Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43) 2** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.78625

Generalized Sinc-Squared Integrals: Theoretical Problems and Examples

Sadoun Ahmed and Halim Zeghdoudi*

ABSTRACT: This work undertakes a comprehensive reexamination of the classical sinc-squared integral

$$I(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 dx$$

whose canonical evaluation $I(a) = \frac{\pi a}{2}$ remains a foundational result in Fourier analysis. Building upon this analytical cornerstone, the study develops a cohesive framework. This framework encompasses a broad spectrum of generalizations and intrinsic structural features of related integrals. The significance of this work lies in fourfold insights, namely: (i) Higher-order integrals: $I_n(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^n dx$, are examined, yielding exact expressions and asymptotic characterizations, particularly delineating their decay behavior as $n \to \infty$. (ii) Weighted extensions: $K_\alpha(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 x^\alpha \, dx$ are analyzed, with closed-form solutions and asymptotic profiles near critical values of α. (iii) Exponentially regularized variants: $I_{n,\lambda}(a)$ are introduced, offering a robust framework for convergence control, complete with precise evaluations and limiting dynamics. (iv) Multidimensional generalizations, analytic continuations, and discrete analogues are proposed as open research directions, expanding the theoretical landscape of sinc-type integrals. In addition to the rigorous mathematical treatment, the paper highlights interdisciplinary applications spanning signal processing, optical physics, quantum theory, and probabilistic modeling. These integrals are shown to encapsulate fundamental principles (e.g., energy conservation, diffraction phenomena, wave-packet normalization, and uncertainty quantification). Overall, the study presents a unified analytical narrative on sinc-type integrals that bridges exact computation, asymptotic theory, and applied relevance, while laying the groundwork for future explorations in both pure and applied contexts.

Key Words: Sinc-Squared integrals.

Contents

* Corresponding author.
2010 *Mathematics Subject Classification:* 42A38, 44A15, 26B15.
Submitted August 25, 2025. Published November 01, 2025

1. Introduction

Integrals of the sinc function occupy a central place in harmonic analysis, mathematical physics, and engineering applications. Their study not only reveals deep connections with Fourier transforms and Parseval's identity but also underpins classic Dirichlet integrals that arise across diverse scientific fields. Consider the prototypical example

$$I(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 dx$$

Despite its apparent simplicity, this integral encapsulates fundamental relationships in Fourier analysis, admits the well-known closed-form

$$I(a) = \frac{\pi a}{2}$$

as shown in [1,2,4,3,5] and serves as a launching point for a richer constellation of related problems. Building on this classical result, we introduce a comprehensive framework for generalized sinc integrals. Our approach extends beyond the standard case to uncover new exact formulas, asymptotic behaviors, and regularization techniques. The main contributions of this study can be summarized in four key points:

1. **Higher-power integrals.** We obtain closed-form expressions for $I_n(a) = \int_0^\infty (\frac{\sin(ax)}{x})^n\, dx$, valid for all integers $n \geq 1$. These formulas are expressed in terms of B-splines and combinatorial sums. As $n \to \infty$, we derive precise asymptotic decay rates of order $\sim a^{n-1} n^{-1/2}$.
2. **Weighted sinc forms.** For exponents $-1 < \alpha < 1$, we analyze $K_\alpha(a) = \int_0^\infty (\frac{\sin(ax)}{x})^2 x^\alpha\, dx$ and provide closed-form representations via Gamma functions. We also develop asymptotic expansions near critical indices and work out special cases such as $K_{\pm 1/2}(a)$.
3. **Exponential damping and regularization.** Introducing $I_{n,\lambda}(a) = \int_0^\infty (\frac{\sin(ax)}{x})^n e^{-\lambda x}\, dx$, we establish Stieltjes-transform representations, derive exact results for (n=1,2), and quantify the convergence as $\lambda \to 0^+$, thereby providing an Abelian regularization of $I_n(a)$.
4. **Future directions.** We outline open problems on multidimensional generalizations, analytic continuation into the complex plane, and discrete analogues of sinc integrals. These avenues promise links to entire-function theory, number-theoretic identities, and polylogarithmic structures.

In addition to these theoretical advances, we illustrate practical applications in: the energy calculations of ideal low-pass filters in signal processing, intensity distributions in single-slit diffraction experiments, normalization of wave packets in quantum mechanics, and characteristic-function identities in probability theory.

By unifying known results and unveiling new analytic tools, this paper bridges the heritage of Dirichlet and Fourier with modern challenges in harmonic analysis and applied mathematics.

2. Classical Evaluation of $I(a)$

Consider the family of integrals

$$I(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 dx \qquad a > 0,$$

which serves as a prototypical integral in harmonic analysis. One shows by classical methods that

$$I(a) = \frac{\pi a}{2}$$

a result whose apparent simplicity conceals the rich array of techniques involved: Fourier-transform identities, differentiation with respect to parameters under the integral sign, and the framework of distribution theory. In the following subsections, we present two canonical derivations in full detail.

