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The third natural representation of the symmetric groups

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ABSTRACT: The intension of this work is to analyze the third natural representation module, $M^3(n) = KS_n x_1 x_2 x_3$ and we concentrate our work on finding exact sequence for the KS_n-submodules.

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1. Introduction

Let K be a field of characteristic P (which may be zero or a prime number) and $x_1, \ldots, x_n$ be linearly independent commuting indeterminate. KS_n denote the group algebra of the symmetric group S_n over K.

We are interesting in the following cyclic KS_n–submodules of $K[x_1, \ldots, x_n]$ which are finite dimensional vector spaces over K.

The rth − natural representation module $M^r(n),\ r = 1, 2, 3$ defined by $M^r(n) = KS_n x_1 x_2 \ldots x_r$ $(0 \leq r \leq n)$.

The permutation module $M(n-r, r),\ n \geq 3$, has a K-basis $B^r(n) = \{x_{i_1} x_{i_2} \ldots x_{i_r} \mid 1 \leq i_1 \leq i_2 \ldots \leq i_r \leq n\}$ and dim $M^r(n) = \binom{n}{r}$.

And $M_0^r(n)$ the KS_n-submodule of $M^r(n)$ which consist of all polynomials of the form

$$\sum_{1 \leq i_1 < \ldots < i_r \leq n} k_{i_1 \ldots i_r} x_{i_1} \ldots x_{i_r}, \quad \text{with} \sum_{1 \leq i_1 < \ldots < i_r \leq n} k_{i_1 \ldots i_r} = 0$$

and $M_0^r(n)$ has a K-basis $B_0^r(n) = \{x_i x_j x_k - x_1 x_2 x_3 \mid (i, j, k) \neq (1, 2, 3)\}$ with dim $= \binom{n}{r} - 1$.

$S_k(n-r, r)$ the KS_n-submodule of $M_k(n-r, r)$ over K generated by

$$\{(x_{i_1} x_{i_2} \ldots x_{i_r})(x_j - x_k) \mid 1 \leq i_1 \leq \ldots \leq i_r \leq n,\ j \neq k\}.$$

In $n \geq 6$ the spect module $S_k(n-r, 1^r)$ generated by $KS_n(x_{i+1} - x_i)$. We can apply the same idea for the concepts appear in [5-8] also [9-13].

2. The third natural representation module of S_n

Throughout this work we will be interesting in the following KS_n-homomorphism.

$h_3 : M(n-3, 3) \to K$ which is defined by

$$h_3\Big(\sum_{i_1 i_2 i_3 \leq n} k_{i_1 i_2 i_3} x_{i_1} x_{i_2} x_{i_3}\Big) = \sum_{i_1 i_2 i_3 \leq n} k_{i_1 i_2 i_3}.$$

$d_2 : M(n-3, 3) \to M(n-2, 2)$ which is defined by

$$d_2(x_i x_j x_s) = \sum_{\beta=1}^{n} \frac{\partial}{\partial x_\beta}(x_i x_j x_s),$$

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and $\overline{d_2}$ is the restriction of d_2 to $M_0(n-3,3)$, i.e. $\overline{d_2} : M_0(n-3,3) \to M_0(n-2,2)$.

The KS_n–homomorphism $\mu_2 : S_k(n-r, 1^r) \to M(n-r,r), r = 2,3$ defined by

$$\mu_2(a(x_{i_1}, x_i, x_j)) = d_2(x_i x_j x_j).$$

Let d_2 be the restriction of the KS_n-homomorphism d_2 to the KS_n-submodule $M_0(n-3,3)$ i.e.

$$\overline{d_2} : M_0(n-3,3) \to M_0(n-2,2)$$

defined by:

$$\overline{d_2}(x_i x_j x_t - x_r x_s x_t) = d_2(x_i x_j x_t - x_r x_s x_t) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_i x_j x_t - x_r x_s x_t).$$

$$= x_i x_j + x_i x_t + x_j x_t - x_r x_s - x_r x_t - x_s x_t \in B_0(n-3,3).$$

Lemma (2.1): If K is a field of characteristic $p \neq 2$, then $M_0(n-2,2)$ is a KS_n-submodule of $\mathrm{Im}(d_2)$.

