $x \in \mathcal{R}$ , are  $h$ -continuous mappings.
- (2) The mappings  $\Gamma_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\Gamma'_r : \mathcal{R} \rightarrow \mathcal{R}$ , where  $\Gamma_r(x) = x + r$  and  $\Gamma'_r(x) = r + x$ , for  $x \in \mathcal{R}$ , are  $h$ -continuous mappings.

**Proof:**

1. Let  $x \in \mathcal{R}$ . Then  $\sum_r(x) = x \cdot r$ . Let  $\mathcal{U}$  be any open set in  $\mathcal{R}$  containing  $x \cdot r$ . Clearly, by Definition 3.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $x$  and  $r$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Thus  $\sum_r(\mathcal{U}_1) = \mathcal{U}_1 \cdot r \subseteq \mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . This proves that  $\sum_r$  is an  $h$ -continuous mapping. Similarly we can prove that the mapping  $\sum'_r$  is also  $h$ -continuous.
2. Let  $x \in \mathcal{R}$  and  $\mathcal{U}$  be any open set in  $\mathcal{R}$  containing  $\Gamma_r(x) = x + r$ . By hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $x$  and  $r$  respectively such that  $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . Thus  $\Gamma_r(\mathcal{U}_1) = \mathcal{U}_1 + r \subseteq \mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . This proves that  $\Gamma_r$  is an  $h$ -continuous mapping. Similarly we can prove that the mapping  $\Gamma'_r$  is also  $h$ -continuous. □

**Theorem 3.11** Let  $\mathcal{R}$  be an  $h$ -topological ring with unity and  $r \in I(\mathcal{R})$ . Then the mappings  $\sum_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\sum'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 3.10) are  $h$ -homeomorphisms.

**Proof:** It is immediately clear that the mappings  $\sum_r$  and  $\sum'_r$  are bijective mappings.  $h$ -continuity of these mappings follows from the preceding theorem. Next, we will prove that these mappings are  $h$ -open. For this, consider an open set  $\mathcal{U}$  in  $\mathcal{R}$ . Then  $\sum_r(\mathcal{U}) = \mathcal{U} \cdot r$ . By Theorem 3.4(2),  $\mathcal{U} \cdot r$  is an  $h$ -open set in  $\mathcal{R}$ . Hence,  $\sum_r$  is an  $h$ -open mapping. In the similar manner, it can be proved that  $\sum'_r$  is also an  $h$ -open mapping. Thus the mappings  $\sum_r$  and  $\sum'_r$  are  $h$ -homeomorphisms. □

**Corollary 3.7** Let  $\mathcal{R}$  be an  $h$ -topological ring. Then the mappings  $\Gamma_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\Gamma'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 3.10) are  $h$ -homeomorphisms.

**Theorem 3.12** Consider a ring homomorphism  $\Omega : \mathcal{R} \rightarrow \mathcal{S}$ , where  $\mathcal{R}$  is an  $h$ -topological ring and  $\mathcal{S}$  is a topological ring. If  $\Omega$  is continuous at zero, then  $\Omega$  is  $h$ -continuous.

**Proof:** Let  $p \in \mathcal{R}$  and  $\mathcal{U}$  be an open neighborhood of  $\Omega(p)$  in  $\mathcal{S}$ . Also,  $\Omega(0) = 0$ . Then  $\mathcal{U} - \Omega(p)$  is open neighborhood of 0 in  $\mathcal{S}$ . By given hypothesis, there exists an open neighborhood  $\mathcal{V}$  of 0 in  $\mathcal{R}$  such that  $\Omega(\mathcal{V}) \subseteq \mathcal{U} - \Omega(p)$ . This implies that  $\Omega(p + \mathcal{V}) \subseteq \mathcal{U}$ . From Corollary 3.2(1),  $p + \mathcal{V} \in \tau_h$ . Thus,  $\Omega$  is  $h$ -continuous. □

**Theorem 3.13** Let  $\Omega$  be a ring homomorphism from an  $h$ -topological ring  $\mathcal{R}$  to an  $h$ -topological ring  $\mathcal{S}$ . If  $\Omega$  is  $h$ -totally continuous at zero, then  $\Omega$  is  $h$ -continuous.

**Proof:** Let  $x \in \mathcal{R}$  and  $\mathcal{U}$  be an open set in  $\mathcal{S}$  containing  $\Omega(x)$ . By Corollary 3.2(2),  $\mathcal{U} - \Omega(x)$  is an  $h$ -open set in  $\mathcal{S}$  containing zero. Since  $\Omega$  is  $h$ -totally continuous at zero, there exists clopen set  $\mathcal{U}_1$  in  $\mathcal{R}$  containing zero such that  $\Omega(\mathcal{U}_1) \subseteq \mathcal{U} - \Omega(x)$ . Thus by Theorem 3.2(2),  $\mathcal{U}_1 + x$  is an  $h$ -open set in  $\mathcal{R}$  containing  $x$ , and as  $\Omega$  is a ring homomorphism,

$$\Omega(\mathcal{U}_1 + x) = \Omega(\mathcal{U}_1) + \Omega(x) \subseteq \mathcal{U} - \Omega(x) + \Omega(x) \subseteq \mathcal{U}.$$

Therefore, the mapping  $\Omega : \mathcal{R} \rightarrow \mathcal{S}$  is  $h$ -continuous. □

**Theorem 3.14** *Consider an  $h$ -topological ring  $\mathcal{R}$ . Then:*

- (1) *An open subring  $\mathcal{S}$  of  $\mathcal{R}$  is also an  $h$ -topological ring;*
- (2)  *$\overline{\mathcal{S}}$  is also an  $h$ -topological ring, where  $\mathcal{S}$  is a subring of  $\mathcal{R}$ .*