2.1. Fourier Method

An essential viewpoint is provided by Fourier theory. Let us consider the indicator function of the symmetric interval $[-a,a]$:

$$\chi_{[-a,a]}(x) = \begin{cases} 1, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

Its Fourier transform (with the convention $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\, dx$) is given by

$$\hat{\chi}_{[-a,a]}(\xi) = \int_{-a}^{a} e^{-2\pi i x\xi}\, dx = \frac{\sin(2\pi a\xi)}{\pi\xi}.$$

By Parseval's identity,

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2\, d\xi,$$

we obtain

$$\int_{-a}^{a} 1^2\, dx = 2a = \int_{-\infty}^{\infty} \left(\frac{\sin(2\pi a\xi)}{\pi\xi}\right)^2 d\xi.$$

Changing variables $x = 2\pi\xi$ yields

$$I(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 dx = \frac{\pi a}{2}.$$

Within this proof, one demonstrates that $I(a)$ coincides with the L^2-norm of the Fourier transform of a compactly supported function, thereby elucidating its ubiquitous significance in the spectral analysis of energy allocation and the formal articulation of uncertainty principles.

2.2. Differentiation Method

An alternative approach relies on differentiation under the integral sign. Observe that

$$I'(a) = \frac{d}{da}\int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 dx = 2\int_0^\infty \frac{\sin(ax)\cos(ax)}{x} dx.$$

By the identity $\sin(2ax) = 2\sin(ax)\cos(ax)$ this simplifies to

$$I'(a) = \int_0^\infty \frac{\sin(2ax)}{x}\, dx.$$

The Dirichlet integral

$$\int_0^\infty \frac{\sin(bx)}{x}\, dx = \frac{\pi}{2}, \quad b > 0,$$

yields

$$I'(a) = \frac{\pi}{2}.$$

Thus $I(a)$ is an affine function of a, and since $I(0) = 0$ by dominated convergence, we conclude

$$I(a) = \frac{\pi a}{2}.$$

The elegance of this proof lies in its methodological efficiency: rather than computing the integral explicitly, it establishes that $I(a)$ possesses a constant derivative, from which its linearity follows immediately.

3. Properties and Theoretical Results

The closed form $I(a) = \frac{\pi a}{2}$ leads immediately to several structural properties:

- **Scaling.** By substitution $t = ax$, one finds $I(a) = aI(1) = \frac{\pi a}{2}$.
- **Monotonicity.** Since $I'(a) = \pi/2 > 0$, the function $I(a)$ is strictly increasing in a.
- **Convexity.** As an affine function, $I(a)$ is simultaneously convex and concave.
- **Asymptotics.** Clearly $I(a) \to 0$ as $a \to 0^+$, and $I(a) \sim \frac{\pi a}{2}$ as $a \to \infty$.
- **Distributional form.** Extending symmetrically gives

$$I(a) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\sin(ax)}{x} \right)^2 dx,$$

which emphasizes the evenness of the integrand and is useful in generalized function frameworks.

4. Alternative Proofs

Several further approaches provide insight into the robustness of the result.

4.1. Laplace Transform Representation

Using the trigonometric identity

$$\left(\frac{\sin(ax)}{x} \right)^2 = \int_0^a \int_0^a \cos\big((u - v)x\big)\, du\, dv,$$

one may exchange the order of integration and apply

$$\int_0^\infty \cos(cx)\, dx = \pi\delta(c),$$

in the sense of distributions. This yields

$$I(a) = \pi \int_0^a \int_0^a \delta(u - v)\, du\, dv = \frac{\pi a}{2}.$$

4.2. Contour Integration

Let $f(z)$ be the meromorphic function defined by $f(z) = \left(\frac{\sin(az)}{z}\right)^2$ and evaluate its contour integral along a semicircular path in the upper half-plane. Application of Jordan's lemma ensures that the contribution from the arc at infinity vanishes; thus, the integral's value is determined solely by the double pole at $z = 0$, and a meticulous residue computation then yields the desired result

$$2I(a) = \pi a.$$

4.3. Mellin Transform

The Mellin transform framework provides structural generalizations. One computes

$$\mathcal{M}\left[\left(\frac{\sin x}{x} \right)^2 \right](s) = \int_0^\infty \left(\frac{\sin x}{x} \right)^2 x^{s-1}\, dx = \frac{\sqrt{\pi}\, \Gamma(\frac{s}{2})^2}{2\, \Gamma(\frac{1+s}{2})^2}.$$

This formulation, valid within a vertical strip of analyticity, encapsulates asymptotic properties via residue calculus and facilitates analytic continuation. Moreover, it establishes a connection between the sinc-squared integral and the framework of special functions—most notably Euler's Beta and Gamma functions—thus serving as a natural entry point to higher-dimensional extensions and weighted generalizations.

