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Certain Subclass of Analytic Functions Defined by $q-$Analogue Generalized Differential Operator

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ABSTRACT: In the present work, we define a subclass of uniformly starlike functions corresponding to the class of uniformly convex functions involving the q-analogue of a generalized differential operator. Furthermore, we discuss coefficient estimates, neighborhoods, partial sums, integral means inequality, and Radii of close-to-convexity and Starlikeness results related to the defined class.

Key Words: Univalent functions, analytic mappings, symmetric points, uniformly convex mappings, uniformly starshaped mappings, Sălăgean derivative, $q-$derivative.

Contents

1. Introduction

Let $\mathscr{A}$ represent the class or category of every mappings v have the representation

$$v(\xi) = \xi + \sum_{t=2}^{\infty} a_t \xi^t \tag{1.1}$$

in the set $\Delta = \{\xi \in \mathbb{C} : |\xi| < 1\}$, which is open unit disc. Let $\mathscr{S}$ be a subcategory of $\mathscr{A}$ with univalent as well as normalized by $v(0) = v'(0) - 1 = 0$. A function $v \in \mathscr{A}$ which is starshaped have the order ς, if v fulfills with $0 \leq \varsigma < 1$, and

$$\Re\left\{\frac{\xi v'(\xi)}{v(\xi)}\right\} > \varsigma, \quad \xi \in \Delta \tag{1.2}$$

and convex mapping has order ς, if v fulfills with $0 \leq \varsigma < 1$, and

$$\Re\left\{1 + \frac{\xi v''(\xi)}{v'(\xi)}\right\} > \varsigma, \quad \xi \in \Delta. \tag{1.3}$$

Also, the classes of starshaped and convex mappings are represented respectively as $\mathscr{S}^*(\varsigma)$ and $\mathscr{K}(\varsigma)$. Let $\mathscr{T}$ be a subcategory of $\mathscr{S}$ consisting mappings of the form

$$v(\xi) = \xi - \sum_{t=2}^{\infty} a_t \xi^t, \quad |a_t| \geq 0 \tag{1.4}$$

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introduced and studied by Silverman [16]. In [14], Sakaguchi introduced a subclass $\mathscr{S}\mathscr{T}_s$ of starshaped mappings with respect to symmetric points as follows:

$$\Re\left\{\frac{2\xi v'(\xi)}{v(\xi)-v(-\xi)}\right\}>0, \qquad \xi\in\Delta$$

and Owa et al. [10] outlined the class $\mathscr{S}\mathscr{T}_s(\upsilon,\varsigma)$ described below:

$$\Re\left\{\frac{(1-\varsigma)\xi v'(\xi)}{v(\xi)-v(\varsigma\xi)}\right\}>\upsilon, \qquad 0\leq\upsilon<1,\ |\varsigma|\leq 1,\ \varsigma\neq 1,\ \xi\in\Delta.$$

It is interesting to noted as

$$\mathscr{S}\mathscr{T}_s(0,-1):=\mathscr{S}\mathscr{T}_s$$

and

$$\mathscr{S}\mathscr{T}_s(\upsilon,-1):=\mathscr{S}\mathscr{T}_s(\upsilon).$$

After that, for $0\leq\omega<1$ and $k\geq 0$, the class $k-\mathscr{U}ST(\omega)$ of $k-$uniformly starshaped of order ω and the class $k-\mathscr{U}CV(\omega)$ of $k-$uniformly convex mappings of order ω have the following definitions [5]:

$$\Re\left\{\frac{\xi v'(\xi)}{v(\xi)}\right\}>k\left|\frac{\xi v'(\xi)}{v(\xi)}-1\right|+\omega$$

and

$$\Re\left\{1+\frac{\xi v''(\xi)}{v'(\xi)}\right\}>k\left|\frac{\xi v''(\xi)}{v'(\xi)}\right|+\omega.$$

The $q-$calculus or quantum calculus begun with straight to the point Jackson [7] within the early 20^{th} century, but Jacobi as wellas Euler worked out this kind of calculus. At present, In order to activate it, the endless prerequisite regarding arithmetic of which simulates $q-$calculus appeared in quantum computing, also discussed as an affiliation amid of physics and science. It needs extensive use in many numerical domains, including the mechanics, quantum hypothesis, basic hypergeometric mappings, and the theory of relativity.
Define the $q-$number $[\kappa]_q$ with respect to $0<q<1$, as

$$[\kappa]_q=\begin{cases}\dfrac{1-q^{\kappa}}{1-q}, & \text{if} \quad \kappa\in\mathbb{C}\setminus\mathbb{N},\\ \sum\limits_{i=0}^{\kappa-1}q^i, & \text{if} \quad \kappa\in\mathbb{N}.\end{cases} \tag{1.5}$$