**Proof:**

- (1) Let  $r_1, r_2 \in \mathcal{S} \subseteq \mathcal{R}$  and  $\mathcal{V}$  be an open set containing  $r_1 + r_2$  in  $\mathcal{R}$ . Since  $\mathcal{R}$  is an  $h$ -topological ring, there exist  $h$ -open sets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$  respectively such that  $\mathcal{V}_1 + \mathcal{V}_2 \subseteq \mathcal{V}$ . Since  $\mathcal{S}$  is open, there exists  $h$ -open neighborhoods  $\mathcal{U}_1 = \mathcal{S} \cap \mathcal{V}_1$  and  $\mathcal{U}_2 = \mathcal{S} \cap \mathcal{V}_2$  in  $\mathcal{S}$  containing  $r_1$  and  $r_2$  respectively such that  $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{V}_1 + \mathcal{V}_2 \subseteq \mathcal{V}$ . Thus  $+$  is  $h$ -continuous. Similarly, we can prove  $h$ -continuity of  $-$  and multiplication operation. Thus,  $\mathcal{S}$  is an  $h$ -topological ring.
- (2) Let  $r_1, r_2 \in \overline{\mathcal{S}}$  and  $\mathcal{V}$  be an open neighborhood of  $r_1 + r_2$ . Since ring addition is  $h$ -continuous, there exists  $h$ -open neighborhoods  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $r_1$  and  $r_2$  respectively such that  $\mathcal{V}_1 + \mathcal{V}_2 \subseteq \mathcal{V}$ . As  $r_1, r_2 \in \overline{\mathcal{S}}$ , every  $h$ -open neighborhood containing  $r_1$  and  $r_2$  intersects  $\mathcal{S}$  and thus  $\mathcal{V}_1 \cap \mathcal{S} \neq \emptyset$  and  $\mathcal{V}_2 \cap \mathcal{S} \neq \emptyset$ . Suppose  $p_1 \in \mathcal{V}_1 \cap \mathcal{S}$  and  $p_2 \in \mathcal{V}_2 \cap \mathcal{S}$ . This implies that  $p_1 + p_2 \in \mathcal{V}$ . Also  $p_1 + p_2 \in \mathcal{S}$  as  $\mathcal{S}$  is subring. Thus,  $p_1 + p_2 \in \mathcal{S} \cap \mathcal{V}$  and hence  $r_1 + r_2 \in \overline{\mathcal{S}}$ . In the similar manner, we can prove  $r_1 \cdot r_2 \in \overline{\mathcal{S}}$ . We can also  $h$ -continuity of  $-$ . Let  $r \in \overline{\mathcal{S}}$  and  $\mathcal{U}$  be an neighborhood of  $-r$ . Then there exists an  $h$ -open neighborhood  $\mathcal{W}$  of  $r$  such that  $-\mathcal{W} \subseteq \mathcal{U}$ . Clearly,  $\mathcal{W}$  intersects  $\mathcal{S}$  and suppose  $q \in \mathcal{W} \cap \mathcal{S}$  and thus  $-q \in \mathcal{U} \cap \mathcal{S}$ . This implies that  $-r \in \overline{\mathcal{S}}$ . Consequently,  $\overline{\mathcal{S}}$  is subring of  $\mathcal{R}$  and by (1),  $\overline{\mathcal{S}}$  is an  $h$ -topological ring. □

**Corollary 3.8** *Consider an  $h$ -topological ring  $\mathcal{R}$ . Then*

- (1) *An open ideal  $\mathcal{S}$  of  $\mathcal{R}$  is also an  $h$ -topological ring;*
- (2)  *$\overline{\mathcal{S}}$  is also an  $h$ -topological ring, where  $\mathcal{S}$  is an ideal of  $\mathcal{R}$ .*

**Definition 3.4** [10] *Let  $(\mathcal{X}, \tau)$  be a topological space. Then  $\mathcal{X}$  is called  $h$ -disconnected if and only if  $\mathcal{X} = \mathcal{A} \cup \mathcal{B}$ , where  $\mathcal{A} \neq \emptyset$  and  $\mathcal{B} \neq \emptyset$  are  $h$ -open sets and  $\mathcal{A} \cap \mathcal{B} = \emptyset$ . Furthermore,  $\mathcal{X}$  is said to be an  $h$ -connected space if  $\mathcal{X}$  is not  $h$ -disconnected space.*

**Theorem 3.15** [10] *Let  $(\mathcal{X}, \tau)$  and  $(\mathcal{Y}, \sigma)$  be two topological spaces. Suppose  $\mathcal{X}$  is  $h$ -connected, then  $h$ -continuous image of  $\mathcal{X}$  is connected.*

**Theorem 3.16** *Consider an  $h$ -topological ring  $\mathcal{R}$  and an  $h$ -connected component  $\mathcal{B}$  of zero. Then  $r + \mathcal{B}$  is a connected component of  $r$ , for every  $r \in \mathcal{R}$ . Also,  $\mathcal{B}$  is a closed ideal.*

**Proof:** From Theorem 3.11 and Theorem 3.15,  $r + \mathcal{B}$  is a connected component of  $r$ . Suppose  $r \in \mathcal{B}$ . Then  $\mathcal{B} \cup (r + \mathcal{B})$  is connected and contains zero and thus  $r + \mathcal{B} \subseteq \mathcal{B}$ . Further  $-\mathcal{B}, r \cdot \mathcal{B}, \mathcal{B} \cdot r$  are also connected subsets of  $\mathcal{R}$  containing zero such that  $-\mathcal{B} \subseteq \mathcal{B}, r \cdot \mathcal{B} \subseteq \mathcal{B}$  and  $\mathcal{B} \cdot r \subseteq \mathcal{B}$ . Thus  $\mathcal{B}$  is a closed ideal. □

#### 4. $h$ -irresolute topological ring

In this section, we establish the primary results concerning irresolute structures in topological rings and examine their implications in ring homomorphisms and ideal structures.

**Definition 4.1** *An  $h$ -irresolute topological ring  $(\mathcal{R}, +, \cdot, \tau)$  is a ring  $(\mathcal{R}, +, \cdot)$  endowed with some topology  $\tau$  such that ring operations are  $h$ -irresolute. Equivalently,*

- (C1)  $(\mathcal{R}, +)$  is an  $h$ -irresolute topological group [7].

(C2) For each  $r_1, r_2 \in \mathcal{R}$  and  $h$ -open set  $\mathcal{U}$  in  $\mathcal{R}$  containing  $r_1 \cdot r_2$ , there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$ , respectively, such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ .

**Remark 4.1** It is easy to verify that conditions (C1) and (C2) are equivalent to the following condition: For each  $r_1, r_2 \in \mathcal{R}$  and each  $h$ -open set  $\mathcal{U}$  in  $\mathcal{R}$  containing  $r_1 - r_2$ , there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$ , respectively, such that  $\mathcal{U}_1 - \mathcal{U}_2 \subseteq \mathcal{U}$ .