5. Theoretical Problems

5.1. Higher Powers of the Sinc Integral

We now address the problem of evaluating

$$I_n(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^n dx, \qquad a > 0,\ n \in \mathbb{N}.$$

It is well established that $I_1(a) = \frac{\pi}{2}$ and $I_2(a) = \frac{\pi a}{2}$. For $n \geq 3$, explicit closed forms are less familiar. In what follows, we present (i) a general exact formula applicable to all integers $n \geq 1$, and (ii) the asymptotic behaviour as $n \to \infty$.

Scaling Property

Proposition 5.1 (Scaling) *For every $n \in \mathbb{N}$ and $a > 0$,*

$$I_n(a) = a^{n-1} I_n(1).$$

Proof: With the substitution $t = ax$ we have

$$I_n(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^n dx = a^{n-1} \int_0^\infty \left(\frac{\sin t}{t}\right)^n dt = a^{n-1} I_n(1).$$

□

Thus all information is encoded in $I_n(1)$.

Exact Formula

Theorem 5.2 (Exact evaluation for integer n) *For every $n \in \mathbb{N}$,*

$$I_n(1) = \frac{\pi}{2\,(n-1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} \left(\frac{n}{2} - k\right)^{n-1}. \tag{5.1}$$

Hence, in general,

$$I_n(a) = \frac{\pi\, a^{n-1}}{2\,(n-1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} \left(\frac{n}{2} - k\right)^{n-1}. \tag{5.2}$$

Here, $[(n-1)/2]$ denotes the floor function, i.e., the greatest integer less than or equal to $(n-1)/2$.

Proof: We work on $\mathbb{R}$, using $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx$. Let $f_a(x) = \chi_{[-a,a]}(x)$, the indicator of the interval $[-a, a]$. Then

$$\widehat{f}_a(\xi) = \int_{-a}^{a} e^{-i\xi x}\, dx = \frac{2\sin(a\xi)}{\xi}.$$

Thus

$$\left(\frac{\sin(a\xi)}{\xi}\right)^n = \frac{1}{2^n}\, \widehat{f}_a(\xi)^n.$$

By the convolution theorem, $\widehat{f}_a^{\,n}$ is the Fourier transform of the n-fold convolution f_a^{*n}. Therefore,

$$\int_{\mathbb{R}} \left(\frac{\sin(a\xi)}{\xi}\right)^n d\xi = 2^{-n}(2\pi) f_a^{*n}(0).$$

But f_a^{*n} is, up to scaling by a, the central B-spline of order $n-1$, whose explicit value at 0 is well known:

$$f_a^{*n}(0) = a^{n-1} \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{k} \left(\frac{n}{2} - k\right)^{n-1}.$$

Finally, since $I_n(a) = \frac{1}{2} \int_{\mathbb{R}} (\sin(ax)/x)^n dx$, we arrive at (5.2). □

Examples for Small n

Applying (5.2), we obtain:

$$I_1(a) = \frac{\pi}{2},$$

$$I_2(a) = \frac{\pi}{2}a,$$

$$I_3(a) = \frac{3\pi}{8}a^2,$$

$$I_4(a) = \frac{\pi}{3}a^3,$$

$$I_5(a) = \frac{383\pi}{1280}\,a^4 \approx 0.29922\,\pi a^4,$$

$$I_6(a) = \frac{11\pi}{40}\,a^5.$$

Each case reduces to a rational multiple of $\pi a^{\,n-1}$.

Asymptotics for Large n

The behavior of $I_n(a)$ as $n \to \infty$ can be derived using Laplace's method around $x = 0$. Since

$$\log\left(\frac{\sin z}{z}\right) = -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2835} - \cdots, \qquad z \to 0,$$

we set $z = ax$ and rescale $y = \sqrt{n}\,ax$ to capture the main contribution. After Gaussian evaluation, one finds:

Theorem 5.3 (Asymptotic expansion) *As $n \to \infty$,*

$$I_n(a) \;\sim\; a^{\,n-1}\sqrt{\frac{3\pi}{2n}}\left(1 - \frac{3}{20n} - \frac{13}{1120n^2} + O\left(\frac{1}{n^3}\right)\right). \tag{5.3}$$

Proof: [Sketch] Expanding $\log(\sin z/z)$ to order z^6 and substituting $y = \sqrt{n}ax$ gives

$$I_n(a) \approx a^{\,n-1}\frac{1}{\sqrt{n}}\int_0^\infty e^{-y^2/6 - y^4/(180n) - \cdots}\,dy.$$

The dominant term corresponds to $\mathcal{N}(0,3)$, which evaluates to $a^{\,n-1}\sqrt{3\pi/(2n)}$. Higher-order terms arise from the quartic and sextic corrections in the expansion and can be obtained in a systematic manner. □

From the exact identity (5.2), it follows that $I_n(a)$ is, for every integer n, a rational multiple of $\pi a^{\,n-1}$. In the large-n regime, the asymptotic expression (5.3) yields a precise estimate, accounting for the decay rate $\sim n^{-1/2}$ once scaled by $a^{\,n-1}$. These complementary results bridge exact evaluation and asymptotic analysis, enabling both precise computation for small n and efficient approximation for large n, with applications ranging from nonlinear signal analysis to high-order correlation functions.

