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Degree Power Product Connectivity Energy of Graph

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ABSTRACT: In this paper we introduce and investigate degree power product connectivity matrix (which is basically a degree based matrix) and the corresponding energy of graph structures. The upper and lower bounds have been obtained. Further we generalize the degree power product connectivity energy to various families of graphs including some regular and semi regular graphs.

Key Words: Degree power product connectivity matrix, spectra and energy.

Contents

1. Introduction

In mathematical chemistry, topological indices and energy of graph are the two topics which attracted the researchers a lot. In 1978, Ivan Gutman introduced the concept of energy of graph by considering the adjacency matrix of the molecular graph. The concept is basically originated from chemistry and he defined the energy as the sum of the absolute values of the eigenvalues of adjacency matrix of the graph. Writing the adjacency matrix for a graph is one of the popular concept and using this one can generate the energy of graph. Topological indices are the graph invarients which are oftenly used to predict the physio-chemical properties of chemical compounds by means of the molecular graphs. In this paper we introduce a new topological index of graph, degree power product connectivity index as

$$DPP(G) = \frac{d_u^2 d_v + d_u d_v^2}{\sqrt{d_u^2 + d_v^2 + d_u d_v}}.$$

(Where d_u and d_v represents the degree of the respective vertices u and v.) Discussing the algebraic properties of graph is indeed a very interesting and attracted the researchers a lot. In recent era, motivated by the topological indices, the researchers introduced many graph matrices which got the importance in mathematical chemistry. Randić matrix [1], atom bond connectivity matrix are the few examples of matrices with respect to topological indices. Motivated by this we introduce a new matrix as

$$DPP(G) = \begin{cases} \frac{d_u^2 d_v + d_u d_v^2}{\sqrt{d_u^2 + d_v^2 + d_u d_v}} & uv \in E \\ 0 & otherwise \end{cases}$$

Let α be the eigenvalues of degree power product connectivity marices, then the degree power product connectivity energy is given by

$$DPPE(G) = \sum_{i=1}^{n} |\alpha_i|.$$

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2. Upper and lower bounds of degree power product connectivity energy of a graph

The following result gives an upper bound for the DPP energy of a graph G.

Theorem 2.1 *Let G be a graph with n vertices. Then*

$$DPPE(G) \leq \sqrt{2n\left[\sum_{i<j}\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right]}$$

.

Proof: Let $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ be the eigenvalues of $DPP(G)$. Now by the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

We let $a_i = 1$ and $b_i = \alpha$. Then

$$\left(\sum_{i=1}^{n} |\alpha_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\alpha_i|^2\right)$$

which implies that

$$[DPPE(G)]^2 \leq 2n\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right]$$

and finally

$$DPPE(G) \leq \sqrt{2n\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right]}$$

which is an upper bound. □

Theorem 2.2 *Let G be a graph with n vertices and let $det(DPP)$ be the determinant of $DPP(G)$, then*

$$DPPE(G) \geq \sqrt{2\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right] + n(n-1)[det(DPP)]^{\frac{2}{n}}}.$$

3. Properties of degree power product connectivity energy of a graph

Proposition 3.1 *The first three coefficients of the polynomial $\phi_{DPP}(G,\alpha)$ are given as follows:*

(i) $a_0 = 1,$
(ii) $a_1 = 0,$
(iii) $a_2 = -\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right].$

Proof: **(i)** From the definition, $\phi_{DPP}(G,\alpha) = det[\alpha I - DPP(G)]$ and then we get $a_0 = 1$ after easy calculations.

(ii) The sum of the determinants of all 1×1 principal submatrices of $DPP(G)$ is equal to the trace of $DPP(G)$. Therefore

$$a_1 = (-1)^1 \cdot \text{trace of } [DPP(G)] = 0.$$

(iii) Similarly we have

$$\begin{aligned}
(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\
&= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\
&= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\
&= -\sum_{i<j} \left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right].
\end{aligned}$$

□

Proposition 3.2 *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $DPP(G)$, then*

$$\sum_{i=1}^{n} {\alpha_i}^2 = 2\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right].$$

Proof: We know that

$$\begin{aligned}
\sum_{i=1}^{n} \alpha^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2 \\
&= 2\sum_{i<j} a_{ij}^2 \\
&= 2\sum_{i<j}\left[\frac{(d_u^2 d_v + d_u d_v^2)^2}{d_u^2 + d_v^2 + d_u d_v}\right].
\end{aligned}$$

□

Theorem 3.1 *Let G be a regular graph of n vertices with regularity r, then*

$$DPPE(G) = \left(\frac{2r^2}{\sqrt{3}}\right) E(G)$$

Theorem 3.2 *Let G be a semiregular graph of degrees $r_1 \ge 1$ and $r_2 \ge 1$.*
Then $DPPE(G) = (\frac{r_1^2 r_2 + r_1 r_2^2}{\sqrt{r_1^2 + r_2^2 + r_1 r_2}})E(G)$.