Proof:

$$d_2(x_1x_3x_4 + x_2x_3x_4 - x_1x_2x_3 - x_1x_2x_4)$$

$$= (x_1x_3 + x_1x_4 + x_3x_4 + x_2x_3 + x_2x_4 + x_3x_4 - x_1x_2 - x_1x_3 - x_2x_3 - x_1x_2 - x_1x_4 - x_2x_4) \in \mathrm{Im}(d_2).$$

Hence, $2x_3x_4 - 2x_1x_2 = 2(x_3x_4 - x_1x_2) \in \mathrm{Im}(d_2)$, which implies that $(x_3x_4 - x_1x_2) \in \mathrm{Im}(d_2)$ since $p \neq 2$.

Thus $KS_n(x_3x_4 - x_1x_2)$ is a KS_n-submodule of $\mathrm{Im}(d_2)$, but $M_0(n-2,2) = KS_n(x_3x_4 - x_1x_2)$. Therefore $M_0(n-2,2)$ is a KS_n-submodule of $\mathrm{Im}(d_2)$ when $p \neq 2$. □

Theorem (2.2): If $p = 3$ and $n \geq 6$ then $\mathrm{Im}(d_2) = d_2(M_0(n-3,3)) = M_0(n-2,2)$.

Proof:

$$M_0(n-2,2) \subseteq \mathrm{Im}(d_2) \quad \text{by Lemma (2.1).}$$

And $\mathrm{Im}(d_2) \subseteq M_0(n-2,2)$ since $d_2(x_i x_j x_k) = x_i x_j + x_i x_k + x_j x_k \in M_0(n-2,2)$.

Thus $\mathrm{Im}(d_2) = M_0(n-2,2)$.

Now $(2x_1x_3x_4 + 2x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4) \in M_0(n-3,3)$, and

$$d_2(2x_1x_3x_4 + 2x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4) = d_2(2x_1x_3x_4) + d_2(2x_2x_3x_4) + d_2(x_1x_2x_3) + d_2(x_1x_2x_4).$$

$$= x_1x_3 + x_1x_4 + 2x_3x_4 + x_2x_3 + x_2x_4 + 2x_3x_4 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2 + x_1x_4 + x_2x_4.$$

Therefore $KS_n(x_3x_4 - x_1x_2)$ is a KS_n–submodule of $d_2(M_0(n-3,3))$. Hence $M_0(n-2,2) = KS_n(x_3x_4 - x_1x_2) \subseteq d_2(M_0(n-3,3)) \subseteq M_0(n-2,2)$.

Thus $d_2(M_0(n-3,3)) = M_0(n-2,2)$. □

Corollary (2.3): If $p = 3$, then the KS_n-homomorphism

$$\overline{d_2} : M_0(n-3,3) \to M_0(n-2,2) is onto.$$

Theorem (2.4): If $p = 3$, then the following sequence of KS_n-modules is exact:

$$0 \longrightarrow S_K(n-3,3) \xrightarrow{\text{incl.}} M_0(n-3,3) \xrightarrow{\overline{d_2}} M_0(n-2,2) \longrightarrow 0 \tag{1}$$

Proof: The sequence (1) is exact since $\overline{d_2}$ is onto by corollary (2.3), and incl. is one to one. And $S_K(n-3,3) \subseteq \ker(\overline{d_2})$, since:

$$\overline{d_2}((x_2 - x_1)(x_4 - x_3)(x_6 - x_5)))$$

$$= d_2(x_2x_4x_6 - x_2x_4x_5 - x_2x_3x_6 + x_2x_3x_5 - x_1x_4x_6 + x_1x_4x_5 + x_1x_3x_6 - x_1x_3x_5)$$

$$= x_2x_4 + x_2x_6 + x_4x_6 - x_2x_4 - x_2x_5 - x_4x_5 - x_2x_3 - x_2x_6 - x_3x_6 +$$

$$x_2x_3 + x_3x_5 + x_2x_5 - x_1x_4 - x_1x_6 + x_4x6 + x_1x_5 + x_4x5 + x_1x_3 + x_1x_6 + x_3x_6 - x_1x_3 - x_1x_5 - x_3x_5 = 0.$$