**Note 3** Throughout this section, we denote the  $h$ -irresolute topological ring  $(\mathcal{R}, +, \cdot, \tau)$  simply by  $\mathcal{R}$ .

**Example 4.1** Let  $(\mathcal{R}, +, \cdot)$  be any ring and suppose  $\tau$  be a discrete or indiscrete topology on  $\mathcal{R}$ . Then  $(\mathcal{R}, +, \cdot, \tau)$  is always an  $h$ -irresolute topological ring.

**Example 4.2**  $Z_4$ , the ring of integers modulo 4 with topology  $\tau = \{\emptyset, \{0, 1, 2\}, Z_4\}$ ,  $\tau_h = P(Z_4)$ . Clearly,  $(Z_4, \oplus, \odot, \tau)$  is an example of  $h$ -irresolute topological ring that is not a topological ring.

Next, we provide an example of not an  $h$ -irresolute topological ring.

**Example 4.3** Consider the ring  $\mathcal{R} = (\mathbb{R}, +, \cdot)$  together with topology  $\tau = \{\emptyset, \mathbb{Q}, \mathbb{Q}^c, \mathbb{R}\}$ , where  $\mathbb{Q}$  and  $\mathbb{Q}^c$  denotes the set of rational and irrational numbers respectively. Let  $\mathbb{Q}$  be an  $h$ -open set in  $\mathcal{R}$  containing zero and  $0 = \sqrt{3} + (-\sqrt{3})$ . Clearly, condition (C1) of Definition 4.1 will not be satisfied.

**Example 4.4** Let  $\mathcal{R} = (\mathbb{Z}, +, \cdot)$  be the ring of integers with  $\tau = \{\emptyset, E, O, \mathbb{Z}\}$ , where  $E$  and  $O$  denotes the set of even integers and set of odd integers respectively. Then  $\tau_h = \{\emptyset, E, O, \mathbb{Z}\}$ . Thus  $(\mathcal{R}, +, \cdot, \tau)$  is an  $h$ -irresolute topological ring.

**Verification:**

(1) Suppose  $x, y \in \mathcal{R}$ .

If  $x + y$  is an even integer, then  $h$ -open sets in  $\mathcal{R}$  containing  $x + y$  are  $E$  and  $\mathbb{Z}$ , then we have the following cases:

Case(i). If both  $x$  and  $y$  are even integers, we select  $h$ -open set  $E$  containing both  $x$  and  $y$  such that  $E + E \subseteq E$  and  $E + E \subseteq \mathbb{Z}$ .

Case(ii). If both  $x$  and  $y$  are odd integers, we select  $h$ -open set  $O$  containing both  $x$  and  $y$  such that  $O + O \subseteq E$  and  $O + O \subseteq \mathbb{Z}$ .

If  $x + y$  is an odd integer, then the only  $h$ -open sets in  $\mathcal{R}$  containing  $x + y$  are  $O$  and  $\mathbb{Z}$ . And we know that sum of two integers will be odd if one of them is even and other is odd. Suppose  $x$  is even and  $y$  is odd, then we have  $h$ -open set  $E$  containing  $x$  and  $h$ -open set  $O$  containing  $y$  such that  $E + O \subseteq O$  and  $E + O \subseteq \mathbb{Z}$ .

(2) If  $x \in \mathcal{R}$  and  $x$  is an even integer, then the  $h$ -open sets in  $\mathcal{R}$  containing  $-x$  are  $E$  and  $\mathbb{Z}$ . And we choose  $h$ -open set  $E$  containing  $x$  such that  $-E \subseteq E$  and  $-E \subseteq \mathbb{Z}$ .

If  $x \in \mathcal{R}$  and  $x$  is an odd integer, then the  $h$ -open sets in  $\mathcal{R}$  containing  $-x$  are  $O$  and  $\mathbb{Z}$ . Then we have  $h$ -open set  $O$  containing  $x$  such that  $-O \subseteq O$  and  $-O \subseteq \mathbb{Z}$ .

(3) If  $x \cdot y$  is an even integer, then  $E$  and  $\mathbb{Z}$  are the only  $h$ -open sets in  $\mathcal{R}$  containing  $x \cdot y$ . Then the following cases arise:

Case(i). If both  $x$  and  $y$  are even integers, then we choose  $h$ -open set  $E$  which contain both  $x$  and  $y$  such that  $E \cdot E \subseteq E$  and  $E \cdot E \subseteq \mathbb{Z}$ .

Case(ii). If one of them is even and other is odd, consider  $x$  is even and  $y$  is odd. Then, for the selection of  $h$ -open sets  $E$  and  $O$  in  $\mathcal{R}$  containing  $x$  and  $y$  respectively such that  $E \cdot O \subseteq E$  and  $E \cdot O \subseteq \mathbb{Z}$ .

If  $x \cdot y$  is an odd integer, then  $O$  and  $\mathbb{Z}$  are the only  $h$ -open sets in  $\mathcal{R}$  containing  $x \cdot y$ . Also  $x \cdot y$  is odd only possible when both  $x$  and  $y$  are odd integers. Then, for the selection of  $h$ -open set  $O$  which contain both  $x$  and  $y$  such that  $O \cdot O \subseteq O$  and  $O \subseteq \mathbb{Z}$

Thus by Remark 4.1, Example 4.2 and Example 4.4 are also examples of an  $h$ -topological ring.

**Note 4**  $\mathcal{J}(\mathcal{R})$  denotes the set of all invertible elements in an  $h$ -irresolute topological ring  $\mathcal{R}$ .

**Theorem 4.1** Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring and  $r \in \mathcal{R}$ . Then following statements are true:

- (1) the mappings  $\phi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\phi'_r : \mathcal{R} \rightarrow \mathcal{R}$ , where  $\phi_r(x) = x \cdot r$  and  $\phi'_r(x) = r \cdot x$  for  $x \in \mathcal{R}$ , are  $h$ -irresolute mappings of the topological space  $\mathcal{R}$  into itself;
- (2) the mappings  $\psi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\psi'_r : \mathcal{R} \rightarrow \mathcal{R}$ , where  $\psi_r(x) = x + r$  and  $\psi'_r(x) = r + x$  for  $x \in \mathcal{R}$ , are  $h$ -irresolute mappings of the topological space  $\mathcal{R}$  into itself.