Note that as $q\to 1^-$, $[\kappa]_q\to\kappa$. Further, define the $q-$fractional $[\kappa]_q!$ as

$$[\kappa]_q!=\begin{cases}1, & \text{if} \quad \kappa=1,\\ \prod\limits_{t=1}^{\kappa}[t]_q, & \text{if} \quad \kappa\in\mathbb{N}\setminus\{1\}.\end{cases} \tag{1.6}$$

Define the $q-$derivative $\mathscr{D}_q v$ of a mapping v by

$$(\mathscr{D}_q)v(\xi)=\begin{cases}\dfrac{v(\xi)-v(\xi q)}{(1-q)\xi}, & \text{if} \quad \xi\neq 0,\\ v'(0), & \text{if} \quad \xi=0\end{cases} \tag{1.7}$$

given that $v'(0)$ exists. It generates from (1.7) that

$$\lim_{q\to 1^-}\mathscr{D}_q v(\xi)=\lim_{q\to 1^-}\frac{v(\xi)-v(\xi q)}{(1-q)\xi}=v'(\xi)$$

for the mapping v, that is differentiable mapping in a subset of $\mathbb{C}$, which is given. Consequently, we possess

$$(\mathscr{D}_q)v(\xi) = 1 + \sum_{t=2}^{\infty}[t]_q a_t \xi^{t-1}. \tag{1.8}$$

Next, we consider the Sălăgean $q-$ differential operator as follows [6]:

$$\begin{aligned}
\mathscr{D}_q^0 v(\xi) =& v(\xi)\\
\mathscr{D}_q^1 v(\xi) =& \xi\left(\mathscr{D}_q v(\xi)\right)\\
&\vdots\\
\mathscr{D}_q^\lambda v(\xi) =& \mathscr{D}_q^1\left(\mathscr{D}_q^{\lambda-1}v(\xi)\right) = \xi\left(\mathscr{D}_q\mathscr{D}_q^{\lambda-1}v(\xi)\right).
\end{aligned}$$

Thus, we have

$$\mathscr{D}_q^\lambda v(\xi) = \xi + \sum_{t=2}^{\infty}[t]_q^\lambda a_t \xi^t. \tag{1.9}$$

We note that if $q \to 1^-$,

$$\mathscr{D}^\lambda v(\xi) = \xi + \sum_{t=2}^{\infty} t^\lambda a_t \xi^t \tag{1.10}$$

is well-known Sălăgean derivative [15]. Let us now

$$\begin{aligned}
\mathscr{D}^0 &= \mathscr{D}_q^\wp v(\xi)\\
\mathscr{D}_{q,\lambda}^{1,\wp} v(\xi) &= (1-\lambda)\mathscr{D}_q^\wp v(\xi) + \lambda\xi(\mathscr{D}_q^\wp v(\xi))'\\
&= \xi + \sum_{t=2}^{\infty}[t]_q^\wp[1+(t-1)\lambda]a_t\xi^t\\
\mathscr{D}_{q,\lambda}^{2,\wp} v(\xi) &= (1-\lambda)\mathscr{D}_{q,\lambda}^{1,\wp} v(\xi) + \lambda\xi(\mathscr{D}_{q,\lambda}^{1,\wp} v(\xi))'\\
&= \xi + \sum_{t=2}^{\infty}[t]_q^\wp[1+(t-1)\lambda]^2a_t\xi^t\\
\cdots \qquad & \cdots \qquad \cdots\\
\mathscr{D}_{q,\lambda}^{\hbar,\wp} v(\xi) &= (1-\lambda)\mathscr{D}_{q,\lambda}^{\hbar-1,\wp} v(\xi) + \lambda\xi(\mathscr{D}_{q,\lambda}^{\hbar-1,\wp} v(\xi))'\\
&= \xi + \sum_{t=2}^{\infty}[t]_q^\wp[1+(t-1)\lambda]^\hbar a_t\xi^t, \qquad \lambda > 0, \quad \hbar \in \mathbb{N}_0
\end{aligned} \tag{1.11}$$

where $[t]_q!$ is represented as (1.6). this may be noted because, whenever $q \to 1^-$, we now possess

$$\begin{aligned}
\lim_{q\to 1^-}\mathscr{D}_{q,\lambda}^{\hbar,\wp} v(\xi) =& \xi + \lim_{q\to 1^-}\sum_{t=2}^{\infty}[t]_q^\wp[1+(t-1)\lambda]^\hbar a_t\xi^t\\
=& \xi + \sum_{t=2}^{\infty} t^\wp[1+(t-1)\lambda]^\hbar a_t\xi^t a_t\xi^t\\
=& \mathscr{D}_\lambda^{\hbar,\wp} v(\xi).
\end{aligned}$$