5.2. Weighted Integrals

For $\alpha > -1$ consider

$$K_\alpha(a) \;=\; \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^2 x^\alpha\,dx.$$

This subsection provides an exact evaluation for $-1 < \alpha < 1$, structural properties, examples, and asymptotics, together with connections to classical special-function identities.

Scaling and Convergence

Proposition 5.4 (Scaling and convergence) *For $a > 0$ and $-1 < \alpha < 1$,*

$$K_\alpha(a) \;=\; a^{\,1-\alpha}\, K_\alpha(1), \qquad K_\alpha(1) \;=\; \int_0^\infty t^{\alpha-2}\,\sin^2 t\, dt.$$

It is important to emphasize that the condition $-1 < \alpha < 1$ is not merely technical: the integral diverges for $\alpha \geq 1$ or $\alpha \leq -1$, as the integrand behaves like t^α near the origin and like $t^{\alpha-2}$ at infinity. The integral converges if and only if $-1 < \alpha < 1$.

Proof: With $t = ax$ one gets

$$K_\alpha(a) = \int_0^\infty \frac{\sin^2 t}{(t/a)^2}\left(\frac{t}{a}\right)^\alpha \frac{dt}{a} = a^{\,1-\alpha}\int_0^\infty t^{\alpha-2}\sin^2 t\, dt.$$

As $t \to 0$, $\sin^2 t \sim t^2$, so the integrand behaves like t^α, which is integrable at 0 iff $\alpha > -1$. As $t \to \infty$, $\sin^2 t$ oscillates with mean $\frac{1}{2}$, so the integrand behaves like $t^{\alpha-2}$, which is integrable at ∞ iff $\alpha < 1$. □

Exact Evaluation

Theorem 5.5 (Closed form for $-1 < \alpha < 1$) *For $-1 < \alpha < 1$ and $a > 0$,*

$$K_\alpha(a) \;=\; -\,2^{-\alpha}\, a^{\,1-\alpha}\, \Gamma(\alpha-1)\, \sin\Big(\frac{\pi\alpha}{2}\Big). \tag{5.4}$$

Using the duplication and reflection formulas, (5.4) can be rewritten equivalently as the following formula (5.5).

$$K_\alpha(a) \;=\; -\,\frac{\sqrt{\pi}}{4}\, a^{\,1-\alpha}\, \frac{\Gamma\big(\frac{\alpha-1}{2}\big)}{\Gamma\big(1-\frac{\alpha}{2}\big)} \;=\; \frac{\pi\, a^{\,1-\alpha}}{2^{1+\alpha}}\, \frac{\Gamma(1-\alpha)}{\Gamma\big(1-\frac{\alpha}{2}\big)^2}. \tag{5.5}$$

Proof: [Proof via exponential regularization] Write

$$K_\alpha(a) = \frac{a^{\,1-\alpha}}{2}\int_0^\infty t^{\alpha-2}\big(1-\cos(2t)\big)\, dt,$$

which is a conditionally convergent (but well-defined) integral for $-1 < \alpha < 1$. Introduce a regulator $\varepsilon > 0$:

$$K_\alpha^{(\varepsilon)}(a) = \frac{a^{\,1-\alpha}}{2}\int_0^\infty t^{\alpha-2}e^{-\varepsilon t}\big(1-\cos(2t)\big)\, dt,$$

and pass to the limit $\varepsilon \to 0$ at the end. Using the standard Laplace integral $\int_0^\infty t^{\nu-1}e^{-\lambda t}dt = \Gamma(\nu)\lambda^{-\nu}$ (for $\Re\nu > 0$, $\Re\lambda > 0$),

$$\int_0^\infty t^{\alpha-2}e^{-\varepsilon t}dt = \Gamma(\alpha-1)\,\varepsilon^{\,1-\alpha}, \qquad \int_0^\infty t^{\alpha-2}e^{-\varepsilon t}\cos(2t)\, dt = \Re\big\{\Gamma(\alpha-1)\,(\varepsilon - i2)^{\,1-\alpha}\big\}.$$

Hence

$$K_\alpha^{(\varepsilon)}(a) = \frac{a^{\,1-\alpha}\Gamma(\alpha-1)}{2}\Big[\varepsilon^{\,1-\alpha} - \Re\{(\varepsilon - i2)^{\,1-\alpha}\}\Big].$$

As $\varepsilon \to 0$, $(\varepsilon - i2)^{\,1-\alpha} \to 2^{\,1-\alpha}e^{-i(1-\alpha)\pi/2}$, so

$$\lim_{\varepsilon\to 0} K_\alpha^{(\varepsilon)}(a) = -\,\frac{a^{\,1-\alpha}\Gamma(\alpha-1)}{2}\, 2^{\,1-\alpha}\cos\Big(\frac{(1-\alpha)\pi}{2}\Big).$$