**Proof:**

1. Let  $\mathcal{U}$  be any  $h$ -open set in  $\mathcal{R}$  containing  $\phi_r(x) = x \cdot r$ , where  $x \in \mathcal{R}$  be arbitrary. Then, by definition of  $h$ -irresolute topological ring, there exists  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $x$  and  $r$  respectively, such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Thus,  $\phi_r(\mathcal{U}_1) = \mathcal{U}_1 \cdot r \subseteq \mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . This proves that  $\phi_r$  is an  $h$ -irresolute mapping. Similarly we can prove that  $\phi'_r$  is an  $h$ -irresolute mapping.
2. Let  $x \in \mathcal{R}$  and let  $\mathcal{U}$  be any  $h$ -open set in  $\mathcal{R}$  containing  $\psi_r(x) = x + r$ . Then, by Definition 4.1, there exists  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $x$  and  $r$  respectively, such that  $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . Thus,  $\psi_r(\mathcal{U}_1) = \mathcal{U}_1 + r \subseteq \mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . This proves that  $\psi_r$  is an  $h$ -irresolute mapping. Similarly we can prove that  $\psi'_r$  is an  $h$ -irresolute mapping.

□

**Corollary 4.1** Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring and  $r \in \mathcal{R}$ . Then following statements are true:

- (1) the mappings  $\phi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\phi'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 4.1(1)) are  $h$ -continuous mappings of the topological space  $\mathcal{R}$  into itself;
- (2) the mappings  $\psi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\psi'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 4.1(2)) are  $h$ -continuous mappings of the topological space  $\mathcal{R}$  into itself.

**Proposition 4.1** Let  $\mathcal{A}$  be an  $h$ -open set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{A} \cdot r$  (or  $r \cdot \mathcal{A}$ ) is  $h$ -open set in  $\mathcal{R}$  for every  $r \in \mathcal{J}(\mathcal{R})$ .

**Proof:** Let  $x \in \mathcal{A} \cdot r$  be an arbitrary element. Then  $x = a \cdot r$  for some  $a \in \mathcal{A}$ . Now  $a = (a \cdot r) \cdot r^{-1} = x \cdot r^{-1} = \phi_{r^{-1}}(x)$ . By Theorem 4.1(1),  $\phi_{r^{-1}} : \mathcal{R} \rightarrow \mathcal{R}$  is  $h$ -irresolute. Thus for the  $h$ -open set  $\mathcal{A}$  containing  $a = \phi_{r^{-1}}(x)$ , there exists an  $h$ -open set  $\mathcal{U}_x$  in  $\mathcal{R}$  containing  $x$  such that  $\phi_{r^{-1}}(\mathcal{U}_x) \subseteq \mathcal{A}$ . This results in  $\mathcal{U}_x \cdot r^{-1} \subseteq \mathcal{A}$  and hence  $\mathcal{U}_x \subseteq \mathcal{A} \cdot r$ . Thus  $\mathcal{A} \cdot r$  is an  $h$ -open set in  $\mathcal{R}$ . Similarly, we can prove that  $r \cdot \mathcal{A}$  is  $h$ -open set. □

**Corollary 4.2** Let  $\mathcal{A}$  be an open set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{A} \cdot r$  (or  $r \cdot \mathcal{A}$ ) is  $h$ -open set in  $\mathcal{R}$  for every  $r \in \mathcal{J}(\mathcal{R})$ .

**Proof:** In view of Proposition 4.1, and by virtue of the fact that every open set is  $h$ -open, the proof is obvious. □

**Proposition 4.2** Let  $\mathcal{A}$  be an  $h$ -open set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{A} + r$  (or  $r + \mathcal{A}$ ) is  $h$ -open set in  $\mathcal{R}$  for every  $r \in \mathcal{R}$ .

**Proof:** (1) Let  $z \in \mathcal{A} + r$  be an arbitrary element. Then  $z = a + r$  for some point  $a$  in  $\mathcal{A}$ . Now  $a = a + r - r = z - r = \psi_{(-r)}(z)$  say and by Theorem 4.1(2),  $\psi_{(-r)}$  is  $h$ -irresolute for  $z \in X$ . Thus, for  $h$ -open set  $S$  containing  $a = \psi_{(-r)}(z)$ ,  $\exists$  an  $h$ -open set  $U_z$  containing  $z$  such that  $\psi_{(-r)}(U_z) \subseteq S$ . This implies that  $U_z + (-r) \subseteq \mathcal{A}$  and hence  $U_z \subseteq \mathcal{A} + r$ . Thus  $\mathcal{A} + r$  is an  $h$ -open set in  $\mathcal{R}$ . Similarly, we can prove that  $r + \mathcal{A}$  is  $h$ -open set in  $\mathcal{R}$ .  $\square$

**Corollary 4.3** *Let  $\mathcal{A}$  be an open set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{A} + r$  (or  $r + \mathcal{A}$ ) is  $h$ -open set in  $\mathcal{R}$  for every  $r \in \mathcal{R}$ .*

**Theorem 4.2** *Let  $(\mathcal{R}, \tau)$  be an  $h$ -irresolute topological ring with unity and  $r \in \mathcal{J}(\mathcal{R})$ . Then following statements are true:*

- (1) *the mappings  $\phi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\phi'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 4.1(1)) are  $h$ -homeomorphisms of the topological space  $\mathcal{R}$  into itself;*
- (2) *the mappings  $\psi_r : \mathcal{R} \rightarrow \mathcal{R}$  and  $\psi'_r : \mathcal{R} \rightarrow \mathcal{R}$  (see Theorem 4.1(2)) are  $h$ -homeomorphisms of the topological space  $\mathcal{R}$  into itself.*

**Proof:**

1. It is clearly evident that mappings  $\phi_r$  and  $\phi'_r$  are bijective mappings.  $h$ -continuity of these mappings follows from statements (1) and (2) (see Corollary 4.1). Next, we will prove that mappings  $\phi_r$  and  $\phi'_r$  are  $h$ -open. Let  $\mathcal{U}$  be any open set in  $\mathcal{R}$ , then  $\phi_r(\mathcal{U}) = \mathcal{U} \cdot r$ . Thus, by Corollary 4.2,  $\mathcal{U} \cdot r$  is an  $h$ -open set in  $\mathcal{R}$ . Hence  $\phi_r : \mathcal{R} \rightarrow \mathcal{R}$  is  $h$ -open mapping. In the similar manner it can be proved that  $\phi'_r : \mathcal{R} \rightarrow \mathcal{R}$  is also  $h$ -open mapping too. Thus, the mappings  $\phi_r$  and  $\phi'_r$  are  $h$ -homeomorphisms.
2. In the similar manner we prove that mappings  $\psi_r$  and  $\psi'_r$  are  $h$ -homeomorphisms.