We noted that for $\wp = 0$, the differential operator $\mathscr{D}^\hbar$ defined by Al-Oboudi [2] and if $\hbar = 0$, we get the Sălăgean differential operator $\mathscr{D}^\wp$ introduced in [15]. Now we define $k - \mathscr{U}\mathscr{S}\mathscr{T}_s(\lambda, q, \hbar, \wp, \omega, \varsigma)$ by the usage of a generalized differential operator $\mathscr{D}_{q,\lambda}^{\hbar,\wp} v(\xi)$ as follows.

Definition 1.1 Assume $0 < q < 1$, $\lambda > 0$, $k \geq 0$, $|\varsigma| \leq 1$, $\varsigma \neq 1$, and $0 \leq \omega < 1$. A mapping $v \in \mathscr{A}$ is allegedly in the class. $k - \mathscr{U}\mathscr{S}\mathscr{T}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$, whether the adopting connection is valid:

$$\Re\left\{\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}\right\} \geq k\left|\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}-1\right|+\omega, \qquad \xi \in \Delta.$$

In addition, a mapping $v \in k - \mathscr{U}\mathscr{S}\mathscr{T}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$ belong the subcategory $k - \widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$ if $v \in \mathscr{T}$.

Firstly, we need the adopting lemmas [3].

Lemma 1.1 *Suppose the number a which is complex as well as the number β which is real, gives the result as*

$$\Re(a) \geq \beta \Leftrightarrow \quad |a-(1+\beta)| \leq |a+(1-\beta)|.$$

Lemma 1.2 *Suppose the number a which is complex as well as the numbers β, ω which are real, gives the result as*

$$\Re\ (a) > \beta|a-1|+\omega \quad \Leftrightarrow \quad \Re\{a(1+\beta e^{i\varrho})-\beta e^{i\varrho}\} > \omega, \quad -\pi < \varrho \leq \pi.$$

2. Bounds for the coefficients

Theorem 2.1 *Suppose $v \in \mathscr{T}$. Then $v \in k - \widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$ iff*

$$\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(k+1)-u_t(k+\omega)\right|a_t \leq 1-\omega, \tag{2.1}$$

for which $u_t = 1 + \varsigma + \cdots + \varsigma^{t-1}$. The estimate is sharp with

$$v(\xi) = \xi - \frac{1-\omega}{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(k+1)-v_t(k+\omega)\right|}\xi^t.$$

Proof: we get from the Definition (1.1) as

$$\Re\left\{\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}\right\} \geq k\left|\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}-1\right|+\omega.$$

Next, by Lemma 1.2, we have

$$\Re\left\{\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma z)}(1+ke^{i\varrho})-ke^{i\varrho}\right\} \geq \omega, \qquad -\pi < \varrho \leq \pi$$

which implies that

$$\Re\left\{\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)} - \frac{ke^{i\varrho}\left[\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right]}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}\right\} \geq \omega. \tag{2.2}$$

Now, suppose that

$$L(\xi) = (1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho}) - ke^{i\varrho}\left[\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right]$$

and

$$M(\xi)=\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi).$$

By virtue of Lemma 1.1, (2.2) acquires

$$|L(\xi)+(1-\omega)M(\xi)|\geq|L(\xi)-(1+\omega)M(\xi)|,\qquad 0\leq\omega<1.$$

After that, we acquire

$$\begin{aligned}|L(\xi)+(1-\omega)M(\xi)| &=\Bigg|(1-\varsigma)\Bigg\{(2-\omega)\xi-\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}(t+v_t(1-\omega))a_t\xi^t\\ &\quad -ke^{i\varrho}\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}(t-v_t)a_t\xi^t\Bigg\}\Bigg|\\ &\geq|1-\varsigma|\Bigg\{(2-\omega)|\xi|-\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t+v_t(1-\omega)|a_t|\xi|^t\\ &\quad -k\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t-v_t|a_t|\xi|^t\Bigg\}.\end{aligned}$$