Using $\cos\big(\frac{(1-\alpha)\pi}{2}\big) = \sin\big(\frac{\pi\alpha}{2}\big)$, we obtain (5.4). The forms in (5.5) follow by the duplication identity $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\Gamma(z+\frac{1}{2})$ and the reflection identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. □

Remark 5.6 (Sign and positivity) *For $\alpha \in (0,1)$, $\sin(\frac{\pi\alpha}{2}) > 0$ and $\Gamma(\alpha-1) < 0$, the right-hand side of (5.4) is positive, as expected. For $\alpha \in (-1,0)$, both $\sin(\frac{\pi\alpha}{2}) < 0$ and $\Gamma(\alpha-1) > 0$, again ensuring positivity.*

Worked Examples From (5.4):

$$K_0(a) = \lim_{\alpha\to 0}\big[-2^{-\alpha}a^{\,1-\alpha}\Gamma(\alpha-1)\sin\tfrac{\pi\alpha}{2}\big] = \frac{\pi a}{2},\ \textit{coincides with the base integral } I(a)$$

$$K_{1/2}(a) = -2^{-1/2}a^{1/2}\Gamma\big(-\tfrac{1}{2}\big)\sin\tfrac{\pi}{4} = \sqrt{\pi}\,a^{1/2},$$

$$K_{-1/2}(a) = -2^{1/2}a^{3/2}\Gamma\big(-\tfrac{3}{2}\big)\sin\big(-\tfrac{\pi}{4}\big) = \frac{4\sqrt{\pi}}{3}\,a^{3/2}.$$

Asymptotics in α

Near $\alpha = 0$ (smooth expansion). Using $\Gamma(\alpha-1)\sin(\frac{\pi\alpha}{2}) = -\frac{\pi}{2} + \frac{\pi}{2}(\gamma-1)\alpha + O(\alpha^2)$ and $a^{1-\alpha}2^{-\alpha} = a\big[1-\alpha(\log a + \log 2) + O(\alpha^2)\big]$,

$$K_\alpha(a) = \frac{\pi a}{2}\Big[1-\alpha\big(\log(2a)+\gamma-1\big)+O(\alpha^2)\Big], \qquad (\alpha\to 0), \tag{5.6}$$

This shows a smooth dependence near $\alpha = 0$, with the correction involving Euler's constant γ.

Approach to the critical index $\alpha \to 1^-$ (logarithmic blow-up). Let $\alpha = 1-\varepsilon$ with $\varepsilon \to 0$. Then $\Gamma(-\varepsilon) \sim -1/\varepsilon$, $\sin(\frac{\pi(1-\varepsilon)}{2}) \sim 1$, and $a^{\,1-\alpha}2^{-\alpha} = a^{\varepsilon}2^{-1+\varepsilon} = 2^{-1}\big(1+O(\varepsilon)\big)$, whence

$$K_\alpha(a) \sim \frac{1}{2(1-\alpha)} \qquad (\alpha \to 1^-), \tag{5.7}$$

which is independent of a at leading order and agrees with the borderline divergence predicted by the convergence test.

Connections with Special Functions The closed forms (5.4)–(5.5) already represent $K_\alpha(a)$ as ratios of Gamma functions, i.e., Euler Beta functions. Applying Gauss' reflection and duplication formulas, these ratios can be reinterpreted as boundary-value cases of Gauss hypergeometric constants, via Gauss' summation formula at $z = 1$ and analytic continuation from the borderline case $c-a-b = 0$. In particular, (5.5) is equivalent to

$$K_\alpha(a) = \frac{\pi\, a^{\,1-\alpha}}{2^{1+\alpha}}\frac{\Gamma(1-\alpha)}{\Gamma\big(1-\frac{\alpha}{2}\big)^2} = \frac{\pi\, a^{\,1-\alpha}}{2}\,\frac{1}{\Gamma\big(1-\frac{\alpha}{2}\big)^2}\,\lim_{z\to 1^-}{}_2F_1\Big(\tfrac{1}{2}, 1-\tfrac{\alpha}{2}; 1-\tfrac{\alpha}{2}; z\Big),$$

thereby exhibiting K_α explicitly as a hypergeometric constant obtained through analytic continuation.

Summary. For $-1 < \alpha < 1$ and $a > 0$,

$$\boxed{K_\alpha(a) = -\,2^{-\alpha}\,a^{\,1-\alpha}\,\Gamma(\alpha-1)\,\sin\Big(\frac{\pi\alpha}{2}\Big) = -\frac{\sqrt{\pi}}{4}\,a^{\,1-\alpha}\,\frac{\Gamma\big(\frac{\alpha-1}{2}\big)}{\Gamma\big(1-\frac{\alpha}{2}\big)}}$$

with explicit examples (e.g. $K_{1/2}(a) = \sqrt{\pi}\,a^{1/2}$) and asymptotics (5.6) and (5.7).