Proof: Consider a semiregular graph of degrees $r_1 \ge 1$ and $r_2 \ge 1$, the $FL-$matrix is given by

$$DPP(G) = \left(\frac{r_1^2 r_2 + r_1 r_2^2}{\sqrt{r_1^2 + r_2^2 + r_1 r_2}}\right)(J - I).$$

$$\alpha_i = \left(\frac{r_1^2 r_2 + r_1 r_2^2}{\sqrt{r_1^2 + r_2^2 + r_1 r_2}}\right)\lambda_i.$$

(Here λ_i represents the eigenvalue with respect to adjacency matrix of the corresponding graph.)
Thus the proof follows. □

4. Degree power product connectivity energy of some standard graphs

Theorem 4.1 *Degree power product connectivity energy of complete graph K_n is*

$$DPPE(K_n) = \left(\frac{4(n-1)^3}{\sqrt{3}}\right).$$

Proof: $\frac{2(n-1)^2}{\sqrt{3}}$ For each and every vertex u in K_n, $d(u) = (n–1)$. Then every ij^{th}-entry of the degree power product connectivity matrix will be

$$\begin{bmatrix} 0 & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & \cdots & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} \\ \frac{2(n-1)^2}{\sqrt{3}} & 0 & \frac{2(n-1)^2}{\sqrt{3}} & \cdots & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} \\ \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & 0 & \cdots & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & \cdots & 0 & \frac{2(n-1)^2}{\sqrt{3}} \\ \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^2}{\sqrt{3}} & \cdots & \frac{2(n-1)^2}{\sqrt{3}} & 0 \end{bmatrix}.$$

Hence the characteristic equation will be

$$\left(\alpha + \left(\frac{2(n-1)^2}{\sqrt{3}}\right)^{n-1}\left(\alpha - \frac{2(n-1)^3}{\sqrt{3}}\right)\right) = 0$$

and therefore the spectrum becomes

$$Spec_{DPP}(K_n) = \begin{pmatrix} -\frac{2(n-1)^2}{\sqrt{3}} & \frac{2(n-1)^3}{\sqrt{3}} \\ n-1 & 1 \end{pmatrix}.$$

Therefore,

$$DPPE(K_n) = \left(\frac{4(n-1)^3}{\sqrt{3}}\right).$$

□

Theorem 4.2 *The degree power product connectivity energy of the crown graph S_n^0 is*

$$DPPE(S_n^0) = \left[\frac{8(n-1)^3}{\sqrt{3}}\right].$$

Proof: Let S_n^0 be the crown graph of order $2n$ with vertex set
$\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The degree power product connectivity matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1 & \dots & \alpha_1 & \alpha_1 \\ 0 & 0 & 0 & \dots & 0 & \alpha_1 & 0 & \dots & \alpha_1 & \alpha_1 \\ 0 & 0 & 0 & \dots & 0 & \alpha_1 & \alpha_1 & \dots & 0 & \alpha_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \alpha_1 & \alpha_1 & \dots & \alpha_1 & 0 \\ 0 & \alpha_1 & \alpha_1 & \dots & \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_1 & 0 & \alpha_1 & \dots & \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 & \alpha_1 & 0 & \dots & \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_1 & \alpha_1 & \alpha_1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Where $\alpha_1 = \frac{2(n-1)^2}{\sqrt{3}}$. In that case the characteristic equation is

$$(\alpha - (\alpha_1))^{n-1} (\alpha + \alpha_1)^{n-1} (\alpha + (n-1)(\alpha_1)) (\alpha - (n-1)(\alpha_1)) = 0$$

implying that the spectrum is

$$Spec_{DPP}(S_n^0) = \begin{pmatrix} -(n-1)(\alpha_1) & (n-1)(\alpha_1) & -\alpha_1 & \alpha_1 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore,

$$DPPE(S_n^0) = \left[\frac{8(n-1)^3}{\sqrt{3}}\right].$$

□

Theorem 4.3 *The degree power product connectivity energy of complete bipartite graph $K_{m\times n}$ is*

$$DPPE(K_{m,n}) = 2\left(\frac{m^{\frac{5}{2}}n^{\frac{3}{2}} + m^{\frac{3}{2}}n^{\frac{5}{2}}}{\sqrt{m^2+n^2+mn}}\right).$$