However, we acquire

$$\begin{aligned}|L(\xi)+(1+\omega)M(\xi)| &=\Bigg|(1-\varsigma)\Bigg\{-\omega\xi-\sum_{t\geq2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}(t+v_t(1-\omega))a_t\xi^t\\ &\quad -ke^{i\varrho}\sum_{t\geq2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}(t-v_t)a_t\xi^t\Bigg\}\Bigg|\\ &\geq|1-\varsigma|\Bigg\{\omega|\xi|-\sum_{t\geq2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t+v_t(1-\omega)|a_t\\ &\quad -k\sum_{t\geq2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t-v_t|a_t|\xi|^t\Bigg\}.\end{aligned}$$

Consequently, we discovered as

$$\begin{aligned}&|L(\xi)+(1-\omega)M(\xi)|-|L(\xi)+(1+\omega)M(\xi)|\\ &\quad\geq|1-\varsigma|\Bigg\{2(1-\omega)|\xi|-\sum_{t\geq2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\\ &\qquad\Big[|t+v_t(1-\omega)|+|t-v_t(1+\omega)|+2k|\,|t-v_t|a_t|\xi|^t\Big]\Bigg\}\\ &\quad\geq2(1-\omega)|\xi|-\sum_{t\geq2}2[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t(k+1)-v_t(k+\omega)|a_t|\xi|^t\geq0.\end{aligned}$$

Or

$$\sum_{t=2}^{\infty}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t(k+1)-v_t(k+\omega)|a_t\leq1-\omega.$$

Conversely, suppose (2.1) holds. Next, we have to state that

$$\Re\left\{\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})-ke^{i\varrho}\left[\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right]}{\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)}\right\}\geq\omega.$$

Taking the values of ξ $(0\le|\xi|=r<1)$ on the positive $x-$ axis, then

$$\Re\left\{\frac{(1-\omega)-\sum\limits_{t\ge2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}[t(1+ke^{i\varrho})-v_t(\omega+ke^{i\varrho})]a_t\xi^{t-1}}{1-\sum\limits_{t\ge2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}v_ta_t\xi^{t-1}}\right\}\ge0.$$

Since $\Re(-e^{i\varrho})\ge-|e^{i\varrho}|=-1$, then

$$\Re\left\{\frac{(1-\omega)-\sum\limits_{t\ge2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}[t(1+k)-v_t(\omega+k]a_tr^{t-1}}{1-\sum\limits_{t\ge2}[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}v_ta_tr^{t-1}}\right\}\ge0.$$

Now obtain the desired outcome, if we take $r\to1^-$,. □

Corollary 2.1 *If $v\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$, then*

$$a_t\le\frac{1-\omega}{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(k+1)-v_t(k+\omega)\right|},$$

where $v_t=1+\varsigma+\cdots+\varsigma^{t-1}$.

3. Neighbourhood properties

Motivated by Goodman [4], Ruscheweyh [12] and Santosh [11], the notion of neighborhoods of analytic functions is introduced in this section. The neighbourhood of the mapping $u\in\mathscr{T}$ is defined as follows:

Definition 3.1 Let $0<q<1$, $\lambda>0$, $k\ge0$, $|\varsigma|\le1$, $\varsigma\ne1$, $0\le\omega<1$, $\upsilon\ge0$ and $u_t=1+\varsigma+\cdots+\varsigma^{\eta-1}$. The $\upsilon-$ neighbourhood defined for the mapping $v\in\mathscr{T}$ and represented as $N_\upsilon(v)$ consisting of all mappings $g(\xi)=\xi-\sum\limits_{t=2}^{\infty}b_t\xi^t\in S$ $(b_t\ge0)$ satisfying

$$\sum_{t\ge2}\frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(k+1)-v_t(k+\omega)\right|}{1-\omega}|a_t-b_t|\le1-\upsilon.$$

Theorem 3.1 *Suppose that $v\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$ and $\Re(\omega)\ne1$. For any complex number ε with $|\varepsilon|<\upsilon$, $(\upsilon\ge0)$, if u fulfills the below requirement:*

$$\frac{f(\xi)+\varepsilon\xi}{1+\varepsilon}\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$$

then $N_\upsilon(v)\subset k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$.