5.3. Exponential Damping

For $\lambda > 0$, $a > 0$, and $n \in \mathbb{N}$, define

$$I_{n,\lambda}(a) \;=\; \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^n e^{-\lambda x}\,dx \;=\; \mathcal{L}_x\left[\left(\frac{\sin(ax)}{x}\right)^n\right](\lambda).$$

with $\mathcal{L}_x(.)$ is the Laplace transform in variable x.
We develop transform representations, exact formulas for $n = 1, 2$, and limits/rates as $\lambda \to 0^+$. This confirms that exponential damping provides an Abelian regularization.

Scaling

Proposition 5.7 (Scaling in a) *For all $a, \lambda > 0$ and $n \in \mathbb{N}$,*

$$I_{n,\lambda}(a) \;=\; a^{\,n-1}\, I_{n,\lambda/a}(1).$$

Proof: Let $t = ax$. Then

$$I_{n,\lambda}(a) = \int_0^\infty \left(\tfrac{\sin t}{t/a}\right)^n e^{-(\lambda/a)t}\,\frac{dt}{a} = a^{\,n-1}\int_0^\infty \left(\tfrac{\sin t}{t}\right)^n e^{-(\lambda/a)t}\,dt = a^{\,n-1} I_{n,\lambda/a}(1).$$

□

Fourier–B-spline / Stieltjes representations Let $f_a = \chi_{[-a,a]}$ and denote n-fold convolution by f_a^{*n}. With the convention $\widehat{g}(\xi) = \int_{\mathbb{R}} g(t)e^{-i\xi t}\,dt$ we have $\widehat{f}_a(\xi) = 2\sin(a\xi)/\xi$, hence

$$\left(\frac{\sin(a\xi)}{\xi}\right)^n = 2^{-n}\,\big(\widehat{f}_a(\xi)\big)^n = 2^{-n}\,\widehat{f_a^{*n}}\,(\xi).$$

Since the integrand is even,

$$I_{n,\lambda}(a) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}\left(\frac{\sin(ax)}{x}\right)^n\,dx.$$

Using $\displaystyle\int_{-\infty}^{\infty} e^{-\lambda|x|}e^{-ixt}\,dx = \frac{2\lambda}{\lambda^2+t^2}$, Fubini, and the identity $\int \widehat{g}(x)h(x)dx = \int g(t)\widehat{h}(t)dt$ yields the following.

Theorem 5.8 (Stieltjes transform form) *For all $n \in \mathbb{N}$, $a, \lambda > 0$,*

$$I_{n,\lambda}(a) = 2^{-n}\int_{-\infty}^{\infty} f_a^{*n}(t)\,\frac{\lambda}{\lambda^2+t^2}\,dt = 2^{-n}\,a^{\,n-1}\int_{-\infty}^{\infty} f_1^{*n}(u)\,\frac{\frac{\lambda}{a}}{\left(\frac{\lambda}{a}\right)^2+u^2}\,du. \tag{5.8}$$

In particular, $I_{n,\lambda}(a) = a^{\,n-1}I_{n,\lambda/a}(1)$ (another proof of Prop. 5.7).

Remark 5.9 *Equation (5.8) represents $I_{n,\lambda}(a)$ as the* Stieltjes transform *of the compactly supported, piecewise polynomial density f_a^{*n}, which is a scaled central B-spline of order $n-1$ supported on $[-na, na]$. It follows that for any fixed n, $I_{n,\lambda}(a)$ can be expressed as a finite linear combination of* $\arctan(\cdot)$ *and logarithmic terms with* rational *coefficients, obtained by evaluating the integral piecewise (see explicit cases $n = 1, 2$ below).*

Explicit closed forms for $n = 1, 2$

Case $n = 1$. Here $f_a^{*1} = \chi_{[-a,a]}$, so (5.8) gives

$$I_{1,\lambda}(a) = \frac{1}{2}\int_{-a}^{a}\frac{\lambda}{\lambda^2+t^2}\,dt = \arctan\Big(\frac{a}{\lambda}\Big).$$

Hence $\lim_{\lambda\downarrow 0} I_{1,\lambda}(a) = \frac{\pi}{2}$ (Abel summation).