$\square$

**Proposition 4.3** *Let  $\mathcal{K}$  be an  $h$ -closed set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{K} \cdot r$  (or,  $r \cdot \mathcal{K}$ ) is  $h$ -closed set in  $\mathcal{R}$  for every  $r \in \mathcal{J}(\mathcal{R})$ .*

**Proof:** Let  $x \in Cl_h(\mathcal{K} \cdot r)$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $x$ , where  $y = x \cdot r^{-1}$ . Then by given hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $x$  and  $r^{-1}$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Since  $x \in Cl_h(\mathcal{K} \cdot r)$  we have  $(\mathcal{K} \cdot r) \cap \mathcal{U}_1 \neq \emptyset$ . Suppose  $s \in (\mathcal{K} \cdot r) \cap \mathcal{U}_1$ , then  $s \cdot r^{-1} \in \mathcal{K} \cap (\mathcal{U}_1 \cdot \mathcal{U}_2) \subseteq \mathcal{K} \cap \mathcal{U}$  which implies  $\mathcal{K} \cap \mathcal{U}$  is non empty set. This gives  $y \in Cl_h(\mathcal{K}) = \mathcal{K}$  since  $\mathcal{K}$  is  $h$ -closed set. Consequently,  $x \in \mathcal{K} \cdot r$  implies that  $Cl_h(\mathcal{K} \cdot r) \subseteq \mathcal{K} \cdot r$ . The inclusion  $\mathcal{K} \cdot r \subseteq Cl_h(\mathcal{K} \cdot r)$  is obvious. Hence  $\mathcal{K} \cdot r$  is  $h$ -closed. Similarly we can prove that  $r \cdot \mathcal{K}$  is  $h$ -closed.  $\square$

**Corollary 4.4** *Let  $\mathcal{K}$  be an  $h$ -closed set in an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{K} + r$  (or  $r + \mathcal{K}$ ) is  $h$ -open set in  $\mathcal{R}$  for every  $r \in \mathcal{R}$ .*

**Theorem 4.3** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be any two subsets of an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then the following statements are true:*

- (1)  $Cl_h(\mathcal{A}) + Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} + \mathcal{B});$
- (2)  $Cl_h(-\mathcal{A}) = -Cl_h(\mathcal{A});$
- (3)  $Cl_h(\mathcal{A}) - Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} - \mathcal{B});$
- (4)  $Cl_h(\mathcal{A}) \cdot Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} \cdot \mathcal{B}).$

**Proof:**

1. Let  $x \in Cl_h(\mathcal{A}) + Cl_h(\mathcal{B})$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $x$ . Then  $x = a + b$ , for some  $a \in Cl_h(\mathcal{A})$  and  $b \in Cl_h(\mathcal{B})$ . By Definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $a$  and  $b$  respectively such that  $\mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . By hypothesis,  $\mathcal{U}_1 \cap \mathcal{A} \neq \emptyset$  and  $\mathcal{U}_2 \cap \mathcal{B} \neq \emptyset$ , there exist  $a_1 \in \mathcal{U}_1 \cap \mathcal{A}$  and  $a_2 \in \mathcal{U}_2 \cap \mathcal{B}$ . Thus  $a_1 + a_2 \in \mathcal{A} + \mathcal{B}$  and  $a_1 + a_2 \in \mathcal{U}_1 + \mathcal{U}_2 \subseteq \mathcal{U}$ . This implies that  $(\mathcal{A} + \mathcal{B}) \cap \mathcal{U} \neq \emptyset$ . Consequently,  $Cl_h(\mathcal{A}) + Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} + \mathcal{B})$ .
2. Let  $x \in Cl_h(-\mathcal{A})$  be an arbitrary element. Let  $z = -x$  and suppose  $\mathcal{U}$  is an  $h$ -open set in  $\mathcal{R}$  containing  $z$ . By Definition 4.1, there exists  $h$ -open set  $\mathcal{U}_1$  containing  $z$  such that  $-\mathcal{U}_1 \subseteq \mathcal{U}$ . Since  $(-\mathcal{A}) \cap \mathcal{U}_1 \neq \emptyset$ , there exists  $a \in (-\mathcal{A}) \cap \mathcal{U}_1$ . Then  $-a \in \mathcal{A} \cap \mathcal{U}$ ,  $\mathcal{A} \cap \mathcal{U} \neq \emptyset$ . Consequently,  $z \in Cl_h(\mathcal{A})$ . Thus  $x \in -Cl_h(\mathcal{A})$ . Conversely, suppose  $y \in -Cl_h(\mathcal{A})$  and let  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $y$ . Then  $y = -x$ , where  $x \in Cl_h(\mathcal{A})$ . Then there exists  $h$ -open  $\mathcal{U}_1$  in  $\mathcal{R}$  containing  $x$  such that  $-\mathcal{U}_1 \subseteq \mathcal{U}$ . As  $\mathcal{A} \cap \mathcal{U}_1 \neq \emptyset$ ,  $a \in \mathcal{A} \cap \mathcal{U}_1$  say. Thus  $-a \in (-\mathcal{A}) \cap \mathcal{U}$ . Consequently,  $y \in Cl_h(-\mathcal{A})$ . Hence,  $Cl_h(-\mathcal{A}) = -Cl_h(\mathcal{A})$ .
3. Inclusion  $Cl_h(\mathcal{A}) - Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} - \mathcal{B})$  results from (2) and (3).
4. Let  $x \in Cl_h(\mathcal{A}) \cdot Cl_h(\mathcal{B})$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $x$ . Then  $x = a \cdot b$ , for some  $a \in Cl_h(\mathcal{A})$  and  $b \in Cl_h(\mathcal{B})$ . By Definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $a$  and  $b$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Since  $\mathcal{U}_1 \cap \mathcal{A} \neq \emptyset$  and  $\mathcal{U}_2 \cap \mathcal{B} \neq \emptyset$ , there exist  $a_1 \in \mathcal{U}_1 \cap \mathcal{A}$  and  $b_1 \in \mathcal{U}_2 \cap \mathcal{B}$ . Thus  $a_1 \cdot b_1 \in \mathcal{A} \cdot \mathcal{B}$  and  $a_1 \cdot b_1 \in \mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . This implies that  $(\mathcal{A} \cdot \mathcal{B}) \cap \mathcal{U} \neq \emptyset$ . Consequently,  $Cl_h(\mathcal{A}) \cdot Cl_h(\mathcal{B}) \subseteq Cl_h(\mathcal{A} \cdot \mathcal{B})$ .