Proof: Evidently $v\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$ if and only if

$$\left|\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})-(ke^{i\varrho}+1+\omega)\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right)}{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})+(1-ke^{i\varrho}-\omega)\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right)}\right|<1,\ -\pi<\varrho\le\pi.$$

For any complex number s $(|s|=1)$, we may write

$$\frac{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})-(ke^{i\varrho}+1+\omega)\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right)}{(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})+(1-ke^{i\varrho}-\omega)\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right)}\ne s.$$

That means,

$$(1-s)(1-\varsigma)\xi\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)\right)'(1+ke^{i\varrho})-(ke^{i\varrho}+1+\omega+s(-1+ke^{i\varrho}+\omega)\times\left(\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\xi)-\mathscr{D}_{q,\lambda}^{\hbar,\wp}v(\varsigma\xi)\right)\neq 0$$

which implies that

$$\xi-\sum_{t=2}^{\infty}\frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left[(t-v_t)(1+ke^{i\varrho}-ske^{i\varrho})-s(t+v_t)-v_t\omega(1-s)\right]}{\omega(s-1)-2s}\xi^t\neq 0.$$

Nevertheless, $v\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)\Leftrightarrow\dfrac{(v*h)}{\xi}\neq 0,\ (\xi\in\Delta-\{0\})$, where

$$h(\xi)=\xi-\sum_{t=2}^{\infty}c_t\xi^t$$

and

$$c_t=\frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left[(t-v_t)(1+ke^{i\varrho}-ske^{i\varrho})-s(t+v_t)-v_t\omega(1-s)\right]}{\omega(s-1)-2s}.$$

Since $\dfrac{v(\xi)+\varepsilon\xi}{1+\varepsilon}\in k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$, noted as

$$|c_t|\leq\frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(1+k)-v_t(k+\omega)\right|}{1-\omega}.$$

Therefore $\xi^{-1}\left(\dfrac{v(\xi)+\varepsilon\xi}{1+\varepsilon}*h(\xi)\right)\neq 0$, gives the similar result to

$$\frac{(v*h)(\xi)}{(1+\varepsilon)\xi}+\frac{\varepsilon}{1+\varepsilon}\neq 0. \tag{3.1}$$

Let us now contemplate that $\left|\dfrac{(v*h)(\xi)}{\xi}\right|<\upsilon$. Using (3.1), we get

$$\left|\frac{(v*h)(\xi)}{(1+\varepsilon)\xi}+\frac{\varepsilon}{1+\varepsilon}\right|\geq\frac{|\varepsilon|}{|1+\varepsilon|}-\frac{1}{|1+\varepsilon|}\left|\frac{(v*h)(\xi)}{\xi}\right|>\frac{|\varepsilon|-\upsilon}{|1+\varepsilon|}\geq 0.$$

Which contradicts to $|\varepsilon|<\upsilon$ after that, we have $\left|\dfrac{(v*h)(\xi)}{\xi}\right|\geq\upsilon$.

Suppose $g(\xi)=\xi-\sum\limits_{t=2}^{\infty}b_t\xi^t\in N_\upsilon(v)$, then

$$\begin{aligned}\upsilon-\left|\frac{(g*h)(\xi)}{\xi}\right| &\leq\left|\frac{((v-g)*h)(\xi)}{\xi}\right|\leq\sum_{t=2}^{\infty}|a_t-b_t|\,|c_t|\,|\xi|^t\\ &<\sum_{t=2}^{\infty}\frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}\left|t(1+k)-v_t(k+\omega)\right|}{1-\omega}|a_t-b_t|\\ &\leq\upsilon.\end{aligned}$$

□

4. Partial sums

In this section, employing a method exploited by Silverman [17], we study the ratio of a mapping $v\in\mathscr{T}$ to its sequence of partial sums $v_m(\xi)=\xi+\sum\limits_{t=2}^{m}a_t\xi^t$.

Theorem 4.1 *If the mapping $v \in \mathscr{T}$ fulfills (2.1) then*

$$\Re\left\{\frac{v(\xi)}{v_m(\xi)}\right\} \geq 1 - \frac{1}{\chi_{m+1}}$$

and

$$\chi_t = \begin{cases} 1, & t = 2, \cdots, m \\ \chi_{m+1}, & t = m+1, m+2, \cdots \end{cases},$$

where

$$\chi_t = \frac{[t]_q^{\wp}[1+(t-1)\lambda]^{\hbar}|t(1+k) - v_t(k+\omega)|}{1-\omega}.$$

The estimate is accurate for each m, with

$$v(\xi) = \xi + \frac{\xi^{m+1}}{\chi_{m+1}}. \tag{4.1}$$

Proof: Assume that as

$$\frac{1+w(\xi)}{1-w(\xi)} = \chi_{m+1}\left\{\frac{v(\xi)}{v_m(\xi)} - \left(1 - \frac{1}{\chi_{m+1}}\right)\right\} = \left\{\frac{1 + \sum_{t=2}^{m} a_t\xi^{t-1} + \chi_{m+1}\sum_{t\geq m+1} a_t\xi^{t-1}}{1 + \sum_{t=2}^{m} a_t\xi^{t-1}}\right\}. \tag{4.2}$$