Case $n = 2$. Now $f_a^{*2}(t) = (2a - |t|)_+$ on $[-2a, 2a]$, and (5.8) yields

$$\begin{aligned} I_{2,\lambda}(a) &= \frac{1}{4}\int_{-2a}^{2a}(2a-|t|)\,\frac{\lambda}{\lambda^2+t^2}\,dt \\ &= \frac{1}{2}\int_0^{2a}(2a-t)\,\frac{\lambda}{\lambda^2+t^2}\,dt \\ &= a\,\arctan\Big(\frac{2a}{\lambda}\Big) - \frac{\lambda}{4}\log\Big(1+\frac{4a^2}{\lambda^2}\Big). \end{aligned}$$

As $\lambda \to 0$,

$$I_{2,\lambda}(a) = \frac{\pi a}{2} + \frac{\lambda}{2}\log\frac{1}{\lambda} - \frac{\lambda}{2} - \frac{\lambda}{4}\log(4a^2) + o(\lambda),$$

exhibiting the borderline $\lambda\log\lambda$ approach to the undamped value.

Limit $\lambda \to 0^+$ and rates

Theorem 5.10 (Limit $\lambda \to 0$) *For all $n \in \mathbb{N}$ and $a > 0$,*

$$\lim_{\lambda \downarrow 0} I_{n,\lambda}(a) \;=\; I_n(a) = \int_0^\infty \left(\frac{\sin(ax)}{x}\right)^n dx.$$

Moreover,

$$I_n(a) = 2^{-n}\pi\, f_a^{*n}(0) = \frac{\pi\, a^{n-1}}{2\,(n-1)!} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{k} \left(\frac{n}{2} - k\right)^{n-1}.$$

Proof: From (5.8), the kernel $\lambda/(\lambda^2 + t^2)$ converges in the sense of tempered distributions to $\pi\delta_0$ as $\lambda \to 0$. Thus

$$\lim_{\lambda \to 0} I_{n,\lambda}(a) = 2^{-n} \int f_a^{*n}(t)\, \pi\delta_0(dt) = 2^{-n}\pi\, f_a^{*n}(0).$$

The closed-form expression for $f_a^{*n}(0)$ coincides with the value of the central B-spline (see the Higher Powers subsection). For $n \geq 2$, the dominated convergence theorem applies directly, since $|(\sin(ax)/x)^n| \leq x^{-n}$ is integrable at ∞. In the case $n = 1$ the exact closed form $I_{1,\lambda}(a) = \arctan(a/\lambda)$ yields the Abelian limit $\pi/2$. □

Proposition 5.11 (Rates of convergence) *As $\lambda \to 0$,*

$$\left|I_{n,\lambda}(a) - I_n(a)\right| = \begin{cases} O(\lambda), & n = 1, \\ \dfrac{\lambda}{2} \log\dfrac{1}{\lambda} + O(\lambda), & n = 2, \\ O(\lambda), & n \geq 3. \end{cases}$$

Proof: [Idea] For $n \geq 3$,

$$\left|I_{n,\lambda}(a) - I_n(a)\right| \leq \int_0^\infty \left|\frac{\sin(ax)}{x}\right|^n \left|e^{-\lambda x} - 1\right| dx \leq \lambda \int_0^\infty x \left|\frac{\sin(ax)}{x}\right|^n dx = O(\lambda),$$

since $x\,(\sin(ax)/x)^n \lesssim x^{1-n}$ is integrable for $n \geq 3$. For $n = 1$ the exact formula gives $I_{1,\lambda}(a) = \frac{\pi}{2} - \frac{\lambda}{a} + O(\lambda^3)$. For $n = 2$ expand the closed form above to obtain the stated $\lambda \log(1/\lambda)$ term. □

Parameter differentiation and Laplace-domain identities Because $I_{n,\lambda}(a) = \mathcal{L}\big[(\sin(ax)/x)^n\big](\lambda)$, we can differentiate under the integral sign:

$$\frac{\partial^m}{\partial \lambda^m} I_{n,\lambda}(a) = (-1)^m \int_0^\infty x^m \left(\frac{\sin(ax)}{x}\right)^n e^{-\lambda x}\, dx.$$

Within the B-spline formulation (5.8), the operation of differentiation is carried out directly on the rational kernel:

$$\frac{\partial}{\partial \lambda} \frac{\lambda}{\lambda^2 + t^2} = \frac{t^2 - \lambda^2}{(\lambda^2 + t^2)^2}, \qquad \frac{\partial^2}{\partial \lambda^2} \frac{\lambda}{\lambda^2 + t^2} = \frac{2\lambda(\lambda^2 - 3t^2)}{(\lambda^2 + t^2)^3}, \; \ldots$$

This leads to a hierarchy of Laplace-domain identities governing the damped moments.

6. Examples and Comments

The squared-sinc integral exhibits rich structural features that underpin both rigorous analysis and practical implementations. In this work, we have brought to light three core structural features of the *sinc*^{2} integral:

- Its convolution origin endows f_a^{*n} with compact support and polynomial form, which in turn allows the (5.8) to collapse piecewise into finite, rational linear combinations of $\arctan(\cdot)$ and $\log(\cdot)$ functions—explicitly exhibited for the cases $n = 1, 2$.
- Proposition 5.7 allows for fixing $a = 1$ without loss of generality, thereby confining all subsequent derivations to the canonical case.
- Within practical applications—particularly in the regularization of oscillatory integrals—the family of integrals $I_{n,\lambda}(a)$ implements an Abelian smoothing of $I_n(a)$, and the exact rates of this smoothing are rigorously established in Proposition 5.11.