□

**Theorem 4.4** *Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring with unity and  $\mathcal{A}$  be a subset of  $\mathcal{R}$ . Then for each  $r \in \mathcal{J}(\mathcal{R})$ , the following hold:*

- (1)  $Cl_h(r \cdot \mathcal{A}) = r \cdot Cl_h(\mathcal{A})$ ;
- (2)  $Int_h(r \cdot \mathcal{A}) = r \cdot Int_h(\mathcal{A})$ .

**Proof:**

1. Let  $x \in Cl_h(r \cdot \mathcal{A})$  and  $y = r^{-1} \cdot x$ . Let  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $y$ . By Definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $x$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . By hypothesis,  $(r \cdot \mathcal{A}) \cap \mathcal{U}_2 \neq \emptyset$ , and let  $s \in (r \cdot \mathcal{A}) \cap \mathcal{U}_2$ . This implies that  $r^{-1} \cdot s \in \mathcal{A} \cap (\mathcal{U}_1 \cdot \mathcal{U}_2) \subseteq \mathcal{A} \cap \mathcal{U}$ . Thus  $\mathcal{A} \cap \mathcal{U} \neq \emptyset$  and hence  $y \in Cl_h(\mathcal{A})$ . Consequently,  $x \in r \cdot Cl_h(\mathcal{A})$ . Conversely, let  $y \in r \cdot Cl_h(\mathcal{A})$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{R}$  containing  $y = r \cdot x$ , where  $x \in Cl_h(\mathcal{A})$ . Then by Definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r$  and  $x$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{U}$ . Since  $x \in Cl_h(\mathcal{A})$ ,  $\mathcal{A} \cap \mathcal{U}_2 \neq \emptyset$  and suppose  $s \in \mathcal{A} \cap \mathcal{U}_2$ . This implies that  $r \cdot s \in (r \cdot \mathcal{A}) \cap (\mathcal{U}_1 \cdot \mathcal{U}_2) \subseteq (r \cdot \mathcal{A}) \cap \mathcal{U}$ . Consequently,  $y \in Cl_h(r \cdot \mathcal{A})$ . Thus,  $r \cdot Cl_h(\mathcal{A}) \subseteq Cl_h(r \cdot \mathcal{A})$ . Hence  $Cl_h(r \cdot \mathcal{A}) = r \cdot Cl_h(\mathcal{A})$ .
2. Let  $y \in Int_h(r \cdot \mathcal{A})$ . Then  $y = r \cdot x$  for some  $x \in \mathcal{A}$ . By definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r$  and  $x$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq Int_h(r \cdot \mathcal{A})$ . This implies that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq r \cdot \mathcal{A}$ . In particular,  $r \cdot \mathcal{U}_2 \subseteq r \cdot \mathcal{A}$ . Since  $\mathcal{U}_2$  is  $h$ -open, it follows that  $y \in r \cdot Int_h(\mathcal{A})$ . Thus,  $Int_h(r \cdot \mathcal{A}) \subseteq r \cdot Int_h(\mathcal{A})$ . Conversely, let  $x \in r \cdot Int_h(\mathcal{A})$ . Since  $r \in \mathcal{J}(\mathcal{R})$ ,  $r^{-1} \cdot x \in Int_h(\mathcal{A})$ . Therefore, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r^{-1}$  and  $x$  respectively such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq Int_h(\mathcal{A})$ . This results in  $r^{-1} \cdot \mathcal{U}_2 \subseteq Int_h(\mathcal{A}) \subseteq \mathcal{A}$  and thus  $\mathcal{U}_2 \subseteq r \cdot \mathcal{A}$ . Consequently,  $x \in Int_h(r \cdot \mathcal{A})$ . Hence  $Int_h(r \cdot \mathcal{A}) = r \cdot Int_h(\mathcal{A})$ .

□

**Corollary 4.5** *Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring and  $\mathcal{A}$  be a subset of  $\mathcal{R}$ . Then for each  $r \in \mathcal{R}$ , the following hold:*

- (1)  $Cl_h(r + \mathcal{A}) = r + Cl_h(\mathcal{A})$

$$(2) \text{Int}_h(r + \mathcal{A}) = r + \text{Int}_h(\mathcal{A}).$$

**Definition 4.2** A collection  $\mathcal{U}_r$  of subsets of an  $h$ -irresolute topological ring  $\mathcal{R}$  is said to be a basis of  $h$ -neighborhoods of  $r \in \mathcal{R}$  if any subset of  $\mathcal{U}_r$  is an  $h$ -neighborhood of  $r$  and any  $h$ -neighborhood of the element  $r$  contains some subset from  $\mathcal{U}_r$ .

**Proposition 4.4** Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring and  $\mathcal{U}_0$  be a basis of  $h$ -neighborhoods of zero of  $\mathcal{R}$ . Then, the following conditions hold:

(N1) For each  $\mathcal{V}, \mathcal{U} \in \mathcal{U}_0$ , there exists  $\mathcal{W} \in \mathcal{U}_0$  such that  $\mathcal{W} \subseteq \mathcal{V} \cap \mathcal{U}$ .

(N2) For each  $\mathcal{U} \in \mathcal{U}_0$ , there exists  $\mathcal{W} \in \mathcal{U}_0$  such that  $\mathcal{W} + \mathcal{W} \subseteq \mathcal{U}$ .

(N3) For each  $\mathcal{U} \in \mathcal{U}_0$ , there exists  $\mathcal{W} \in \mathcal{U}_0$  such that  $-\mathcal{W} \subseteq \mathcal{U}$ .

(N4) For each  $\mathcal{U} \in \mathcal{U}_0$  there exists  $\mathcal{W} \in \mathcal{U}_0$  such that  $\mathcal{W} \cdot \mathcal{W} \subseteq \mathcal{U}$ .

(N5) For each  $\mathcal{U} \in \mathcal{U}_0$  and any element  $s \in \mathcal{R}$  there exists  $\mathcal{W} \in \mathcal{U}_0$  such that  $s \cdot \mathcal{W} \subseteq \mathcal{U}$  and  $\mathcal{W} \cdot s \subseteq \mathcal{U}$ .