Then, from (4.2), we have

$$w(\xi) = \frac{\chi_{m+1}\sum_{t\geq m+1} a_t\xi^{t-1}}{2\left(1 + \sum_{t=2}^{m} a_t\xi^{t-1}\right) + \chi_{m+1}\sum_{t\geq m+1} a_t\xi^{t-1}}$$

and

$$|w(\xi)| \leq \frac{\chi_{m+1}\sum_{t\geq m+1} a_t}{2 - 2\sum_{t=2}^{m} a_t - \chi_{m+1}\sum_{t\geq m+1} a_t}.$$

Next, $|w(\xi)| \leq 1$ if

$$2\chi_{m+1}\sum_{t\geq m+1} a_t \leq 2\left(1 - \sum_{t=2}^{m} a_t\right).$$

It suggests that

$$\sum_{t=2}^{m} a_t + \chi_{m+1}\sum_{t\geq m+1} a_t \leq 1. \tag{4.3}$$

It serves as sufficient evidence that the left hand side of (4.3) is bounded above by $\sum_{t=2}^{\infty}\chi_t a_t$, that means,

$$\sum_{t=2}^{m}(\chi_t - 1)a_t + \sum_{t\geq m+1}(\chi_t - \chi_{m+1})a_t \geq 0.$$

For $z = re^{i\pi/t}$, we have to find that the sharp result.

$$\frac{v(\xi)}{v_m(\xi)} = 1 + \frac{\xi^m}{\chi_{m+1}}.$$

By considering $\xi \to 1^-$, we get

$$\frac{v(\xi)}{v_m(\xi)} = 1 - \frac{1}{\chi_{m+1}}.$$

□

We now express bounds for $\dfrac{v_m(\xi)}{v(\xi)}$.

Theorem 4.2 *If the mapping $v \in \mathscr{T}$ fulfills (2.1), then*

$$\Re\left\{\frac{v_m(\xi)}{v(\xi)}\right\} \geq \frac{\chi_{m+1}}{1+\chi_{m+1}}.$$

The estimate is sharp for (4.1).

Proof: It is customary to confirm that

$$\frac{1+w(\xi)}{1-w(\xi)} = (1+\chi_{m+1})\left\{\frac{v_m(\xi)}{v(\xi)} - \frac{\chi_{m+1}}{1+\chi_{m+1}}\right\} = \left\{\frac{1+\sum\limits_{t=2}^{m} a_t\xi^{t-1} - \chi_{m+1}\sum\limits_{t\geq m+1} a_t\xi^{t-1}}{1+\sum\limits_{t\geq 2} a_t\xi^{t-1}}\right\},$$

where

$$w(\xi) = \frac{(1+\chi_{m+1})\sum\limits_{t\geq m+1} a_t\xi^{t-1}}{-2\left(1+\sum\limits_{t=2}^{m} a_t\xi^{t-1}\right) - (1-\chi_{m+1})\sum\limits_{t\geq m+1} a_t\xi^{t-1}}.$$

It follows that

$$|w(\xi)| \leq \frac{(1+\chi_{m+1})\sum\limits_{t\geq m+1} a_t}{2-2\sum\limits_{t=2}^{m} a_t + (1-\chi_{m+1})\sum\limits_{t\geq m+1} a_t} \leq 1$$

and hence

$$\sum_{t=2}^{m} a_t + \chi_{m+1}\sum_{t\geq m+1} a_t \leq 1. \tag{4.4}$$

It is enough to express that LHS of (4.4) is bounded above by $\sum\limits_{t\geq 2} \chi_t a_t$, which is equivalent to

$$\sum_{t=2}^{m}(\chi_t - 1)a_t + \sum_{t\geq m+1}(\chi_t - \chi_{m+1})a_t \geq 0.$$

□

Theorem 4.3 *If the function v of the form (1.1) fulfills (2.1), then*

$$\Re\left\{\frac{v'(\xi)}{v'_m(\xi)}\right\} \geq 1 - \frac{m+1}{\chi_{m+1}} \tag{4.5}$$

and

$$\Re\left\{\frac{v'_m(\xi)}{v'(\xi)}\right\} \geq \frac{\chi_{m+1}}{1+m+\chi_{m+1}},$$

where

$$\chi_t \geq \begin{cases} 1, & t = 1, 2, \cdots, m \\ t\dfrac{\chi_{m+1}}{m+1}, & t = m+1, m+2, \cdots \end{cases}.$$

These estimates are sharp (4.1).