Proof: The degree power product connectivity matrix of complete bipartite graph $K_{m\times n}$ is

$$\left(\frac{m^2n+mn^2}{\sqrt{m^2+n^2+mn}}\right)\begin{bmatrix} 0_{m\times m} & J_{m\times n} \\ J_{n\times m} & 0_{n\times n} \end{bmatrix}.$$

$$Spec_{DPP}(K_{m,n}) = \begin{pmatrix} \left(\frac{m^{\frac{5}{2}}n^{\frac{3}{2}}+m^{\frac{3}{2}}n^{\frac{5}{2}}}{\sqrt{m^2+n^2+mn}}\right) & 0 & -\left(\frac{m^{\frac{5}{2}}n^{\frac{3}{2}}+m^{\frac{3}{2}}n^{\frac{5}{2}}}{\sqrt{m^2+n^2+mn}}\right) \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

$$DPPE(K_{m,n}) = 2\left(\frac{m^{\frac{5}{2}}n^{\frac{3}{2}} + m^{\frac{3}{2}}n^{\frac{5}{2}}}{\sqrt{m^2+n^2+mn}}\right).$$

□

Definition 4.1 [1] The cocktail party graph, denoted by $K_{n\times 2}$, is graph having vertex set $V = \cup_{i=1}^{n}\{u_i, v_i\}$ and the edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < j \le n\}$

Theorem 4.4 *The degree power product connectivity energy of Cocktail party graph $K_{n\times 2}$ is*

$$DPPE(K_{n\times 2}) = \frac{16(n-1)^3}{\sqrt{3}}$$

Proof: Let $K_{n\times 2}$ be a Cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The degree power product connectivity matrix is $2\sqrt{2n-2} + (2n-2)^{\frac{2}{3}}$

$$DPPE(K_{n\times 2}) = \left(\frac{8(n-1)^2}{\sqrt{3}}\right)\begin{pmatrix} (J-I)_{n\times n} & (J-I)_{n\times n} \\ (J-I)_{n\times n} & (J-I)_{n\times n} \end{pmatrix}.$$

Characteristic equation is

$$\alpha^n \left(\alpha + \frac{8(n-1)^2}{\sqrt{3}}\right)^{n-1} \left(\alpha + \frac{8(n-1)^3}{\sqrt{3}}\right) = 0$$

Hence, spectrum is

$$Spec_{DPP}(K_{n\times 2}) = \begin{pmatrix} -\frac{8(n-1)^2}{\sqrt{3}} & 0 & \frac{8(n-1)^3}{\sqrt{3}} \\ n-1 & n & 1 \end{pmatrix}.$$

Therefore,

$$DPPE(K_{n\times 2}) = \frac{16(n-1)^3}{\sqrt{3}}$$

□

Theorem 4.5 *The degree power product connectivity energy of star graph $K_{1,n-1}$ is*

$$DPPE(K_{1,n-1}) = 2\sqrt{\frac{n^5-n^4-2n^3+2n^2+n-1}{n^2-n+1}}.$$

Proof: Let $K_{1,n-1}$ be the star graph with vertex set $V=\{v_0,v_1...v_{n-1}\}$. The degree power product connectivity matrix is

$$DPP(K_{1,n-1}) = \frac{n^2-1}{\sqrt{n^2-n+1}}\begin{pmatrix} 0_{1\times 1} & J_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1\times n-1}\end{pmatrix}.$$

Characteristic equation is

$$(\alpha)^{n-2}\left(\alpha^2 - \frac{n^5-n^4-2n^3+2n^2+n-1}{n^2-n+1}\right)$$

spectrum is $Spec_{DPP}(K_{1,n-1}) = \begin{pmatrix} \sqrt{\frac{n^5-n^4-2n^3+2n^2+n-1}{n^2-n+1}} & 0 & -\sqrt{\frac{n^5-n^4-2n^3+2n^2+n-1}{n^2-n+1}} \\ 1 & n-2 & 1\end{pmatrix}$.
Therefore, $DPPE(K_{1,n-1}) = 2\sqrt{\frac{n^5-n^4-2n^3+2n^2+n-1}{n^2-n+1}}$. □

5. Degree power product connectivity energy of cubic graphs of order 10

There are 21 cubic graphs of order 10. They are represented in [1]. The eigen values and the inverse sum indeg energy of cubic graphs of order 10 are given in the following table.