**Proof:** Condition (N1) follows from the definition of a basis of neighborhoods of an element in a topological space. The conditions (N2) and (N3) results from the fact that  $\mathcal{U}_0$  is a basis of  $h$ -neighborhoods of zero in  $\mathcal{R}$ , from conditions (C1) and (C2) (see Definition 4.1) and from condition (N1) with the consideration of the fact that  $0 + 0 = 0$  and  $0 = -0$ . Since  $\mathcal{U}_0$  is a basis of  $h$ -neighborhoods of zero of an  $h$ -irresolute topological ring  $\mathcal{R}$ . The fulfillment of conditions (N4) and (N5) follows from condition (C3) (see Definition 4.1) with regard to  $0 \cdot 0 = 0$  and  $s \cdot 0 = 0 \cdot s = 0$  for any  $s \in \mathcal{R}$ .  $\square$

**Theorem 4.5** Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring. Then  $\mathcal{R}$  has a basis of  $h$ -neighborhoods of zero consisting of symmetric  $h$ -open neighborhoods.

**Proof:** Let  $\mathcal{U}_0$  be a basis of  $h$ -neighborhoods of zero in  $\mathcal{R}$ . Then for every  $\mathcal{U} \in \mathcal{U}_0$  there exists an  $h$ -open set  $V_{\mathcal{U}}$  in  $\mathcal{R}$  such that  $0 \in V_{\mathcal{U}} \subseteq \mathcal{U}$ . By Proposition 4.1, the set  $-V_{\mathcal{U}}$  is  $h$ -open in  $\mathcal{R}$  containing zero. Since intersection of two  $h$ -open sets is  $h$ -open, then  $V_{\mathcal{U}} \cap (-V_{\mathcal{U}})$  is a symmetric  $h$ -open neighborhood of zero such that  $V_{\mathcal{U}} \cap (-V_{\mathcal{U}}) \subseteq \mathcal{V} \subseteq \mathcal{U}$ . Thus from condition (N2) follows that the family  $\mathcal{U}'_0 = \{V_{\mathcal{U}} \cap (-V_{\mathcal{U}}) \mid \mathcal{U} \in \mathcal{U}_0\}$  is a basis of symmetric  $h$ -open neighborhoods of zero in  $\mathcal{R}$ .  $\square$

**Theorem 4.6** Let  $\mathcal{A}$  be a subset of an  $h$ -irresolute topological ring  $\mathcal{R}$  and  $\mathcal{U}_0$  be a basis of  $h$ -neighborhoods of zero. Then  $Cl_h(\mathcal{A}) = \bigcap_{\mathcal{U} \in \mathcal{U}_0} (\mathcal{A} + \mathcal{U})$ .

**Proof:** Let  $x \in Cl_h(\mathcal{A})$  and  $\mathcal{U} \in \mathcal{U}_0$ . Then, by Proposition 4.4(3), there exists  $\mathcal{U}' \in \mathcal{U}_0$  such that  $-\mathcal{U}' \subseteq \mathcal{U}$ . Since  $\mathcal{U}' \in \mathcal{U}_0$ , there exists  $h$ -open set  $\mathcal{V}$  containing zero such that  $\mathcal{V} \subseteq \mathcal{U}'$ . By Proposition 4.2,  $x + \mathcal{V}$  is  $h$ -open set containing  $x$  and thus  $(x + \mathcal{V}) \cap \mathcal{A} \neq \emptyset$ . Then  $x \in \mathcal{A} - \mathcal{V} \subseteq \mathcal{A} - \mathcal{U}' \subseteq \mathcal{A} + \mathcal{U}$ . Therefore,  $Cl_h(\mathcal{A}) \subseteq \mathcal{A} + \mathcal{U}$ . Hence it follows that  $Cl_h(\mathcal{A}) \subseteq \bigcap_{\mathcal{U} \in \mathcal{U}_0} (\mathcal{A} + \mathcal{U})$ . Now, let  $y \in \bigcap_{\mathcal{U} \in \mathcal{U}_0} (\mathcal{A} + \mathcal{U})$  and let  $\mathcal{U}_1$  be an  $h$ -neighborhood of zero in  $\mathcal{R}$ . Let's choose an  $h$ -neighborhood  $\mathcal{U} \in \mathcal{U}_0$  of zero such that  $-\mathcal{U} \subseteq \mathcal{U}_1$ . Since  $y \in \mathcal{A} + \mathcal{U}$ , then  $\mathcal{A} \cap (y + \mathcal{U}_1) \supseteq \mathcal{A} \cap (y - \mathcal{U}) \neq \emptyset$ . Thus  $\bigcap_{\mathcal{U} \in \mathcal{U}_0} (\mathcal{A} + \mathcal{U}) \subseteq Cl_h(\mathcal{A})$ .  $\square$

**Corollary 4.6** Let  $\mathcal{V}$  and  $\mathcal{W}$  be  $h$ -neighborhoods of zero of an  $h$ -irresolute topological ring  $\mathcal{R}$  such that  $\mathcal{W} + \mathcal{W} \subseteq \mathcal{V}$ . Then  $Cl_h(\mathcal{W}) \subseteq \mathcal{V}$ .

**Proof:** It is clear that the collection  $\mathcal{U}_0$  of all  $h$ -neighborhoods of zero in  $\mathcal{R}$  is a basis of  $h$ -neighborhoods of zero in  $\mathcal{R}$ . By Definition 4.2, for  $h$ -neighborhood  $\mathcal{W}$  of zero, there exists  $\mathcal{U} \in \mathcal{U}_0$  such that  $\mathcal{U} \subseteq \mathcal{W}$ . In the view of Theorem 4.6,

$$Cl_h(\mathcal{W}) = \bigcap_{\mathcal{U} \in \mathcal{U}_0} (\mathcal{W} + \mathcal{U}) \subseteq \mathcal{W} + \mathcal{U} \subseteq \mathcal{W} + \mathcal{W} \subseteq \mathcal{V}.$$

Thus,  $Cl_h(\mathcal{W}) \subseteq \mathcal{V}$ . □

**Theorem 4.7** *Let  $\psi$  be a ring homomorphism from an  $h$ -irresolute topological ring  $\mathcal{R}$  to an  $h$ -irresolute topological ring  $\mathcal{S}$ . If  $\psi$  is  $h$ -irresolute at zero, then  $\psi$  is  $h$ -irresolute.*