We prove the reverse inclusion by counting the dimensions:

$$\dim_K \ker d_2 = \dim_K M_0(n-3,3) - \dim_K M_0(n-2,2)$$

$$= \binom{n}{3} - 1 - \left(\binom{n}{2} - 1\right) = \binom{n}{3} - \binom{n}{2} = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$$

$$= \frac{n(n-1)(n-2) - 3n(n-1)}{6} = \frac{n(n-1)(n-5)}{6} = \dim_K S_K(n-3,3).$$

Hence $\ker(\overline{d_2}) = S_K(n-3,3) = \mathrm{Im}(\mathrm{incl})$. □

Corollary (2.5): If $p = 3$ then

$$M_0(n-3,3)/S_K(n-3,3) \cong M_0(n-2,2).$$

Lemma (2.6): If $p \neq 2$ and $p \neq 3$ then the KS_n-homomorphism

$$d_2 : M(n-3,3) \to M(n-2,2) is onto.$$

Proof: Since $p \neq 2$, then $M_0(n-2,2) \subseteq \mathrm{Im}(d_2)$ by lemma (2.1). And if $p \neq 3$, then $\mathrm{Im}(d_2) \nsubseteq M_0(n-2,2)$, since

$$d_2(x_1x_2x_3) = x_1x_2 + x_1x_3 + x_2x_3 \notin M_0(n-2,2).$$

Thus $M_0(n-2,2) \subsetneq \mathrm{Im}(d_2)$. But $\dim_K M_0(n-2,2) = \binom{n}{2} - 1 < \binom{n}{2} = \dim_K M(n-2,2)$, so

$$\dim_K \mathrm{Im}(d_2) \leq \dim_K M(n-2,2) = \binom{n}{2}.$$

Thus $\dim_K \mathrm{Im}(d_2) = \dim_K M(n-2,2)$. Hence $\mathrm{Im}(d_2) = M(n-2,2)$, since $\mathrm{Im}(d_2)$ is a KS_n–submodule of $M(n-2,2)$. Thus d_2 is onto. □

Remark (2.7): If $p = 2$, then $\mathrm{Im}(d_2) = \mathrm{Im}(\mu_2)$. Therefore:

$$\dim_K \mathrm{Im}(d_2) = \dim_K \mathrm{Im}(\mu_2) = \binom{n-1}{2} \neq \binom{n}{2} = \dim_K M(n-2,2).$$

Thus d_2 is not onto when $p = 2$.

Theorem (2.8): If $p \neq 2$ and $p \neq 3, n \geq 6$ then the following sequence of KS_n-modules is exact:

$$0 \longrightarrow S_K(n-3,3) \xrightarrow{\text{incl.}} M(n-3,3) \xrightarrow{d_2} M(n-2,2) \longrightarrow 0 \qquad (2)$$

Proof: Since incl. is one to one and d_2 is onto, by Remark (2.7). Then, we prove the exactness only on $M(n-3,3)$.

$S_K(n-3,3) \subseteq \ker(d_2)$, since:

$$d_2((x_2 - x_1)(x_4 - x_3)(x_6 - x_5)) = 0.$$

As we proved in theorem (2.4), we prove the reverse inclusion by counting the dimensions:

$$\dim_K \ker(d_2) = \dim_K M(n-3,3) - \dim_K M(n-2,2) = \binom{n}{3} - \binom{n}{2}$$

$$= \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} = \frac{n(n-1)(n-2) - 3n(n-1)}{6} = \frac{n(n-1)(n-5)}{6} = \dim_K S_K(n-3,3).$$

Thus $\ker(d_2) = S_K(n-3,3) = \mathrm{Im}(\mathrm{incl})$. Hence the sequence (2) is exact. □

Corollary (2.9): If $p \neq 2$ and $p \neq 3$, then $M(n-3,3)/S_K(n-3,3) \cong M(n-2,2)$.