Graph	Eigen values	ISI Energy
G_1	-20.7850, -20.7850, -16.2284, -10.3925, -10.3925, 0.0000, 10.3925, 10.3925, 26.6209, 31.1775,	157.1669
G_2	-25.5260, -20.7850, -16.6829, -10.3925, -3.8277, 0.0000, 10.3925, 13.3350, 22.3091, 31.177	154.4281
G_3	-25.7746, -18.7266, -15.9524, -10.3925, -4.6251, -1.5433, 7.5396, 12.9592, 25.3382, 31.1775	154.0291
, G_4	-29.2404, -20.7850, -10.3925, -5.5009, -4.3047, -0.0000, 0.0000, 13.9563, 25.0897, 31.1775 ,	140.4471
G_5	-20.7850, -20.7850, -18.0003, -10.3925, -4.3047, 0.0000, 0.0000, 18.0003, 25.0897, 31.1775	148.5351
G_6	-31.1775, -16.8154, -16.8154, -6.4229, -6.4229, 6.4229, 6.4229, 16.8154, 16.8154, 31.1775	155.3083
G_7	-27.2079, -23.9316, -16.8154, -6.4229, -3.9696, 6.4229, 10.3925, 13.5391, 16.8154, 31.1775	156.6949
G_8	-26.6209, -20.7850, -20.7850, -10.3925, -0.0000, 10.3925, 10.3925, 10.3925, 16.2284, 31.1775	157.1669
G_9	-26.9805, -20.7850, -15.9222, -12.2906, -3.6093, 5.3596, 10.3925, 13.1265, 19.5315, 31.177	159.1754

G_{10}	-27.2079, -27.2079, -6.4229, -6.4229, -3.9696, -3.9696, 10.3925, 16.8154, 16.8154, 31.177	150.4017
G_{11}	-27.2079, -19.3384, -16.8154, -6.4229, -3.9696, -2.6408, 6.4229, 16.8154, 21.9792, 31.1775	152.7900
G_{12}	-20.7850, -20.7850, -20.7850, -10.3925, -10.3925, 10.3925, 10.3925, 10.3925, 20.7850, 31.1775	166.2800
G_{13}	-27.7163, -23.3517, -13.4489, -5.7674, -4.4274, 0.0000, 8.3341, 13.6081, 21.5920, 31.1775	149.4234
G_{14}	-20.7850, -20.7850, -15.9222, -15.9222, -3.6093, -3.6093, 10.3925, 19.5315, 19.5315, 31.1775	161.2660
G_{15}	-28.1561, -18.7266, -18.7266, -4.6251, -4.6251, -2.0155, 12.9592, 12.9592, 19.7791, 31.1775	153.7502
G_{16}	-31.1775, -20.7850, -10.3925, -10.3925, -0.0000, 0.0000, 10.3925, 10.3925, 20.7850, 31.1775	145.4950
G_{17}	-20.7850, -20.7850, -20.7850, -20.7850, 10.3925, 10.3925, 10.3925, 10.3925, 10.3925, 31.1775	166.2800
G_{18}	-25.8699, -20.7850, -10.3925, -10.3925, -3.0052, 0.0000, 0.0000, 10.3925, 28.8751, 31.177	140.8902
G_{19}	-25.6989, -16.8154, -16.8154, -15.2001, -6.4229, 6.4229, 6.4229, 16.8154, 20.1140, 31.1775	161.9055
G_{20}	-20.7850, -20.7850, -10.3925, -10.3925, -10.3925, 0.0000, 0.0000, 10.3925, 31.1775, 31.1775	145.4950
G_{21}	-31.1775, -10.3925, -10.3925, -10.3925, 0, 0, 0, 0, 31.1775 31.1775	124.71

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