Proof: For v given by (1.1), we may write

$$\frac{1+w(\xi)}{1-w(\xi)} = \chi_{m+1}\left\{\frac{v'(\xi)}{v'_m(\xi)} - \left(1-\frac{m+1}{\chi_{m+1}}\right)\right\}$$
$$= \left\{1+\sum_{t=2}^{m} ta_t\xi^{t-1} + \frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1} ta_t\xi^{t-1} 1 + \sum_{t=2}^{m} a_t\xi^{t-1}\right\},$$

where

$$w(\xi) = \frac{\frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1} ta_t\xi^{t-1}}{2+2\sum_{t=2}^{m} ta_t\xi^{t-1} + \frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1} ta_t\xi^{t-1}}.$$

Then, we have

$$|w(\xi)| \leq \frac{\frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1} ta_t}{2-2\sum_{t=2}^{m} ta_t + \frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1} ta_t}.$$

From the above inequality, we get

$$|w(\xi)| \leq 1 \Leftrightarrow \sum_{t=2}^{m} ta_t + \frac{\chi_{m+1}}{m+1}\sum_{t\geq m+1}^{t} a_t \leq 1, \tag{4.6}$$

since the LHS of (4.6) is bounded above by $\sum_{t\geq 2}\chi_t a_t$.

By using the same method as before, we also obtain (4.5). □

5. Integral Means Result

Motivated by an integral means work of Silverman [18] many have discussed integral means results for various subclasses of $\mathscr{T}$. In that line inspired by the works of Ahuja et al. [1] and Magesh et al. [9] in the following theorem we find integral mean inequality for the functions in the class $k-\widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega)$.

For analytic mappings u and v in Δ, u is said to be subordinate to v if There exists an analytic mapping w such that

$$w(0) = 0, \qquad |w(\xi)| < 1 \quad \text{and} \quad u(\xi) = v(w(\xi)), \qquad \xi \in \Delta. \tag{5.1}$$

This subordination will be denoted here by

$$u \prec v, \qquad \xi \in \Delta$$

or, conventionally, by

$$u(\xi) \prec v(\xi), \qquad \xi \in \Delta.$$

Specifically, when v is univalent in Δ,

$$u \prec v \qquad (\xi \in \Delta) \;\Leftrightarrow\; u(0) = v(0) \quad \text{and} \quad u(\Delta) \subset v(\Delta).$$

Lemma 5.1 *[8] If the functions u and v are analytic in Δ with $u \prec v$ then*

$$\int_0^{2\pi} \left|u(re^{i\theta})\right|^{\varkappa} d\theta \leq \int_0^{2\pi} \left|v(re^{i\theta})\right|^{\varkappa} d\theta, \quad \varkappa > 0, \quad \xi = re^{i\theta} \quad \text{and} \quad 0 < r < 1. \tag{5.2}$$

We now determine the integral means inequality for the functions in the class.

Theorem 5.1 *If* $v \in k - \widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q,\lambda,\hbar,\wp,\varsigma,\omega),$ *and* v_2 *is defined by*

$$v_2(\xi) = \xi - \frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}\xi^2 \tag{5.3}$$

then for $\xi = re^{i\theta}$ *and* $0<r<1,$ *we have*

$$\int_0^{2\pi} \left|v(re^{i\theta})\right|^{\varkappa} d\theta \le \int_0^{2\pi} \left|v_t(re^{i\theta})\right|^{\varkappa} d\theta, \qquad \varkappa > 0. \tag{5.4}$$

Proof: Let v of the form (1.4) and

$$v_2(\xi) = \xi - \frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}\xi^2,$$

Consequently, we have to demonstrate that

$$\int_0^{2\pi} \left|1-\sum_{t=1}^{\infty} a_t \xi^{t-1}\right|^{\varkappa} d\theta \le \int_0^{2\pi} \left|1-\frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-u_2(k+\omega)|}\xi\right|^{\varkappa} d\theta.$$

By Lemma 5.1, it suffices to show that
$1-\sum_{t=1}^{\infty} a_t\xi^{t-1} \prec 1 - \frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}\xi.$

If we define the function $w(\xi)$ as follows:

$$w(\xi) = \sum_{t=2}^{\infty} \frac{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}{1-\omega} a_t \xi^{t-1}. \tag{5.5}$$

Based on the equation provided above

$$w(0) = 0. \tag{5.6}$$

Again from (5.5), we have

$$|w(\xi)| \le \sum_{t=2}^{\infty} \frac{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-u_2(k+\omega)|}{1-\omega} |a_t||\xi|^{t-1}.$$