Theorem (2.10): If $p \neq 2$ and $p \neq 3$, then $d_2(M_0(n-3,3)) = M_0(n-2,2)$.

Proof: Clearly $d_2(M_0(n-3,3)) \subseteq M_0(n-2,2)$. We prove the reverse inclusion by counting the dimensions.

$$\dim_k d_2(M_0(n-3,3)) = \dim_k M_0(n-2,2) - \dim_k \ker(d_2) \text{ since } \ker(d_2) = S_K(n-3,3)$$

$$\begin{aligned}\dim_k M_0(n-3,3) &= \binom{n}{3} - 1 - \frac{n(n-1)(n-5)}{6} = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)(n-5)}{6} \\ &= \frac{n(n-1)(n-2-n+5)}{6} - 1 = \frac{n(n-1)(3)}{6} - 1 \\ &= \frac{n(n-1)}{2} - 1 = \binom{n}{2} - 1 = \dim_k M_0(n-2,2).\end{aligned}$$

Theorem (2.11): If $p = 3$ then $\dim_k \ker(d_2) = \dim_k S_K(n-3,3) + 1$.

Proof: If $p = 3$ then $\mathrm{Im}(d_2) = M_0(n-2,2)$, by theorem (2.2)

$$\begin{aligned}\dim_k \ker(d_2) &= \dim_k M_0(n-3,3) - \dim_k \mathrm{Im}(d_2) = \binom{n}{3} - \left(\binom{n}{2} - 1\right) = \binom{n}{3} - \binom{n}{2} + 1 \\ &= \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} + 1 = \frac{n(n-1)(n-5)}{6} + 1 = \dim_k S_K(n-3,3) + 1.\end{aligned}$$

Theorem (2.12): If $p \nmid (n-4)$ then $d_2 g_3 = I_{S_K(n-2,2)}$, where $g_3 : S_K(n-2,2) \to M(n-3,3)$

Since $g_3(((x_2 - x_1)(x_4 - x_3))) = g_3(x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3) = x_2x_4(x^{(n)} - x_2x_4) - x_2x_3(x^{(n)} - x_2 - x_3) - x_1x_4(x^{(n)} - x_1 - x_4) + x_1x_3(x^{(n)} - x_1 - x_3) = x_1x_2x_4 + x_2x_3x_4 + x_2x_4x_5 + \cdots + x_2x_4x_n - x_1x_2x_3 - x_2x_3x_4 - x_2x_3x_5 - \cdots - x_2x_3x_n - x_1x_2x_4 - x_1x_3x_4 - x_1x_4x_5 - \cdots - x_1x_4x_n + x_1x_2x_3 + x_1x_3x_4 + x_1x_3x_5 + \cdots + x_1x_3x_n$ Then

$$\begin{aligned}d_2g_3(((x_2 - x_1)(x_4 - x_3))) &= -x_4x_5 - x_4x_6 - \cdots - x_4x_n + x_3x_5 + x_3x_6 + \cdots + x_3x_n \\ &+ x_4x_5 + x_4x_6 + \cdots + x_4x_n - x_3x_5 - x_3x_6 - \cdots - x_3x_n \\ &- x_2x_n + x_1x_5 + x_1x_6 + \cdots + x_1x_n + x_2x_5 + x_2x_6 + \cdots + x_2x_n - x_1x_5 \\ -x_1x_6 - \cdots - x_1x_n &+ (n-4)x_2x_4 - (n-4)x_2x_3 - (n-4)x_1x_4 + (n-4)x_1x_3 \\ &= (n-4)(x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3) = (n-4)(x_2 - x_1)(x_4 - x_3)\end{aligned}$$

Thus $\frac{1}{(n-4)} d_2g_3(((x_2 - x_1)(x_4 - x_3))) = (x_2 - x_1)(x_4 - x_3)$ since $p \nmid (n-4)$

i.e. $d_2g_3 = I_{S_K(n-2,2)}$.