**Proof:** Let  $x \in \mathcal{R}$  and let  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{S}$  containing  $\psi(x)$ . Then, due to Proposition 4.2,  $\mathcal{U} - \psi(x)$  is  $h$ -open set in  $\mathcal{S}$  containing zero. Since  $\psi$  is an  $h$ -irresolute at zero, there exists  $h$ -open set  $\mathcal{U}_1$  in  $\mathcal{R}$  containing zero such that  $\psi(\mathcal{U}_1) \subseteq \mathcal{U} - \psi(x)$ . Consequently,  $\mathcal{U}_1 + x$  is an  $h$ -open set in  $\mathcal{R}$  containing  $x$ , and as  $\psi$  is a ring homomorphism,

$$\psi(\mathcal{U}_1 + x) = \psi(\mathcal{U}_1) + \psi(x) \subseteq \mathcal{U} - \psi(x) + \psi(x) \subseteq \mathcal{U}.$$

Therefore, the mapping  $\psi : \mathcal{R} \rightarrow \mathcal{S}$  is  $h$ -irresolute. □

**Corollary 4.7** *Let  $\psi$  be a homomorphism from an  $h$ -irresolute topological ring  $\mathcal{R}$  to  $h$ -irresolute topological ring  $\mathcal{S}$ , and if  $\psi$  is  $h$ -totally continuous at zero, then  $\psi$  is  $h$ -irresolute.*

**Corollary 4.8** *Let  $\psi$  be a homomorphism from an  $h$ -irresolute topological ring  $\mathcal{R}$  to  $h$ -irresolute topological ring  $\mathcal{S}$ , and if  $\psi$  is  $h$ -irresolute at zero, then  $\psi$  is  $h$ -continuous.*

**Definition 4.3** *Let  $(\mathcal{R}, +, \cdot, \tau)$  be an  $h$ -irresolute topological ring. A subset  $\mathcal{S}$  of  $\mathcal{R}$  is called a subring of an  $h$ -irresolute topological ring  $\mathcal{R}$  if  $\mathcal{S}$  is a subring of  $\mathcal{R}$  and  $\mathcal{S}$  is endowed with the topology  $\tau' = \{\mathcal{V} \cap \mathcal{S} \mid \mathcal{V} \in \tau\}$ , induced by the topology  $\tau$ .*

**Theorem 4.8** *Every subring  $\mathcal{S}$  of an  $h$ -irresolute topological ring  $\mathcal{R}$  is an  $h$ -irresolute topological ring.*

**Proof:** Let  $r_1, r_2 \in \mathcal{S}$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{S}$  containing  $r_1 - r_2$ . Then  $\mathcal{U} = \mathcal{V} \cap \mathcal{S}$ , where  $\mathcal{V}$  is  $h$ -open set in  $\mathcal{R}$  containing  $r_1 - r_2$ . By Definition 4.1, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$  respectively such that  $\mathcal{U}_1 - \mathcal{U}_2 \subseteq \mathcal{V}$ . Then  $\mathcal{W}_1 = \mathcal{U}_1 \cap \mathcal{S}$  and  $\mathcal{W}_2 = \mathcal{U}_2 \cap \mathcal{S}$  are  $h$ -open sets in  $\mathcal{S}$  containing  $r_1$  and  $r_2$  such that  $\mathcal{W}_1 - \mathcal{W}_2 \subseteq (\mathcal{U}_1 - \mathcal{U}_2) \cap \mathcal{S} \subseteq \mathcal{V} \cap \mathcal{S} = \mathcal{U}$ . Now, suppose  $r_1, r_2 \in \mathcal{S}$  and  $\mathcal{U}$  be an  $h$ -open set in  $\mathcal{S}$  containing  $r_1 \cdot r_2$ . Then  $\mathcal{U} = \mathcal{V} \cap \mathcal{S}$ , where  $\mathcal{V}$  is  $h$ -open set in  $\mathcal{R}$  containing  $r_1 \cdot r_2$ . By hypothesis, there exist  $h$ -open sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  in  $\mathcal{R}$  containing  $r_1$  and  $r_2$ , respectively, such that  $\mathcal{U}_1 \cdot \mathcal{U}_2 \subseteq \mathcal{V}$ . Then  $\mathcal{W}_1 = \mathcal{U}_1 \cap \mathcal{S}$  and  $\mathcal{W}_2 = \mathcal{U}_2 \cap \mathcal{S}$  are  $h$ -open sets in  $\mathcal{S}$  containing  $r_1$  and  $r_2$  such that  $\mathcal{W}_1 \cdot \mathcal{W}_2 \subseteq (\mathcal{U}_1 \cdot \mathcal{U}_2) \cap \mathcal{S} \subseteq \mathcal{V} \cap \mathcal{S} = \mathcal{U}$ . Hence,  $\mathcal{S}$  is an  $h$ -irresolute topological ring. □

**Theorem 4.9** *Let  $\mathcal{R}$  be an  $h$ -irresolute topological ring. Then  $h$ -closure of a subring of  $\mathcal{R}$  is also a subring.*

**Proof:** Let  $\mathcal{S}$  be any subring of an  $h$ -irresolute topological ring  $\mathcal{R}$ . Then  $\mathcal{S} - \mathcal{S} \subseteq \mathcal{S}$  and  $\mathcal{S} \cdot \mathcal{S} \subseteq \mathcal{S}$ . By statements (3) and (4) (see Theorem 4.3),  $Cl_h(\mathcal{S}) - Cl_h(\mathcal{S}) \subseteq Cl_h(\mathcal{S} - \mathcal{S}) \subseteq Cl_h(\mathcal{S})$  and  $Cl_h(\mathcal{S}) \cdot Cl_h(\mathcal{S}) \subseteq Cl_h(\mathcal{S} \cdot \mathcal{S}) \subseteq Cl_h(\mathcal{S})$ . Thus  $Cl_h(\mathcal{S}) - Cl_h(\mathcal{S}) \subseteq Cl_h(\mathcal{S})$  and  $Cl_h(\mathcal{S}) \cdot Cl_h(\mathcal{S}) \subseteq Cl_h(\mathcal{S})$ . Hence,  $Cl_h(\mathcal{S})$  is a subring. □

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