Since, $\xi = re^{i\theta}$ and $0<r<1,$ and using (2.1), confidently, we have from the above inequality

$$|w(\xi)| \le \sum_{t=2}^{\infty} \frac{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-u_2(k+\omega)|}{1-\omega} |a_t| \le 1 \tag{5.7}$$

From (5.5), we have

$$1-\sum_{t=2}^{\infty} |a_t|\xi^{t-1} = 1 - \frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}\; w(\xi). \tag{5.8}$$

Since $w(\xi)$ is analytic in Δ, therefore in view of equations (5.1), (5.5), (5.6), and (5.8); inequality (5.7); and the subordination principle,

$$1-\sum_{t=1}^{\infty} a_t\xi^{t-1} \prec 1 - \frac{1-\omega}{[2]_q^{\wp}[1+\lambda]^{\hbar}\,|2(k+1)-v_2(k+\omega)|}\xi.$$

Since, the function on the both sides of the above relation are analytic in Δ, therefore, in view of Lemma 5.1 and equation (5.3), we get assertion (5.4). This ends the theorem proof. 5.1. □

6. Radii of close-to-convexity and Starlikness

Theorem 6.1 *Let $v \in k - \widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$. Then $v(\xi)$ is close-to-convex of order $\rho (0 \le \rho < 1)$ in $|\xi| < r_1$, where*

$$r_1 = \inf_t \left[\frac{(1-\rho)\Theta}{\rho(1-\omega)} \right]^{\frac{1}{t}}, t \ge 2 \tag{6.1}$$

here $\Theta = [t]_q^{\wp} [1 + (t-1)\lambda]^{\hbar} |t(k+1) - u_t(k+\omega)|$ and $v_t = 1 + \varsigma + \cdots + \varsigma^{t-1}$. The result is sharp.

Proof: We must show that

$$|v'(\xi) - 1| \le 1 - \rho, \quad \text{for } |\xi| < r_1.$$

From (1.4), we have

$$|v'(\xi) - 1| \le \sum_{t=2}^{\infty} t a_t \xi^{t-1}.$$

Thus $|v'(\xi) - 1| \le 1 - \rho$, if

$$\sum_{t=2}^{\infty} \left(\frac{t}{1-\rho} \right) a_t \xi^{t-1} \le 1. \tag{6.2}$$

But by Theorem 2.1, (6.2) will be true if $\left(\frac{t}{1-\rho}\right) \xi^{t-1} \le \frac{\Theta}{1-\omega}$

$$\Rightarrow \ |\xi| \le \left(\frac{(1-\rho)\Theta}{t(1-\omega)} \right)^{\frac{1}{t-1}}, \quad t \ge 2. \tag{6.3}$$

□

Theorem 6.2 *If $v \in k - \widetilde{\mathscr{U}\mathscr{S}\mathscr{T}}_s(q, \lambda, \hbar, \wp, \varsigma, \omega)$ then $v(\xi)$ is starshaped of order $\rho (0 \le \rho < 1)$ in $|\xi| < r_2$, where*

$$r_2 = \inf_t \left[\frac{(1-\rho)\Theta}{(t-\rho)(1-\omega)} \right]^{\frac{1}{t}}, t \ge 2 \tag{6.4}$$

here Θ and u_t are defined in Theorem 6.1. The result is sharp.

Proof: It is enough to demonstrate

$$\left| \frac{\xi v'(\xi)}{v(\xi)} - 1 \right| \le 1 - \rho, \quad \text{for } |\xi| < r_2.$$

we have

$$\left| \frac{z v'(\xi)}{v(\xi)} - 1 \right| \le \frac{\sum_{t=2}^{\infty} (t-1) a_t \xi^{t-1}}{1 - \sum_{t=2}^{\infty} a_t \xi^{t-1}}.$$

Thus $\left| \frac{z v'(\xi)}{v(\xi)} - 1 \right| \le 1 - \rho$, if

$$\sum_{t=2}^{\infty} \left(\frac{t-\rho}{1-\rho} \right) a_t \xi^{t-1} \le 1. \tag{6.5}$$

But, by Theorem 2.1, (6.5) will be true if

$$\left(\frac{t-\rho}{1-\rho} \right) \xi^{t-1} \le \frac{\Theta}{1-\omega}$$

that is, if $|\xi| \le \left[\frac{(1-\rho)\Theta}{(t-\rho)(1-\omega)} \right]^{\frac{1}{t}}$. □

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Competing interests

The author declares no competing interests.

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