Lemma (2.13): If $p = 2$, then $\overline{d_2}(M_0(n-3,3)) = S_k(n-2,2)$

Proof: $d_2(x_1x_2x_4 + x_1x_2x_3) = x_1x_2 + x_1x_4 + x_2x_4 + x_1x_2 + x_1x_3 + x_2x_3 = x_1x_4 + x_2x_4 + x_1x_3 + x_2x_3 = (x_2 + x_1)(x_4 + x_3)$ But $(x_2 + x_1)(x_4 + x_3)$ generates $S_k(n-2,2)$ over KS_n. And $(x_1x_2x_4 + x_1x_2x_3)$ generates $M_0(n-3,3)$ over KS_n. Thus $\mathrm{Im}\overline{d_2} = S_K(n-2,2)$ i.e. $\overline{d_2}$ is onto.

Corollary (2.14): If $p = 2$, then $\mathbb{M}_0(n-3,3)/\ker(\bar{d}_2) \cong \mathcal{S}_{k(n-2)}$.

Remark (2.15): Let $\overline{g}_3$ denotes the restriction of the KS_n-homomorphism g_3 to $S(n-2,2)$, i.e. $\overline{g}_3 = g_3\Big|_{S_k(n-2,2)} : S_k(n-2,2) \to M_0(n-3,3)$, defined by $\overline{g}_3\Big((x_2 - x_1)(x_4 - x_3)\Big) = g_3(x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3) = (x_2x_4(x^{(n)} - x_2 - x_3) - x_2x_3(x^{(n)} - x_2 - x_3) - x_1x_4(x^{(n)} - x_1 - x_4) + x_1x_3(x^{(n)} - x_1 - x_3))$.

Theorem (2.16) If $p = 2$ and n is odd then the following sequence of KS_n-modules is exact and splits

$$0 \longrightarrow \mathrm{Im}(\mu_3) \xrightarrow{\text{incl.}} M_0(n-3,3) \xrightarrow{\overline{d}_2} S_k(n-2,2) \longrightarrow 0 \tag{3}$$

proof Since $p = 2$, then by lemma (2.13) $\overline{d}_2$ is onto. And incl. is one to one.

$$\mathrm{Im}(\mu_3) = \mathrm{Im}(d_3) \quad [3]$$

$$\mathrm{Im}(d_3) = \ker(d_2).$$

Thus $\mathrm{Im}(\mu_3) = \ker(d_2)$. But $\ker(d_2) = \ker(\overline{d}_2)$, since $\ker(\overline{d}_2) \subseteq \ker(d_2)$ and:

$$\dim_K \ker(d_2) = \dim_K M(n-3,3) - \dim_K \mathrm{Im}(d_2) = \binom{n}{3} - \binom{n-1}{2}.$$

And

$$\dim_K \ker(\overline{d}_2) = \dim_K M_0(n-3,3) - \dim_K S_k(n-2,2) = \binom{n}{3} - 1 - \left(\binom{n-1}{2} - 1\right) = \binom{n}{3} - \binom{n-1}{2}.$$

Hence $\mathrm{Im}(\mathrm{incl.}) = \mathrm{Im}(\mu_3) = \ker(d_2) = \ker(\overline{d}_2)$. Thus the sequence (3) is exact. Now, since $p = 2$ and n is odd then there is a KS_n-homomorphism

$$\overline{g}_3 : S_k(n-2,2) \longrightarrow M_0(n-3,3)$$

such that

$$\overline{d}_2\overline{g}_3 = d_2\Big|_{M_0(n-3,3)} g_3\Big|_{S_k(n-2,2)} = I_{S_k(n-2,2)},$$

as we proved in theorem (2.12). Thus the exact sequence (3) splits when $p = 2$ and n is odd.
Corollary (2.17) If $p = 2$ and n is odd then

$$M_0(n-3,3) = \mathrm{Im}(\mu_3) \oplus \mathrm{Im}(g_3).$$

proof : Since $p = 2$ and n is odd then by theorem (2.16) the exact sequence (1) splits, and as in the proof of theorem (2.16) there is a KS_n-homomorphism

$$\overline{g}_3 : S_k(n-2,2) \to M_0(n-3,3)$$

such that $\overline{d}_2\overline{g}_3 = I_{S_k(n-2,2)}$, thus $M_0(n-3,3)$ is decomposable into the direct sum of $\ker(\overline{d}_2)$ and $\mathrm{Im}(\overline{g}_3)$. i.e.