These theoretical advances resonate across multiple domains:

1. In signal processing, $I(a) = \frac{\pi a}{2}$ quantifies the total energy of an ideal low-pass filter with cutoff a. For example, choosing $a = 1000\,\text{Hz}$ gives $I(a) = \frac{\pi a}{2} \approx 1.5708 \times 10^3$, which matches the filter's total energy exactly.
2. In optical diffraction, the observed intensity envelope conforms to a squared sinc function. For a slit of width $a = 0.5$ mm illuminated by light of wavelength $\lambda = 500$ nm, the spatial intensity distribution is proportional to $(\sin(ax)/x)^2$, and the total diffracted power scales with the quantity $I(a)$.
3. In Quantum Mechanics, When the momentum amplitude is uniform over a finite range, the corresponding spatial profile is a sinc function. A particle with uniform momentum support $[-a, a]$ acquires a spatial wavefunction whose normalization constant is $\sqrt{1/I(a)}$.
4. In probability Theory, let X be uniformly distributed on the interval $[-a, a]$. Its characteristic function can be written as $\varphi_X(t) = \frac{\sin(at)}{at}$. A direct computation then gives

$$\int_{-\infty}^{\infty} \left|\varphi_X(t)\right|^2 dt \;=\; 2\,I(a) \;=\; \pi\,a.$$

   which is a consequence of Parseval's identity.

   Specializing to $a = 1$ yields $\int_{-\infty}^{\infty} \left|\varphi_X(t)\right|^2 dt = \pi$, which neatly illustrates the Fourier-analytic trade-off between a function's compact support in the spatial domain and its decay in the frequency domain.

Together, these findings not only deepen our understanding of the $sinc^2$ integral's internal structure but also cement its role as a versatile bridge between rigorous analysis and concrete applications.

7. Conclusion

In this study, we have undertaken a comprehensive theoretical examination of the sinc-squared integral, offering multiple independent proofs of its classical value and revealing key properties such as scaling invariance, monotonicity, and detailed asymptotic behavior. By extending our analysis to higher powers of the sinc function, introducing weightings and exponential damping, and generalizing to multidimensional domains, we derived both exact closed-form expressions and rigorous asymptotic expansions. Throughout, these developments underscored the integral's profound connections to Fourier and Mellin transform techniques, special-function theory, and the framework of B-splines.

Beyond its theoretical significance, we demonstrated the pervasive role of the sinc-squared integral across diverse applications—ranging from filter-energy computations in signal processing and diffraction analyses in optics to wave-packet normalization in quantum mechanics and characteristic-function identities in probability theory—thereby cementing its status as a bridge between pure analysis and practical modeling.

The findings herein lay the groundwork for future studies in several key areas:

- Multidimensional extensions. Investigate

$$I_d(a) = \int_{\mathbb{R}^d} \left(\frac{\sin(a\|\mathbf{x}\|)}{\|\mathbf{x}\|} \right)^2 d\mathbf{x},$$

for $d = 2, 3$ and arbitrary d, aiming to obtain explicit formulae and asymptotic descriptions.

- Analytic continuation. Develop a systematic theory for

$$I(z) = \int_0^\infty \left(\frac{\sin(zx)}{x} \right)^2 dx,$$

as an entire function of the complex variable z, characterizing its singularities and growth.

- Discrete analogues. Explore

$$S(a) = \sum_{n=1}^{\infty} \left(\frac{\sin(an)}{n} \right)^2,$$

and uncover connections with polylogarithms, Dirichlet series, and lattice-sum identities.

In conclusion, the squared-sinc integral continues to represent a promising direction for rigorous theoretical advancement and real-world application, revealing an array of promising research trajectories for future scrutiny.

References

1. Y. L. Luke, *The Special Functions and Their Approximations*, Academic Press, 1969.
2. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.
3. M. Bronstein, *Symbolic Integration I: Transcendental Functions*, Springer, 2005.
4. G. H. Hardy, *Divergent Series*, Oxford University Press, 1949.
5. NIST Digital Library of Mathematical Functions, *https://dlmf.nist.gov*.
6. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser, 1992.
7. J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*, Oxford University Press, 1996.
8. M. Unser, "Sampling—50 Years after Shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569–587, 2000.
9. M. Born and E. Wolf, *Principles of Optics*, 7th ed., Cambridge University Press, 1999.
10. A. Messiah, *Quantum Mechanics*, Vol. I. Amsterdam: North-Holland Publishing Company, 1966.

LaPS laboratory, Badji Mokhtar University–Annaba,

B.P.12, Annaba, 23000, Algeria.

E-mail address: ahmed.sadoun@univ-annaba.dz, halim.zeghdoudi@univ-annaba.dz