$$M_0(n-3,3) = \ker(\overline{d}_2) \oplus \mathrm{Im}(\overline{g}_3).$$

But $\ker(\overline{d}_2) = \mathrm{Im}(\mu_3)$, since the sequence (3) is exact. Thus $M_0(n-3,3) = \mathrm{Im}(\mu_3) \oplus \mathrm{Im}(g_3)$.
Theorem (2.18) The following sequence of KS_n–modules is exact

$$0 \longrightarrow M_0(n-3,3) \xrightarrow{\text{incl.}} M(n-3,3) \xrightarrow{h_3} K \longrightarrow 0 \tag{4}$$

And it is split if and only if $p \nmid \binom{n}{3}$.
Proof: Since incl. is one to one and for each $k \in K$ we have $h_3(kx_1x_2x_3) = k$ which means that h_3 is onto. And $\ker(h_3) = \{x \in M(n-3,3) : h_3(x) = 0\} = \Big\{\sum_{1\le i<j<l\le n} k_{ijl}x_ix_jx_l : \sum_{1\le i<j<l\le n} k_{ijl} = 0\Big\} = M_0(n-3,3) = \mathrm{Im}(\mathrm{incl.})$. Thus the sequence (2) is exact. Now, assume that $p \nmid \binom{n}{3}$ and let $f_3 : K \to M(n-3,3)$ defined by $f_3(k) = \frac{k}{\binom{n}{3}}T^{(n)}$. f_3 is a KS_n–homomorphism, since for each $\sum_{\tau\in S_n} k_\tau\tau \in KS_n$, we have

$$\sum_{\tau\in S_n} k_\tau\tau f_3(k) = \Big(\sum_{\tau\in S_n} k_\tau\tau\Big)\frac{k}{\binom{n}{3}}T^{(n)} = \sum_{\tau\in S_n} k_\tau\frac{k}{\binom{n}{3}}T^{(n)} = \sum_{\tau\in S_n}\frac{k_\tau k}{\binom{n}{3}}T^{(n)} = f_3\Big(\sum_{\tau\in S_n} k_\tau\tau k\Big).$$

Now $h_3f_3(1) = h_3\Big(\frac{1}{\binom{n}{3}}T^{(n)}\Big) = \frac{1}{\binom{n}{3}}h_3(T^{(n)}) = \frac{1}{\binom{n}{3}}\binom{n}{3} = 1$. i.e. $h_3f_3 = 1_K$, thus the exact sequence (2) splits. Conversely, suppose that the exact sequence (2) splits. Let $f : K \to M(n-3,3)$ be a

KS_n–homomorphism, and let $f(1) = \sum_{1\le i<j<l\le n} k_{ijl}x_ix_jx_l$. Since $f(1)$ is invariant under each transposition $T = (x_rx_s)$ of S_n such that $1 \le r < s \le n$. i.e. $Tf(1) = f(T1) = f(1)$ for each transposition of S_n, then

$$0 = f(T1) - Tf(1) = \sum_{1\le i<j<l\le n} k_{ijl}(x_ix_jx_l - Tx_ix_jx_l).$$

But if $i = r$, $j = s$ then $x_ix_jx_l - Tx_ix_jx_l = x_sx_rx_l - x_rx_sx_l = 0$, and if $i = r$, $l = s$ then $x_ix_jx_l - Tx_ix_jx_l = x_sx_jx_i - x_ix_jx_s = 0$, and if $j = r$, $l = s$ then $x_ix_jx_l - Tx_ix_jx_l = x_ix_sx_r - x_ix_rx_s = 0$, and if i, j, l, r, s are distinct integers then $x_ix_jx_l - Tx_ix_jx_l = x_ix_jx_l - x_ix_jx_l = 0$. Hence:

$$0 = f(T1) - Tf(1) = \sum_{\substack{1\le i<j\le n \\ j\ne r,s}} k_{ijr}(x_rx_ix_j - Tx_rx_ix_j) + \sum_{\substack{1\le i<j\le n \\ j\ne r,s}} k_{ijs}(x_sx_ix_j - Tx_sx_ix_j).$$

$$= \sum_{\substack{1\le i<j\le n \\ j\ne r,s}} k_{ijr}(x_rx_ix_j - x_sx_ix_j) + \sum_{\substack{1\le i<j\le n \\ j\ne r,s}} k_{ijs}(x_sx_ix_j - x_rx_ix_j).$$

$$= \sum_{\substack{1\le i<j\le n \\ j\ne r,s}} (k_{ijr} - k_{ijs})(x_rx_ix_j - x_sx_ix_j), \quad (1 \le r < s \le n).$$

Equating the coefficients we have $k_{ijr} - k_{ijs} = 0$, such that $1 \le i, j, r, s \le n$ and i, j, r, s are distinct, which implies that $k_{ijr} = k_{ijs} = k$ (say), such that $1 \le i, j, r, s \le n$ and i, j, r, s are distinct. Hence

$$f(1) = \sum_{1\le i<j<l\le n} k_{ijl}x_ix_jx_l = k \sum_{1\le i<j<l\le n} x_ix_jx_l = kT^{(n)}.$$

Now, since the exact sequence (4) splits, then there exist a KS_n–homomorphism $f_3 : K \to M(n-3, 3)$ such that $h_3f_3 = 1_K$ and $f_3(1) = kT^{(n)}$.

Hence $1 = h_3f_3(1) = h_3(kT^{(n)}) = kh_3(T^{(n)}) = k\binom{n}{3} = k\frac{n(n-1)(n-2)}{6}$.
which implies that

$$p \nmid \frac{n(n-1)(n-2)}{6}, \quad \text{i.e. } p \nmid \binom{n}{3}.$$

Corollary (2.19) If p doesn't divide $\binom{n}{3}$, then $M(n-3, 3)$ is the direct sum of $M_0(n-3, 3)$ and $KT^{(n)}$.
proof : From theorem (2.18) and its proof we have $M(n-3, 3)$ is decomposable into the direct sum of $\ker(h_3)$ and $\mathrm{Im}(f_3)$, i.e.

$$M(n-3, 3) = \ker(h_3) \oplus \mathrm{Im}(f_3).$$

But $\ker(h_3) = M_0(n-3, 3)$, since the sequence (4) is exact, and $\mathrm{Im}(f_3) = KT^{(n)}$, as we proved in corollary (2.17). Hence

$$M(n-3, 3) = M_0(n-3, 3) \oplus KT^{(n)}.$$

Corollary (2.20) If p divides $\binom{n}{3}$ then $M_0(n-3, 3)$ is not the direct summand of $M(n-3, 3)$.
Proof : Assume that $M_0(n-3, 3)$ is the direct summand of $M(n-3, 3)$ when $p \mid \binom{n}{3}$. This means that there exists a KS_n–submodule L of $M(n-3, 3)$ such that

$$M(n-3, 3) = M_0(n-3, 3) \oplus L.$$

This means that the exact sequence (4) splits, then by theorem (2.18) we have $p \nmid \binom{n}{3}$, and this contradicts the assumption. Therefore $M_0(n-3, 3)$ is not the direct summand of $M(n-3, 3)$ when $p \mid \binom{n}{3}$.
Theorem (2.21) If K is a field of char. $p = 2$, then the following sequence of KS_n-modules is exact

$$0 \longrightarrow S(n-3, 1, 1, 1) \xrightarrow{\mu_3} M(n-3, 3) \xrightarrow{d_2} M^2(n) \xrightarrow{h_2} K \longrightarrow 0.$$

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